# Appendixes

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## *Appendix 1*

# Let's Revive an Old Idea

There are several ways to introduce the trigonometric functions: we can define them as ratios of sides in a right triangle, or in terms of the $x$- and $y$-coordinates of a point $P$ on the unit circle, or as "wrapping functions" from the reals to some subset of the reals, or again as certain power series of the independent variable. Each approach has its merits, but clearly not all are equally suitable in the classroom. As I have mentioned in the Preface, the so-called "New Math" has imposed on trigonometry the language and formalities of abstract set theory—certainly not the best way to motivate the beginning student. I would like to suggest that we go back to an old idea: interpret the trigonometric functions as *projections*. To preempt criticism, let me say from the outset that nothing in this approach is new, but it reflects a shift in emphasis from the abstract to the practical. Let's not forget that trigonometry is, first and foremost, a *practical* discipline, born out of and deeply rooted in applications.

In figure 100, let $P(x, y)$ be a point on the unit circle, and let the angle between the positive $x$-axis and $OP$ be $\theta$ (measured in degrees or radians). We define $\cos\theta$ and $\sin\theta$ as the *projections of OP on the x- and y-axes*, respectively. Since $OP = 1$, these projections are simply the $x$- and $y$-coordinates of the point $P$:

$$\cos\theta = OR = x, \quad \sin\theta = PR = y.$$

The tangent function is defined as $\tan\theta = y/x$, but this too can be viewed as a projection: referring again to fig. 100, draw the vertical tangent line to the unit circle at the point $S(1, 0)$, and call this line the $t$-axis. Extend $OP$ until it meets the $t$-axis at $Q$. We have $\tan\theta = y/x = PR/OR$. But triangles $OPR$ and $OQS$ are similar, and therefore $PR/OR = QS/OS$. Recalling that $OS = 1$ and denoting the line segment $QS$ by $t$, we have

$$\tan\theta = QS = t.$$

Thus $\tan\theta$ *is the projection of OP on the t-axis*. The function $\cot\theta$ can be similarly defined as the projection of $OP$ on the horizontal tangent line to the circle at $T(0, 1)$ (fig. 101); we have

$$\cot\theta = QT = c.$$

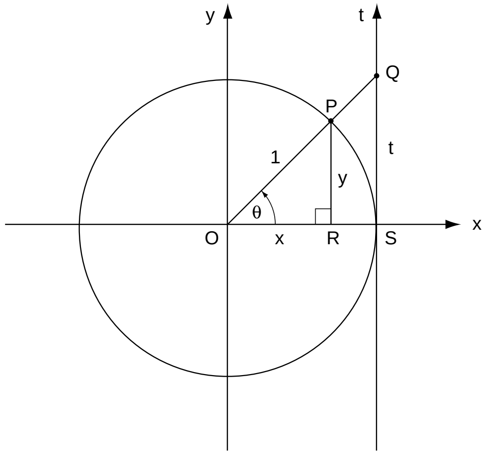


**FIG. 100.** The cosine, sine, and tangent functions as projections of the unit circle: $OR = x = \cos\theta$, $RP = y = \sin\theta$, $SQ = t = \tan\theta$.

So far the segments *OR*, *PR*, and *QS* were nondirected, but let us now think of them as directed line segments. This immediately leads us to conclude that $\tan\theta$ is positive for $0^\circ < \theta < 90^\circ$ (i.e., in quadrant I) and negative for $270^\circ < \theta < 360^\circ$ (quadrant IV). If $\theta$ is in quadrant II, we project *OP backward* until it meets the *t*-axis at *Q* (fig. 102); since triangles *OPR* and *OP′R′* are congruent, we have $\tan\theta = PR/OR = P'R'/OR' = SQ/1 = t$, so that $\tan\theta$ is now the negative line segment *SQ*. If $\theta$ is in quadrant III, projecting *OP* backward gives us again a positive value for *SQ*; we have here a geometric demonstration of the identity $\tan(\theta + 180^\circ) = \tan\theta$.

As for $\sec\theta$ and $\csc\theta$, they too can be interpreted (indeed defined) as projections: again let *P* be a point on the unit circle (fig. 103); draw the tangent line to the circle at *P* and extend it

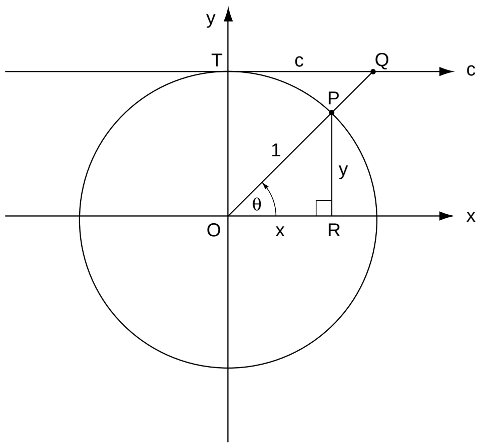


**FIG. 101.** The cotangent function as a projection: $TQ = c = \cot\theta$.

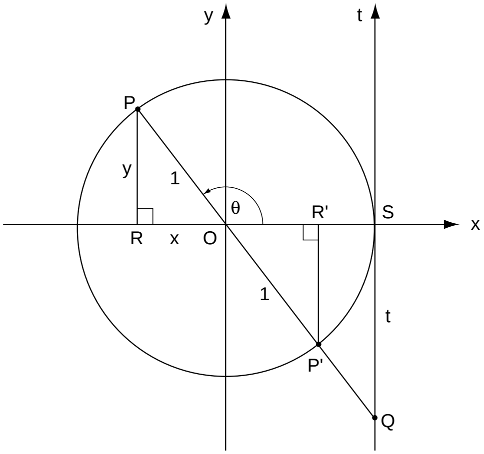


**FIG. 102.** The case when $\theta$ is in quadrant II.

until it meets the $x$- and $y$-axes at $M$ and $N$, respectively. We have $\angle OPM = 90^\circ$, so that triangles $OPR$ and $OMP$ are similar; hence $\sec\theta = 1/x = OP/OR = OM/OP = OM/1$, so that the line segment $OM$ represents the value of $\sec\theta$ (again, it is a directed line segment, being negative when $P$ is in quadrants II and III). Similarly, the line segment $ON$ represents the value of $\csc\theta$. Moreover, since $M$ and $N$ always lie outside the circle, we see that the range of $\sec\theta$ and $\csc\theta$ is $(-\infty, -1] \cup [1, \infty)$.

Viewing all six trigonometric functions as projections allows us to see, quite literally, how these functions vary with $\theta$: we only need to follow the various line segments as $P$ moves around the circle. For example, to an observer watching from above, $\cos\theta$ would appear as the to-and-fro motion of the "shadow"

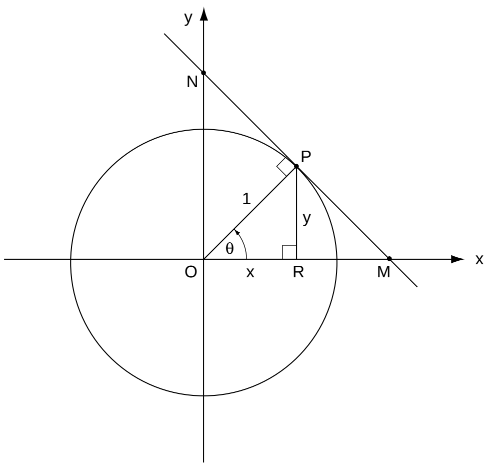


**FIG. 103.** The secant and cosecant functions as projections: $OM = 1/x = \sec\theta$, $ON = 1/y = \csc\theta$.

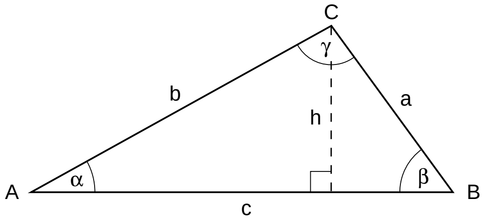


**FIG. 104.** The Law of Sines and the Law of Cosines as projections.

of $P$ on the $x$-axis. This is even more dramatically illustrated in the case of $\tan\theta$: as $\theta$ increases, the shadow of $P$ on the $t$-axis (point $Q$) rises slowly at first, than at an ever-increasing rate until it disappears at infinity as $\theta$ approaches 90°. Like the sweeping beam of a strobe light illuminating a dark wall, we have here a vivid graphic illustration of the peculiar behavior of $\tan\theta$ near its asymptotes.[1]

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The concept of projection can be put to good use not only in defining the trigonometric functions. Consider any triangle $ABC$ (fig. 104). Drop the perpendicular $h$ from $C$ to $AB$. The projections of the sides $a$ and $b$ on $h$—let's call them the "vertical projections"—must of course be equal; hence

$$a \sin\beta = b \sin\alpha, \tag{1}$$

or $a/\sin\alpha = b/\sin\beta$. Repeating this argument for sides $b$ and $c$, we get $b/\sin\beta = c/\sin\gamma$; thus $a/\sin\alpha = b/\sin\beta = c/\sin\gamma$, which is the Law of Sines.

On the other hand, the projections of $a$ and $b$ on $AB$ (the "horizontal projections") must add up to the length of $AB$, that is, to $c$; we thus have

$$c = a\cos\beta + b\cos\alpha, \tag{2}$$

with similar equations for $a$ and $b$ (note that the formula is valid even if one angle, say $\alpha$, is obtuse, in which case $\cos\alpha$ will be negative). It would be entirely appropriate to call equation (2) the Law of Cosines (a name usually given to the more familiar formula $c^2 = a^2 + b^2 - 2ab\cos\gamma$)—all the more so because it involves *two* cosines, thus justifying the plural "s." As an immediate consequence of equation (2) we have

$$c \le a + b, \tag{3}$$

with equality if, and only if, $\alpha = \beta = 0°$. This is the famous *triangle inequality*.

Because it involves five variables—two angles and three sides—the usefulness of equation (2) for solving triangles is rather limited. We can, however, use equations (1) and (2) together to reduce the number of variables. Squaring equation (2), we have

$$\begin{aligned} c^2 &= a^2\cos^2\beta + b^2\cos^2\alpha + 2ab\cos\alpha\cos\beta \\ &= a^2(1-\sin^2\beta) + b^2(1-\sin^2\alpha) + 2ab\cos\alpha\cos\beta \\ &= a^2 + b^2 - (a\sin\beta)(a\sin\beta) - (b\sin\alpha)(b\sin\alpha) \\ &\quad + 2ab\cos\alpha\cos\beta. \end{aligned}$$

Using equation (1), we can write this as

$$\begin{aligned} &= a^2 + b^2 - (a\sin\beta)(b\sin\alpha) \\ &\quad - (a\sin\beta)(b\sin\alpha) + 2ab\cos\alpha\cos\beta \\ &= a^2 + b^2 + 2ab(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= a^2 + b^2 + 2ab\cos(\alpha+\beta). \end{aligned}$$

But $\cos(\alpha+\beta) = \cos(180^\circ - \gamma) = -\cos\gamma$; we thus get

$$c^2 = a^2 + b^2 - 2ab\cos\gamma, \tag{4}$$

which is the familiar form of the Law of Cosines. Thus the sine and the cosine laws simply express the fact that in a triangle, the perpendicular from any vertex to the opposite base is the vertical projection of either of the adjacent sides, and the base is the sum of their horizontal projections.

## NOTE

1. Regrettably, many textbooks plot the graphs of $\tan\theta$ and $\cot\theta$ based on just a few scattered values of $\theta$ (in one case, as few as three!), arbitrarily chosen between $-90^\circ$ and $90^\circ$. Surely no real understanding of the peculiar behavior of these functions can be gained this way.

# *Appendix 2*

## Barrow's Integration of $\sec\phi$

We give here in modern notation Isaac Barrow's proof (1670) that $\int_0^\phi \sec t\,dt = \ln\tan(45^\circ + \phi/2)$ (see p. 178). This proof marks the first use of the technique of decomposition into partial fractions.[1]

We begin with

$$\begin{aligned}\sec\phi &= \frac{1}{\cos\phi} = \frac{\cos\phi}{\cos^2\phi}\\ &= \frac{\cos\phi}{1-\sin^2\phi} = \frac{\cos\phi}{(1+\sin\phi)(1-\sin\phi)}\\ &= \frac{1}{2}\left(\frac{\cos\phi}{1+\sin\phi} + \frac{\cos\phi}{1-\sin\phi}\right).\end{aligned}$$

Therefore,

$$\int \sec\phi\,d\phi = \frac{1}{2}\int\left(\frac{\cos\phi}{1+\sin\phi} + \frac{\cos\phi}{1-\sin\phi}\right)d\phi.$$

The first term inside the integral is of the form $u'/u$, where $u$ is a function of $\phi$, and the second term is of the form $-u'/u$; using the formula $\int(u'/u)\,d\phi = \ln|u(\phi)| + C$, we have

$$\int \sec\phi\,d\phi = \frac{1}{2}[\ln|1+\sin\phi| - \ln|1-\sin\phi|] + C.$$

Using the familiar division property of logarithms, this becomes

$$= \frac{1}{2}\ln\left|\frac{1+\sin\phi}{1-\sin\phi}\right| + C.$$

We multiply and divide the expression inside the logarithm by $(1+\sin\phi)$; in the denominator we get $(1-\sin\phi)(1+\sin\phi) = 1-\sin^2\phi = \cos^2\phi$, and so

$$\int \sec\phi\,d\phi = \frac{1}{2}\ln\frac{(1+\sin\phi)^2}{\cos^2\phi} + C.$$

Using the power property of logarithms, this becomes

$$= \ln\left|\frac{1+\sin\phi}{\cos\phi}\right| + C.$$

Writing $\phi = 2(\phi/2)$ and using the double-angle formulas for sine and cosine, we get

$$= \ln\left|\frac{1 + 2\sin\phi/2\cos\phi/2}{\cos^2\phi/2 - \sin^2\phi/2}\right| + C$$

$$= \ln\left|\frac{(\cos\phi/2 + \sin\phi/2)^2}{(\cos\phi/2 + \sin\phi/2)(\cos\phi/2 - \sin\phi/2)}\right| + C$$

$$= \ln\left|\frac{\cos\phi/2 + \sin\phi/2}{\cos\phi/2 - \sin\phi/2}\right| + C.$$

Finally, dividing the numerator and denominator of the expression inside the logarithm by $\cos\phi/2$, we get

$$= \ln\left|\frac{1 + \tan\phi/2}{1 - \tan\phi/2}\right| + C$$

$$= \ln\left|\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right| + C.$$

Turning now to the definite integral, we have

$$\int_0^\phi \sec t\,dt = \ln\left|\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right| - \ln\tan\frac{\pi}{4}.$$

But $\ln\tan\pi/4 = \ln 1 = 0$, so we finally have

$$\int_0^\phi \sec t\,dt = \ln\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$

(we have dropped the absolute value sign because in the relevant range of $\phi$, namely $-\pi/2 < \phi < \pi/2$, $\tan(\pi/4 + \phi/2)$ is positive).

Today one solves this integral by the substitution $u = \tan t/2$, $du = [(1/2)\sec^2 t/2]\,dt$, but it is still a tough nut to crack for beginning calculus students.

## Source

1. This derivation is based on the article, "An Application of Geography to Mathematics: History of the Integral of the Secant" by V. Frederick Rickey and Philip M. Tuchinsky, in the *Mathematics Magazine*, vol. 53, no. 3 (May 1980).

# *Appendix 3*

## Some Trigonometric Gems

"Beauty is in the eye of the beholder," says an old proverb. I have collected here a sample of trigonometric formulas that will appeal to anyone's sense of beauty. Some of these formulas are easy to prove, others will require some effort on the reader's behalf. My selection is entirely subjective: trigonometry abounds in beautiful formulas, and no doubt the reader can find many others that are equally appealing.

### 1. Finite Formulas

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^4\alpha - \cos^4\alpha = \sin^2\alpha - \cos^2\alpha$$

$$\sec^2\alpha + \csc^2\alpha = \sec^2\alpha\csc^2\alpha$$

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$$

$$\tan(45^\circ+\alpha)\tan(45^\circ-\alpha) = \cot(45^\circ+\alpha)\cot(45^\circ-\alpha) = 1$$

$$\sin(\alpha+\beta+\gamma) + \sin\alpha\sin\beta\sin\gamma$$
$$= \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\gamma\cos\alpha + \sin\gamma\cos\alpha\cos\beta$$

Let $f(\alpha,\beta) = \cos^2\alpha + \sin^2\alpha\cos 2\beta$;

then $f(\alpha,\beta) = f(\beta,\alpha)$.

Let $g(\alpha,\beta) = \sin^2\alpha - \cos^2\alpha\cos 2\beta$;

then $g(\alpha,\beta) = g(\beta,\alpha)$.

In the following relations, let $\alpha+\beta+\gamma = 180^\circ$:

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\alpha/2\cos\beta/2\cos\gamma/2$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = -4\cos 3\alpha/2\cos 3\beta/2\cos 3\gamma/2$$

$$\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\alpha/2\sin\beta/2\sin\gamma/2$$

$$\cos^2 2\alpha + \cos^2 2\beta + \cos^2 2\gamma - 2\cos 2\alpha \cos 2\beta \cos 2\gamma = 1$$

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$ [1]

$$0 < \sin\alpha + \sin\beta + \sin\gamma \leq (3\sqrt{3})/2,$$

with equality if, and only if, $\alpha = \beta = \gamma = 60^\circ$.

In any acute triangle,

$$\tan\alpha + \tan\beta + \tan\gamma \geq 3\sqrt{3},$$

with equality if, and only if, $\alpha = \beta = \gamma = 60^\circ$.

In any obtuse triangle,

$$-\infty < \tan\alpha + \tan\beta + \tan\gamma < 0.$$

## 2. Infinite Formulas[2]

$$\sin x = x - x^3/3! + x^5/5! - + \cdots$$

$$\cos x = 1 - x^2/2! + x^4/4! - + \cdots$$

$$\sin x = x(1 - x^2/\pi^2)(1 - x^2/4\pi^2)(1 - x^2/9\pi^2)\cdots$$

$$\cos x = (1 - 4x^2/\pi^2)(1 - 4x^2/9\pi^2)(1 - 4x^2/25\pi^2)\cdots$$

$$\tan x = 8x[1/(\pi^2 - 4x^2) + 1/(9\pi^2 - 4x^2) + 1/(25\pi^2 - 4x^2) + \cdots]$$

$$\sec x = 4\pi[1/(\pi^2 - 4x^2) - 3/(9\pi^2 - 4x^2) + 5/(25\pi^2 - 4x^2) - + \cdots]$$

$$(\sin x)/x = \cos x/2 \cos x/4 \cos x/8 \cdots$$

$$(1/4)\tan \pi/4 + (1/8)\tan \pi/8 + (1/16)\tan \pi/16 + \cdots = 1/\pi$$

$$\tan^{-1} x = x - x^3/3 + x^5/5 - + \cdots, \qquad -1 < x < 1.$$

### Notes

1. The companion formula

$$\cot\alpha + \cot\beta + \cot\gamma = \cot\alpha\cot\beta\cot\gamma$$

holds true only for $\alpha + \beta + \gamma = 90^\circ$.

2. For a sample of Fourier series, see figure 95, p. 206. Numerous other trigonometric series can be found in *Summation of Series*, collected by L. B. W. Jolley (1925; rpt. New York: Dover, 1961), chaps. 14 and 16.

# *Appendix 4*

## Some Special Values of $\sin\alpha$

$$\sin 0^\circ = 0 = \frac{\sqrt{0}}{2}, \quad \sin 30^\circ = \frac{1}{2} = \frac{\sqrt{1}}{2}, \quad \sin 45^\circ = \frac{\sqrt{2}}{2},$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \sin 90^\circ = 1 = \frac{\sqrt{4}}{2}.$$

$$\sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4},$$

$$\sin 75^\circ = \frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

$$\sin 18^\circ = \frac{-1+\sqrt{5}}{4},$$

$$\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4},$$

$$\sin 54^\circ = \frac{1+\sqrt{5}}{4},$$

$$\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

The last four values are related to a regular pentagon. For example, the side of a regular pentagon inscribed in a unit circle is $2\sin 36^\circ$, its diagonal is $2\sin 72^\circ$, and their ratio is $2\sin 54^\circ$. These values are also related to the "golden section": the ratio in which a line segment must be divided if the whole segment is to the longer part as the longer part is to the shorter. This ratio, denoted by $\phi$, is equal to $(1+\sqrt{5})/2 \approx 1.618$, that is, to $2\sin 54^\circ$.

Repeated use of the half-angle formula for the sine leads to the following expressions, where $n = 1, 2, 3, \ldots$:

$$\sin\frac{45^\circ}{2^n} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2}$$

($n+1$ nested square roots)

$$\sin\frac{15^\circ}{2^n} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}}}{2}$$

(n + 2 nested roots)

$$\sin\frac{18^\circ}{2^n} = \frac{\sqrt{8-2\sqrt{8+2\sqrt{8+\cdots+2\sqrt{10+2\sqrt{5}}}}}}{4}$$

(n + 2 nested roots)