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$(\sin x)/x$

I call our world Flatland, not because we call it so, but to make its nature clear to you, my happy readers, who are privileged to live in Space.
—Edwin A. Abbott, *Flatland* (1884)

Students of calculus encounter the function $(\sin x)/x$ early in their study, when it is shown that $\lim_{x\to 0}(\sin x)/x = 1$; this result is then used to establish the differentiation formulas $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. Once this has been done, however, the function is soon forgotten, and the student rarely sees it again. This is unfortunate, for this simple-looking function not only has some remarkable properties, but it also shows up in many applications, sometimes quite unexpectedly.

We note, to begin with, that the function is defined for all values of x except 0; but we also know that as x gets smaller and smaller, the ratio $(\sin x)/x$—provided x is measured in radians—tends to 1. This provides us with a simple example of a *removable singularity*: we can simply *define* the value of $(\sin 0)/0$ to be 1, and this definition will assure the continuity of the function near $x = 0$.

Let us denote our function by $f(x)$ and plot it for various values of x; the resulting graph is shown in figure 61. Two features make this graph distinct from that of the function $g(x) = \sin x$: first, it is symmetric about the y-axis; that is, $f(-x) = f(x)$ for all values of x (in the language of algebra, $f(x)$ is an *even function*, so called because the simplest functions with this property are of the form $y = x^n$ for even values of n). By contrast, the function $g(x) = \sin x$ has the property that $g(-x) = -g(x)$ for all x (functions with this property are called *odd functions*, for example $y = x^n$ for odd values of n). To prove that $f(x) = (\sin x)/x$ is even, we simply note that $f(-x) = \sin(-x)/(-x) = (-\sin x)/(-x) = (\sin x)/x = f(x)$.

Second, unlike the graph of $\sin x$, whose up-and-down oscillations are confined to the range from -1 to 1 (that is, the sine

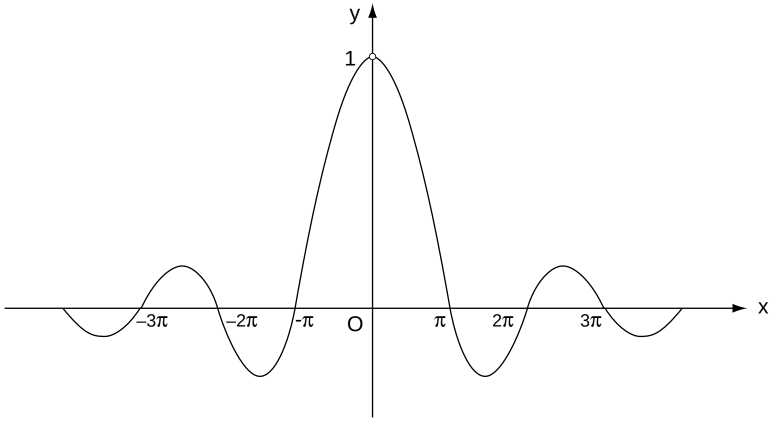

FIG. 61. The graph of $(\sin x)/x$.

wave has a constant amplitude 1), the graph of $(\sin x)/x$ represents *damped* oscillations whose amplitude steadily decreases as $|x|$ increases. Indeed, we may think of $f(x)$ as a sine wave squeezed between the two *envelopes* $y = \pm 1/x$. We now wish to locate the *extreme points* of $f(x)$—the points where it assumes its maximum or minimum values. And here a surprise is awaiting us. We know that the extreme points of $g(x) = \sin x$ occur at all odd multiples of $\pi/2$, that is, at $x = (2n+1)\pi/2$. So we might expect the same to be true for the extreme points of $f(x) = (\sin x)/x$. This, however, is not the case. To find the extreme point, we differentiate $f(x)$ using the quotient rule and equate the result to zero:

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = 0. \tag{1}$$

Now if a ratio is equal to zero, then the numerator itself must equal to zero, so we have $x\cos x - \sin x = 0$, from which we get

$$\tan x = x. \tag{2}$$

Equation (2) cannot, unfortunately, be solved by a closed formula in the same manner as, say, a quadratic equation can; it is a *transcendental equation* whose roots can be found graphically as the points of intersection of the graphs of $y = x$ and $y = \tan x$ (fig. 62). We see that there is an infinite number of these points, whose x-coordinates we will denote by x_n. As x increases in absolute value, these points rapidly approach the asymptotes of $\tan x$, that is, $(2n+1)\pi/2$; these, of course, are the extreme points of $\sin x$. This is to be expected, since as $|x|$ increases, $1/|x|$ decreases at a rate that itself is decreasing, so

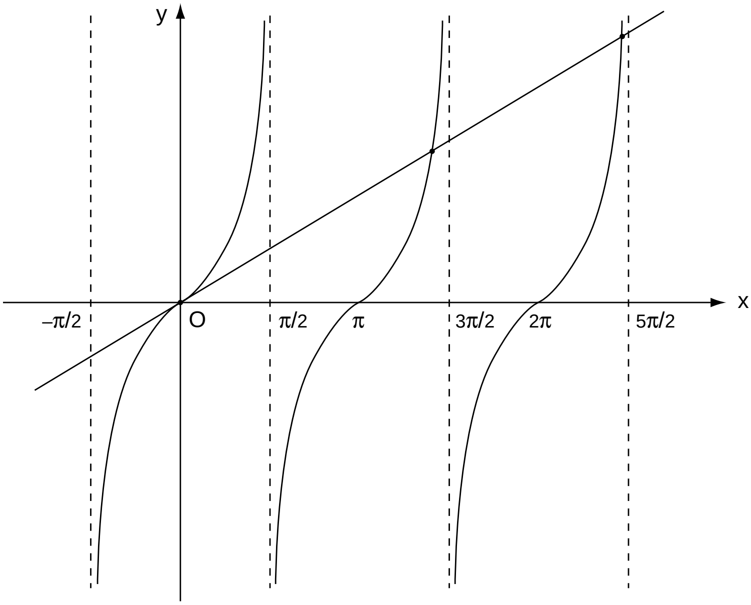

Fig. 62. The roots of $\tan x = x$.

that its effect on the variation of $\sin x$ steadily diminishes. The first few values of x_n are given in table 2.

The peculiar behavior of the extreme points of $(\sin x)/x$ is in marked contrast to another kind of damped oscillations, those represented by the function $e^{-x}\sin x$. Here the extreme points are shifted leftward by a *constant* amount $\pi/4$ relative to those of $\sin x$, as the reader can easily verify.

Having explored the general shape of the graph of $f(x)$, the next question of interest is to find the area under the graph from, say, $x = 0$ to some other x. This area is given by the definite integral

$$\int_0^x \frac{\sin t}{t}\,dt.$$

Table 2

n	x_n	$f(x_n)$
0	0.00	1.000
1	$4.49 = 2.86\pi/2$	−0.217
2	$7.73 = 4.92\pi/2$	0.128
3	$10.90 = 6.94\pi/2$	−0.091
4	$14.07 = 8.96\pi/2$	0.071

where we have denoted the variable of integration by t to distinguish it from the upper limit x. In order to evaluate this integral, we would first seek to find the indefinite integral, or antiderivative, of $(\sin x)/x$. Alas, this is a futile attempt! It is one of the curious facts of calculus that the antiderivatives of many simple-looking functions cannot be expressed in terms of the "elementary functions," that is, polynomials and ratios of polynomials, exponential and trigonometric functions and their inverses, and any finite combination of these functions. The function $(\sin x)/x$ belongs to this group, as do $(\cos x)/x$, e^x/x, and e^{x^2}. This, of course, does not mean that the antiderivatives of these functions do not exist—it only means that they cannot be expressed in "closed form" in terms of the elementary functions. Indeed, the above integral, regarded as a function of its upper limit x, *defines* a new, "higher" function known as the *sine integral* and denoted by $\operatorname{Si}(x)$:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$$

Although we cannot express $\operatorname{Si}(x)$ in terms of the elementary functions, we can nevertheless compute its values and plot them on a graph (fig. 63). This is done by writing the sine function as a power series, $\sin x = x - x^3/3! + x^5/5! - + \cdots$, dividing each term by x, and then integrating term by term. The result is

$$\operatorname{Si}(x) = x - x^3/3 \cdot 3! + x^5/5 \cdot 5! - + \cdots,$$

a series that converges for all x.

As we let the upper limit x increase without bound, will the area under the graph approach a limit? The answer is yes; it can be shown that this limit is $\pi/2$;[1] in other words,

$$\operatorname{Si}(\infty) = \int_0^\infty \frac{\sin x}{x} dx = \pi/2. \tag{3}$$

This important integral is known as the *Dirichlet integral*, after the German mathematician Peter Gustav Lejeune Dirichlet (1805–1859). An unexpected by-product of this integral is obtained by replacing $\sin x$ with $\sin kx$, where k is a constant, and then making the substitution $u = kx$. We find that the new integral has the value $\pi/2$ or $-\pi/2$, depending on whether k is positive or negative (in the latter case the upper limit becomes $-\infty$, so that the further substitution $v = -u$ will result in $-\pi/2$). We thus have the following result:

$$(2/\pi)\int_0^\infty \frac{\sin kx}{x} dx = \begin{cases} 1 & \text{for } k > 0 \\ 0 & \text{for } k = 0 \\ -1 & \text{for } k < 0 \end{cases}. \tag{4}$$

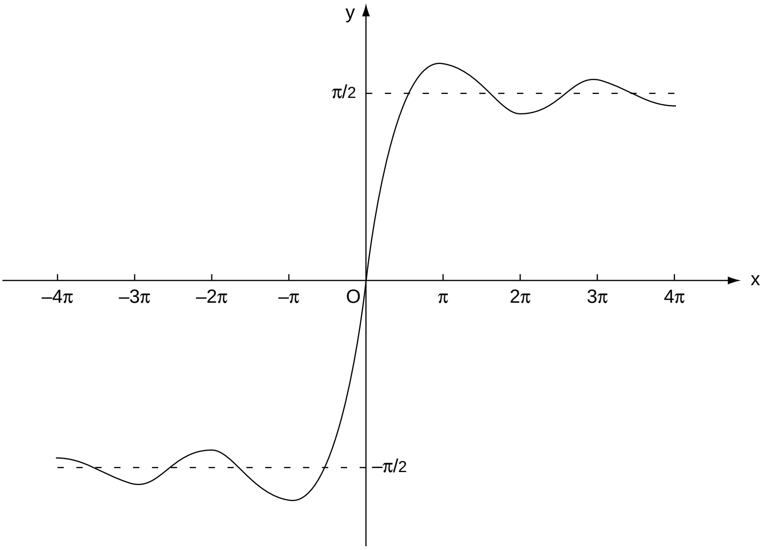

FIG. 63. The graph of $\mathrm{Si}(x) = \int_0^x (\sin t)/t\, dt$.

But the expression on the right, considered as a function of k, is the "sign function" shown in figure 64. We have here one of the simplest examples of an integral representation of a function; the need for such a representation often arises in applied mathematics. The integral on the left is known as *Dirichlet's discontinuity factor*.

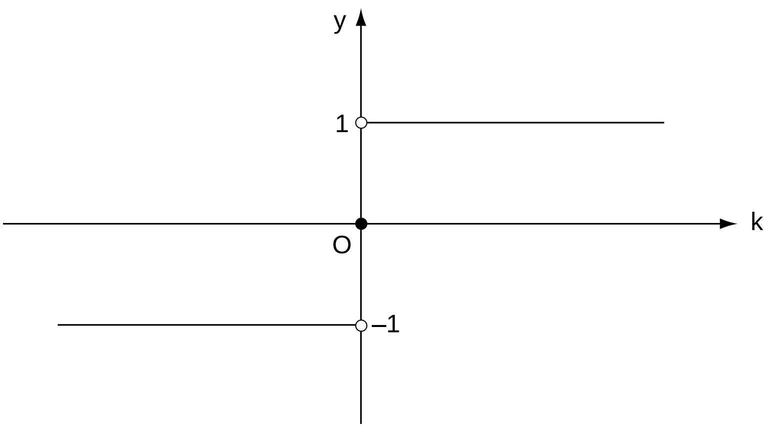

FIG. 64. The graph of sign x.

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Of the many occurrences of the function $(\sin x)/x$, we will consider here one taken from geography. Early in school we learn that our earth is round, though it took many centuries before this fact was universally accepted (the last of the flat-earth believers finally gave up when images from spacecraft showed the earth to be round). Indeed, to the uninitiated it is not at all immediately obvious that we live on a round world—certainly most of our daily experiences could more naturally be explained on the basis of a flat earth. It is only indirectly, chiefly through astronomical observations, that we know the earth is round.

In his classic mathematical novel *Flatland*, Edwin A. Abbott describes the world of two-dimensional, antlike creatures who can move forward and backward and left and right, but not up and down. If these "flatlanders" were to inhabit our earth, they would be unaware of its sphericity: from their viewpoint the earth would seem as flat as a tabletop. But one day they decide to explore their world, intent on discovering its underlying geometry. Starting at the North Pole and using a stretched rope as a compass, they draw circles around the pole with ever larger radii. Then they measure the circumference of each circle and express it in terms of its radius. Back at home, they put to a test what they had learned in school—that the ratio of the circumference of a circle to its radius is the same for all circles, about 6.28. For small circles they find, to their delight, that this indeed seems to be the case. But as the circles get larger, reassurance turns into doubt, then disappointment: our flatlanders find that the circumference-to-radius ratio is not constant after all.

To see the reason for this, let us take advantage of the privilege granted to us, humans, by being three-dimensional creatures: *we* know that our world is round. Let us denote the radius of the earth—assumed to be a perfect sphere—by R. To find the circumference of a circle around the North Pole, we need to know its radius, and this depends on the geographic latitude of the circle. If, for simplicity, we measure the latitude not from the equator, as is done in geography, but from the North Pole, then the radius of a circle of latitude θ is $r = R\sin\theta$ (see fig. 65), and its circumference is

$$c = 2\pi R \sin\theta. \qquad (5)$$

This result, of course, is entirely satisfactory to us three-dimensionals, but to our two-dimensional earth dwellers it is totally meaningless. They have no idea that they live on a curved surface, and if someone told them that their flat

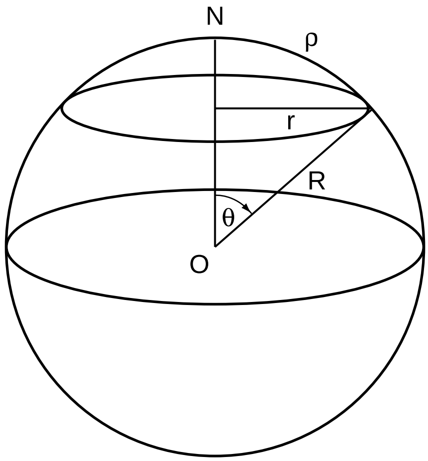

Fig. 65. Circle of latitude θ on the globe.

world is actually spherical, they would be puzzled indeed. For them a quantity such as R, taken from the third dimension and not being capable of direct measurement, is as meaningless as for an elementary school pupil to find the volume of a four-dimensional "sphere."

To make the formula meaningful, we must express it in terms of variables that our inhabitants can measure. Indeed, the most important variable, from their point of view, is the radius of the circle, *as measured on the surface of the earth*. Let us denote this radius by the Greek letter ρ (rho). If we measure θ in radians, we have $\rho = R\theta$, hence $R = \rho/\theta$. Substituting this expression in equation (5), we get

$$c = 2\pi\rho\frac{\sin\theta}{\theta}. \tag{6}$$

Thus the circumference depends not only on the radius, but also on the latitude.

Before contemplating the consequences of this formula, we may wonder how our inhabitants would measure the latitude θ when they are unaware of the sphericity of their world. They might get a clue by watching the sky above them: they might notice, as mariners in ancient times did, that the entire celestial sphere appears to rotate once every 24 hours around one star that seems to be standing still—the North Star. Moreover, the height of the North Star above the horizon steadily decreases as they travel southward; in fact, they find that the angle θ between the North Star and the zenith—the point on the celestial sphere directly above the observer—is proportional to the distance ρ from the North Pole (as follows from the equation $\rho = R\theta$).

And now our inhabitants are ready to put to a test what they had learned in their geometry class. For small latitudes (angular distances from the North Pole), they will find that the ratio c/ρ does indeed appear to be constant, or nearly so, as table 3 shows. Their surveyors might at first dismiss the small discrepancies from constancy as due to errors of measurement, but it will soon become clear that the ratio c/ρ is *not* constant but decreases with θ, as table 4 shows. (The 4 in the last entry reflects the fact that the distance from the pole to the equator is exactly one-fourth the circumference of the equator.) Had our inhabitants extended the table further—that is, into the southern hemisphere—the ratio c/ρ would continue to decrease, until it becomes zero at 180°(the South Pole). Still unaware that their world is round, they will have lost any remaining faith in what they had learned about the constancy of the circumference-to-radius ratio.[2] But perhaps some wise flatlander might interpret these findings differently and conclude that the world they live in is actually curved. That wise flatlander would go down in history as the discoverer of the third dimension.

We can actually draw a map of the world as the flatlanders would see it. Known as an azimuthal equidistant map, it shows all "straight line" distances and directions from a fixed, preselected point, located at the center of the map, to any other point on the globe. (A "straight line" between two points on a sphere is an arc of the *great circle* connecting them—the circle passing through the two points and having its center at the center of the sphere [fig. 66]; it represents the shortest distance between the points.) Figure 67 shows such a map centered on San Francisco; we see that the direct route from San Francisco to Moscow passes over the North Pole, and that Moscow is closer

Table 3

θ	c/ρ
0°	6.283
1°	6.283
2°	6.282
3°	6.280
4°	6.278
5°	6.275

Note: in using equation (6), all angles must first be converted to radians ($1° = \pi/180$ radians).

Table 4

θ	c/ρ
10°	6.251
20°	6.156
30°	6.000
40°	5.785
50°	5.516
60°	5.196
70°	4.833
80°	4.432
90°	4.000

to San Francisco than San Francisco is to Rio de Janeiro. Notice that Africa, Antarctica, and Australia appear extremely distorted, both in shape and size; this is because a circle of radius ρ centered at the fixed point has circumference $2\pi\rho$ on the map, while on the globe its circumference is $2\pi\rho(\sin\theta)/\theta$, where θ has the same meaning as in figure 65, but with the fixed point replacing the North Pole. Concentric circles around the fixed point are thus exaggerated by the ratio 1: $[(\sin\theta)/\theta]$, or $\theta/\sin\theta$, relative to their true size on the globe. This exaggeration factor increases with θ and becomes infinite when $\theta = 180°$, that is, at *the antipode* of the fixed point (the point opposite to it on the globe). On an azimuthal equidistant map, the entire outer boundary represents the antipode of the central point; it marks the edge of the universe of our flatlanders—the farthest point they can reach in any direction. They discover that their universe, though unbounded, is finite.

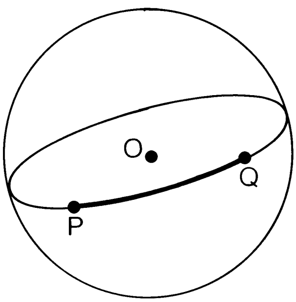

FIG. 66. Arc of a great circle.

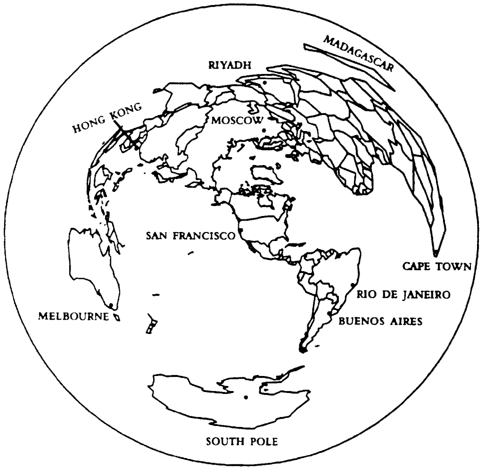

Fig. 67. Azimuthal equidistant world map centered at San Francisco.

Notes and Sources

1. The proof is not elementary; see Richard Courant, *Differential and Integral Calculus* (London: Blackie & Son, 1956), vol. 1, pp. 251–253 and 444–450; for an alternative proof using integration in the complex plane, see Erwin Kreyszig, *Advanced Engineering Mathematics* (New York: John Wiley, 1979), pp. 735–736.

2. A similar situation arises in connection with the area of a "circle" of radius ρ (actually a spherical cap). This area is given by $A = 2\pi Rh$, where h is the height of the cap (the distance from its base to the surface of the sphere). Now $h = R(1 - \cos\theta) = 2R\sin^2(\theta/2)$, so that $A = 4\pi R^2 \sin^2(\theta/2) = 4\pi(\rho/\theta)^2 \sin^2(\theta/2) = \pi\rho^2\{[\sin(\theta/2)]/(\theta/2)\}^2$. A "correction factor" of $\{[\sin(\theta/2)]/(\theta/2)\}^2$ is thus needed if we want to find the ratio A/ρ^2.

Go to Chapter 11