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tan *x*

In his very numerous memoires, and especially in his great work, Introductio in analysin infinitorum *(1748), Euler displayed the most wonderful skill in obtaining a rich harvest of results of great interest. . . . Hardly any other work in the history of Mathematical Science gives to the reader so strong an impression of the genius of the author as the* Introductio.
—E. W. Hobson, *"Squaring the Circle": A History of the Problem* (1913)

Of the numerous functions we encounter in elementary mathematics, perhaps the most remarkable is the tangent function. The basic facts are well known: $f(x) = \tan x$ has its zeros at $x = n\pi\,(n = 0, \pm 1, \pm 2, \ldots)$, has infinite discontinuities at $x = (2n + 1)\pi/2$, and has period π (a function $f(x)$ is said to have a period P if P is the smallest number such that $f(x + P) = f(x)$ for all x in the domain of the function). This last fact is quite remarkable: the functions $\sin x$ and $\cos x$ have the common period 2π, yet their ratio, $\tan x$, reduces the period to π. When it comes to periodicity, the ordinary rules of the algebra of functions may not be valid: the fact that two functions f and g have a common period P does not imply that $f + g$ or fg too have the same period.[1]

As we saw in chapter 2, the tangent function has its origin in the "shadow reckoning" of antiquity. During the Renaissance it was resurrected—though without calling it "tangent"—in connection with the fledging art of perspective. It is a common experience that an object appears progressively smaller as it moves away from the observer. The effect is particularly noticeable when viewing a tall structure from the ground: as the angle of sight is elevated, features that are equally spaced vertically, such as the floors of a building, appear to be progressively shortened; and conversely, equal increments in the angle of elevation intercept the structure at points that are increasingly farther apart. A study by the famed Nürnberg artist Albrecht Dürer (1471–1528), one of the founders of perspective, clearly shows this effect (fig. 75).[2]

Dürer and his contemporaries were particularly intrigued by the extreme case of this phenomenon when the angle of elevation approaches 90° and the height seems to increase without limit. More intriguing still was the behavior of parallel lines in the plane: as they recede from the viewer they seem to get ever closer, ultimately converging on the horizon at a point called the

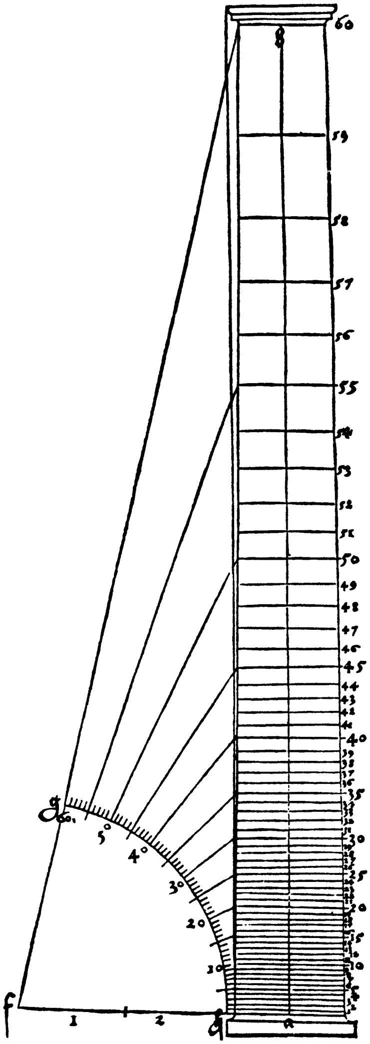

FIG. 75. A study by Albrecht Dürer.

"vanishing point." All these features can be traced to the behavior of $\tan x$ near 90°. Today, of course, we say that $\tan x$ *tends* to infinity as x approaches 90°, whereas *at* 90° it is undefined; but such subtleties were unknown to past generations, and until quite recently one could still find the statement "$\tan 90^\circ = \infty$" in many trigonometry textbooks.

But let us return to mundane matters. Around 1580 Viète stated a beautiful theorem that, unfortunately, has all but disappeared from today's textbooks: the Law of Tangents. It says that in any triangle,

$$\frac{a+b}{a-b} = \frac{\tan(\alpha+\beta)/2}{\tan(\alpha-\beta)/2}. \tag{1}$$

This theorem follows from the Law of Sines ($a/\sin\alpha = b/\sin\beta = c/\sin\gamma$) and the identities $\sin\alpha \pm \sin\beta = 2\sin(\alpha \pm \beta)/2 \cdot \cos(\alpha \mp \beta)/2$, but in Viète's time it was regarded as an independent theorem.[3] It can be used to solve a triangle when two sides and the included angle are given (the *SAS* case). Normally one would use the Law of Cosines ($c^2 = a^2 + b^2 - 2ab\cos\gamma$) to find the missing side, and then find one or the other of the remaining angles using the Law of Sines. However, because the cosine law involves addition and subtraction, it does not lend itself easily to logarithmic computations—practically the only means of solving triangles (or most other computations) before the hand-held calculator became available. With the tangent law one could avoid this difficulty: since one angle, say γ, is given, one could find $(\alpha+\beta)/2$, and with the help of equation (1) and a table of tangents find $(\alpha-\beta)/2$; from these two results the angles α and β are found, and finally the two missing sides from the sine law. With a calculator, of course, this is no longer necessary, which may explain why the Law of Tangents has lost much of its appeal. Still, its elegant, symmetric form should be a good enough reason to resurrect it from oblivion, if not as a theorem then at least as an exercise. And for those who enjoy mathematical *misteakes*—correct results derived incorrectly (such as 1ɸ/ɸ4 = 1/4)—the Law of Tangents provides ample opportunity: start with the right side of equation (1), "cancel" the 1/2 and the "tan" and replace Greek letters with corresponding Latin ones, and you get the left side.[4]

Other formulas involving $\tan x$ are just as elegant; for example, if α, β, and γ are the three angles of any triangle, we have

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \cdot \tan\beta \cdot \tan\gamma. \tag{2}$$

This formula can be proved by writing $\gamma = 180^\circ - (\alpha+\beta)$ and using the addition formula for the tangent. The formula is remarkable not only for its perfect symmetry, but also because it leads to an unexpected result. A well-known theorem from algebra says that if $x_1, x_2, \ldots, x_n$ are any positive numbers, then their arithmetic mean is never smaller than their geometric mean; that is,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Moreover, the two means are *equal* if, and only if, $x_1 = x_2 = \cdots = x_n$. Let us assume that our triangle is acute, so that all three angles have positive tangents. Then the theorem says that

$$\frac{\tan\alpha + \tan\beta + \tan\gamma}{3} \geq \sqrt[3]{\tan\alpha \cdot \tan\beta \cdot \tan\gamma}.$$

But in view of equation (2) this inequality becomes

$$\frac{\tan\alpha \cdot \tan\beta \cdot \tan\gamma}{3} \geq \sqrt[3]{\tan\alpha \cdot \tan\beta \cdot \tan\gamma}.$$

Cubing both sides gives

$$\tan\alpha \cdot \tan\beta \cdot \tan\gamma \geq \sqrt{27} = 3\sqrt{3}.$$

Thus in any acute triangle, the product (and sum) of the tangents of the three angles is never less than $3\sqrt{3} \approx 5.196$; and this minimum value is attained if, and only if, $\alpha = \beta = \gamma = 60^\circ$, that is, when the triangle is equilateral.

If the triangle is obtuse, then one of the three angles has a negative tangent, in which case the theorem does not apply; however, since the obtuse angle can vary only from 90° to 180°, its tangent ranges over the interval $(-\infty, 0)$, while the other two tangents remain positive and finite. Thus the product of the three tangents can assume any negative value.

Summing up, in any acute triangle we have $\tan\alpha \cdot \tan\beta \cdot \tan\gamma \geq 3\sqrt{3}$ (with equality if and only if the triangle is equilateral), and in any obtuse triangle we have $-\infty < \tan\alpha \cdot \tan\beta \cdot \tan\gamma < 0$.

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The tangent of a multiple of an angle provides us with another source of interesting formulas. In chapter 8 we obtained the formula

$$\tan(n+1)\alpha/2 = \frac{\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha}{\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha}.$$

We can make this slightly more useful by writing $\alpha/2 = \beta$ and $n + 1 = m$; then

$$\tan m\beta = \frac{\sin 2\beta + \sin 4\beta + \cdots + \sin 2(m-1)\beta}{\cos 2\beta + \cos 4\beta + \cdots + \cos 2(m-1)\beta}.$$

But even in this form the formula has only limited usefulness, because it expresses the tangent of a multiple of an angle in terms of the sines and cosines of other angles. It would be desirable if we could express $\tan n\alpha$ in terms of the *tangent* of α (and α alone, not its multiples). Fortunately, this can be done.

We start with the familiar addition formula for the tangent,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}.$$

From this we get

$$\tan 2\alpha = \tan(\alpha+\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

$$\tan 3\alpha = \tan(2\alpha+\alpha) = \frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha \cdot \tan\alpha}$$

$$= \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha},$$

and so on. After a few steps a pattern begins to emerge: we discover that the coefficients are the same as those appearing in the expansion of $(1+x)^n$ in powers of x—the familiar *binomial coefficients*—except that they alternate between the numerator and denominator in a zigzag pattern (beginning with the first term in the denominator) and their signs alternate in pairs.[5] Figure 76, taken from an early nineteenth-century trigonometry textbook, shows the pattern up to $\tan 7\alpha$. Taking the signs into account, we can arrange the coefficients in a "Pascal tangent triangle":

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & -1 & & & \\
 & & 1 & & 3 & & -3 & & -1 & & \\
 & 1 & & 4 & & -6 & & -4 & & 1 & \\
1 & & 5 & & -10 & & -10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}$$

(the top entry is 1 because $\tan 0 = 0 = 0/1$). We notice that all entries in the first two diagonal lines (counting from left to right) are positive, those of the next two lines are negative, and so on.

The presence of the binomial coefficients in a formula that appears to have nothing to do with the expansion of $(1+x)^n$ is

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$$\tan a = \tan a$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

$$\tan 4a = \frac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a}$$

$$\tan 5a = \frac{5 \tan a - 10 \tan^3 a + \tan^5 a}{1 - 10 \tan^2 a + 5 \tan^4 a}$$

$$\tan 6a = \frac{6 \tan a - 20 \tan^3 a + 6 \tan^5 a}{1 - 15 \tan^2 a + 15 \tan^4 a - \tan^6 a}$$

$$\tan 7a = \frac{7 \tan a - 35 \tan^3 a + 21 \tan^5 a - \tan^7 a}{1 - 21 \tan^2 a + 35 \tan^4 a - 7 \tan^6 a}$$

$$\tan 8a = \&c.$$

3

§ 45. If we consider the foregoing formulæ for the sine and cosine of the multiple angles expressed wholly in terms of the sines and cosines of the simple angles, and their successive powers, both in relation to the order in which these powers, and to that in which their coefficients, occur, we shall perceive, that: for every corresponding multiple of the sine and cosine, beginning at the first term of the cosine, thence passing to the first term of the sine, then from the second term of the cosine to the second of the sine, and so on to the end; we have all the terms of the binomial in regular order, as well for the powers of cosine *a*, and sine *a*, as for their numeric coefficients; with this difference only, that a regular change of the signs, +, and —, takes place separately, in each of the series.

The same law holds good in the case of the tangents, as far as regards the coefficients; and the powers of the tangents follow in a regular order, from the numerator to the denominator, alternately.

FIG. 76. Expansion of tan $n\alpha$ in powers of tan α. From an early nineteenth-century trigonometry book.

a result of De Moivre's theorem (see p. 83),

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha, \tag{3}$$

where $i = \sqrt{-1}$. If we expand the left side of equation (3) according to the binomial theorem and equate the real and imaginary parts with those on the right, we get expressions for $\cos n\alpha$ and $\sin n\alpha$ in terms of $\cos^{n-k}\alpha \cdot \sin^k \alpha$, where $k = 0, 1, 2, \ldots, n$; from these the formula for $\tan n\alpha$ easily follows:

$$\tan n\alpha = \frac{n\tan\alpha - {}^nC_3\tan^3\alpha + {}^nC_5\tan^5\alpha - + \cdots}{1 - {}^nC_2\tan^2\alpha + {}^nC_4\tan^4\alpha - + \cdots}, \tag{4}$$

where the symbol ${}^nC_k$—also denoted by $\binom{n}{k}$—stands for

$${}^nC_k = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ \ldots\ \cdot(n-k+1)}{k!}. \tag{5}$$

For example, ${}^4C_3 = (4\cdot 3\cdot 2)/(1\cdot 2\cdot 3) = 4$. Note that the expression on the right side of equation (5) is also equal to $n!/[k!\cdot(n-k)!]$, so we have ${}^nC_k = {}^nC_{n-k}$ (in the example just given, ${}^4C_1 = 4!/(3!\cdot 1!) = 4 = {}^4C_3$). Because of this, the binomial coefficients are symmetric—they are the same whether one expands $(1+x)^n$ in ascending or descending powers of x.

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Because the expansion of $(1+x)^n$ for positive integral n involves $(n+1)$ terms, the series appearing in the numerator and denominator of equation (4) are finite sums. But all six trigonometric functions can also be represented by *infinite* expressions, specifically power series and infinite products. The power series for $\sin x$ and $\cos x$ are

$$\sin x = x - x^3/3! + x^5/5! - + \cdots$$

and

$$\cos x = 1 - x^2/2! + x^4/4! - + \cdots.$$

These series were already known to Newton, but it was the great Swiss mathematician Leonhard Euler (1707–1783) who used them to derive a wealth of new results. Euler regarded power series as "infinite polynomials" that obey the same rules of algebra as do ordinary, finite polynomials. Thus, he argued, just as a polynomial of degree n can be written as a product of n (not necessarily different) factors of the form $(1 - x/x_i)$, where x_i are the roots, or zeros, of the polynomial,[6] so can the function $\sin x$ be written as an *infinite product*

$$\sin x = x(1 - x^2/\pi^2)(1 - x^2/4\pi^2)(1 - x^2/9\pi^2)\cdots. \tag{6}$$

Here each quadratic factor $(1 - x^2/n^2\pi^2)$ is the product of the two linear factors $(1 - x/n\pi)$ and $(1 + x/n\pi)$ resulting from the zeros of $\sin x$, $x_n = \pm n\pi$, whereas the single factor x results from the zero $x = 0$.[7] One surprising consequence of equation (6) results if we substitute $x = \pi/2$:

$$1 = (\pi/2) \cdot (1 - 1/4) \cdot (1 - 1/16) \cdot (1 - 1/36) \cdot \cdots .$$

Simplifying each term and solving for $\pi/2$, we get the infinite product

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots . \tag{7}$$

This famous formula is named after John Wallis (see p. 51), who discovered it in 1655 through a daring interpolation process.[8]

The infinite product for $\cos x$ is

$$\cos x = (1 - 4x^2/\pi^2)(1 - 4x^2/9\pi^2)(1 - 4x^2/25\pi^2) \cdots , \tag{8}$$

where $x_i = \pm\pi/2, \pm 3\pi/2, \ldots$ are the zeros of $\cos x$ (here again each quadratic factor is the product of two linear factors). If we now divide equations (6) by (8), we get an analytic expression for $\tan x$:

$$\tan x = \frac{x(1 - x^2/\pi^2)(1 - x^2/4\pi^2)(1 - x^2/9\pi^2)\cdots}{(1 - 4x^2/\pi^2)(1 - 4x^2/9\pi^2)(1 - 4x^2/25\pi^2)\cdots}. \tag{9}$$

This expression, however, is rather cumbersome. To simplify it, let us use a technique familiar from the integral calculus—the decomposition of a rational function into *partial fractions*. We express the right side of equation (9) as an infinite sum of fractions, each with a denominator equal to one *linear* factor in the denominator of equation (9):

$$\tan x = \frac{A_1}{1 - 2x/\pi} + \frac{B_1}{1 + 2x/\pi} + \frac{A_2}{1 - 2x/3\pi} + \frac{B_2}{1 + 2x/3\pi} + \cdots . \tag{10}$$

To find the coefficients of this decomposition, let us "clear fractions": we multiply both sides of equation (10) by the product of all denominators (that is, by $\cos x$), and equate the result to the numerator of equation (9) (that is, to $\sin x$):

$$\begin{aligned} x(1 - x/\pi)(1 + x/\pi)(1 - x/2\pi)(1 + x/2\pi)\cdots \\ = A_1(1 + 2x/\pi)(1 - 2x/3\pi)(1 + 2x/3\pi)\cdots \\ + B_1(1 - 2x/\pi)(1 - 2x/3\pi)(1 + 2x/3\pi)\cdots \\ + \cdots . \end{aligned} \tag{11}$$

Note that in each term on the right side of equation (11) exactly one factor is missing—the denominator of the corresponding coefficient in equation (10) (just as it is when finding an ordinary common denominator).

Now equation (11) is an identity in x—it holds true for any value of x we choose to put in it. To find A_1, let us choose $x = \pi/2$; this will "annihilate" all terms except the first, giving us

$$(\pi/2) \cdot (1/2) \cdot (3/2) \cdot (3/4) \cdot (5/4) \cdot \cdots$$
$$= A_1 \cdot 2 \cdot (2/3) \cdot (4/3) \cdot (4/5) \cdot (6/5) \cdot \cdots.$$

Solving for A_1, we get

$$A_1 = (\pi/2) \cdot (1/2)^2 \cdot (3/2)^2 \cdot (3/4)^2 \cdot (5/4)^2 \cdot \cdots$$
$$= (\pi/2) \cdot [(1/2) \cdot (3/2) \cdot (3/4) \cdot (5/4) \cdot \cdots]^2.$$

But the expression inside the brackets is exactly the reciprocal of Wallis's product, that is, $2/\pi$; we thus have

$$A_1 = (\pi/2) \cdot (2/\pi)^2 = 2/\pi.$$

To find B_1 we follow the same process except that now we put $x = -\pi/2$ in equation (11); this gives us $B_1 = -2/\pi = -A_1$. The other coefficients are obtained in a similar way;[9] we find that $A_2 = 2/(3\pi) = -B_2$, $A_3 = 2/(5\pi) = -B_3$, and in general

$$A_i = \frac{2}{(2i-1)\pi} = -B_i.$$

Putting these coefficients back into equation (10) and combining the terms in pairs, we get our grand prize, the decomposition of $\tan x$ into partial fractions:

$$\tan x = 8x\left[\frac{1}{\pi^2 - 4x^2} + \frac{1}{9\pi^2 - 4x^2} + \frac{1}{25\pi^2 - 4x^2} + \cdots\right]. \tag{12}$$

This remarkable formula shows directly that $\tan x$ is undefined at $x = \pm\pi/2, \pm 3\pi/2, \ldots$; these, of course, are precisely the vertical asymptotes of $\tan x$.

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Now that we have spent so much labor on establishing equation (12), let us reap some benefits from it. Since the equation holds for all x except $(2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$, let us put in it some special values. We start with $x = \pi/4$:

$$\tan \pi/4 = 1 = 8(\pi/4)[1/(\pi^2 - \pi^2/4) + 1/(9\pi^2 - \pi^2/4) + 1/(25\pi^2 - \pi^2/4) + \cdots] = (8/\pi)[1/3 + 1/35 + 1/99 + \cdots].$$

Each term inside the brackets is of the form $1/[4(2n-1)^2 - 1] = 1/(4n-3)(4n-1) = (1/2)[1/(4n-3) - 1/(4n-1)]$, $n = 1, 2, 3, \ldots$; we thus have

$$1 = (4/\pi)[1 - 1/3 + 1/5 - 1/7 + - \cdots],$$

from which we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots. \tag{13}$$

This famous formula was discovered in 1671 by the Scottish mathematician James Gregory (1638–1675), who derived it from the power series for the inverse tangent, $\tan^{-1} x = x - x^3/3 + x^5/5 - + \cdots$, from which equation (13) follows by substituting $x = 1$. Leibniz discovered the same formula independently in 1674, and it is often named after him.[10] It was one of the first results of the newly invented differential and integral calculus, and it caused Leibniz much joy.

The remarkable thing about the Gregory-Leibniz series—as also Wallis's product—is the unexpected connection between π and the integers. However, because of its very slow rate of convergence, this series is of little use from a computational point of view: it requires 628 terms to approximate π to just two decimal places—an accuracy far worse than that obtained by Archimedes, using the method of exhaustion, two thousand years earlier. Nevertheless, the Gregory-Leibniz series marks a milestone in the history of mathematics as the first of numerous infinite series involving π to be discovered in the coming years.

Next, let us use equation (12) with $x = \pi$:

$$\tan \pi = 0 = 8\pi[1/(-3\pi^2) + 1/(5\pi^2) + 1/(21\pi^2) + \cdots].$$

Canceling out $8/\pi$ and moving the negative term to the left side of the equation, we get

$$1/5 + 1/21 + 1/45 + \cdots = 1/3.$$

Perhaps somewhat disappointingly, we arrived at a series that does not involve π.[11] But excitement returns when we try to put

$x = 0$ in equation (12). At first we merely get the indeterminate equation $0 = 0$, but we can go around this difficulty by dividing both sides of the equation by x and then letting *x approach zero*. On the left side we get

$$\lim_{x \to 0} \frac{\tan x}{x} = \left(\lim_{x \to 0} \frac{\sin x}{x}\right) \cdot \left(\lim_{x \to 0} \frac{1}{\cos x}\right) = 1 \cdot 1 = 1.$$

Thus equation (12) becomes

$$1 = (8/\pi^2)\left[\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots\right]$$

or

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots. \tag{14}$$

This last formula is as remarkable as the Gregory-Leibniz series, but we can derive from it an even more interesting result. We will again do this in a nonrigorous way, in the spirit of Euler's daring forays into the world of infinite series (a more rigorous proof will be given in chapter 15). Our task is to find the sum of the reciprocals of the squares of *all* positive integers, even and odd; let us denote this sum by S:[12]

$$\begin{aligned} S &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ &= \left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right) \\ &= \left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right) + \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right) \\ &= \frac{\pi^2}{8} + \frac{1}{4}S. \end{aligned}$$

From this we get $(3/4)S = \pi^2/8$, and finally

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{15}$$

Equation (15) is one of the most celebrated formulas in all of mathematics; it was discovered by Euler in 1734 in a flash of ingenuity that would defy every modern standard of rigor. Its discovery solved one of the great mysteries of the eighteenth century: it had been known for some time that the series converges, but the value of its sum eluded the greatest mathematicians of the time, among them the Bernoulli brothers.[13]

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We consider one more infinite series discovered by Euler. We begin with the double-angle formula for the cotangent,

$$\cot 2x = \frac{1-\tan^2 x}{2\tan x} = \frac{\cot x - \tan x}{2}.$$

Starting with an arbitrary angle $x \neq n\pi/2$ and applying the formula repeatedly, we get

$$\begin{aligned}\cot x &= \frac{1}{2}(\cot x/2 - \tan x/2)\\ &= \frac{1}{4}(\cot x/4 - \tan x/4) - \frac{1}{2}\tan x/2\\ &= \frac{1}{8}(\cot x/8 - \tan x/8) - \frac{1}{4}\tan x/4 - \frac{1}{2}\tan x/2\\ &= \cdots\\ &= \frac{1}{2^n}(\cot x/2^n - \tan x/2^n)\\ &\quad - \frac{1}{2^{n-1}}\tan x/2^{n-1} - \cdots - \frac{1}{2}\tan x/2.\end{aligned}$$

As $n \to \infty$, $[\cot(x/2^n)]/2^n$ tends to $1/x$,[14] so we get

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n}$$

or

$$\frac{1}{x} - \cot x = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \frac{1}{8}\tan\frac{x}{8} + \cdots. \qquad (16)$$

This little-known formula is one more of hundreds of formulas involving infinite processes to emerge from Euler's imaginative mind. And behind it a surprise is hiding: if we put in it $x = \pi/4$, we get

$$4/\pi - 1 = (1/2)\tan\pi/8 + (1/4)\tan\pi/16 + \cdots.$$

Replacing the 1 on the left side with $\tan\pi/4$, moving all the tangent terms to the right side, and dividing the equation by 4, we get

$$\frac{1}{\pi} = \frac{1}{4}\tan\frac{\pi}{4} + \frac{1}{8}\tan\frac{\pi}{8} + \frac{1}{16}\tan\frac{\pi}{16} + \cdots. \qquad (17)$$

Equation (17) must surely rank among the most beautiful in mathematics, yet it hardly ever shows up in textbooks. Moreover, the series on the right side converges extremely rapidly (note that the coefficients *and* the angles decrease by a factor of 1/2 with each term), so we can use equation (17) as an efficient means to approximate π: it takes just twelve terms to obtain π to six decimal places, that is, to one-millionth; four more terms will increase the accuracy to one billionth.[15]

We have followed Euler's spirit in handling equations such as (6) and (9) as if they were finite expressions, subject to the rules of ordinary algebra. Euler lived in an era of carefree mathematical exploration when formal manipulation of infinite series was a normal practice; the questions of convergence and limit were not yet fully understood and were thus by and large ignored. Today we know that these questions are crucial to all infinite processes, and that ignoring them can lead to false results.[16] To quote George F. Simmons in his excellent calculus textbook, "These daring speculations are characteristic of Euler's unique genius, but we hope that no student will suppose that they carry the force of rigorous proof."[17]

Notes and Sources

1. A simple example of this is given by the functions $f(x) = \sin x$, $g(x) = 1 - \sin x$. Each has period 2π, yet their sum, $f(x) + g(x) = 1$, being a constant, has any real number as period.

2. On Dürer's mathematical work, see Julian Lowell Coolidge, *The Mathematics of Great Amateurs* (1949; rpt. New York: Dover, 1963), chap. 5, and Dan Pedoe, *Geometry and the Liberal Arts* (New York: St. Martin's Press, 1976), chap. 2.

3. An equivalent form of the law, in which the left side of the equation is replaced by $(\sin\alpha + \sin\beta)/(\sin\alpha - \sin\beta)$, was already known to Regiomontanus around 1464, but curiously he did not include it—nor any other applications of the tangent function—in his major treatise, *On Triangles* (see p. 44). As for other discoverers of the Law of Tangents, see David Eugene Smith, *History of Mathematics* (1925; rpt. New York: Dover, 1958), vol. 2, pp. 611 and 631.

4. Trigonometry abounds in such examples. We have seen one in chapter 8 in connection with the summation formula for $\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha$. Another example is the identity $\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\cdot\sin(\alpha-\beta)$, which can be "proved" by writing the left side as $\sin(\alpha^2 - \beta^2) = \sin[(\alpha+\beta)\cdot(\alpha-\beta)] = \sin(\alpha+\beta)\cdot\sin(\alpha-\beta)$.

5. We can take the signs into account by considering the expansion of $(1 + ix)^n$, where $i = \sqrt{-1}$.

6. This is equivalent to the more familiar factorization into factors of the form $(x - x_i)$. For example, the zeros of the polynomial $f(x) =$

$x^2 - x - 6$ are -2 and 3, so we have $f(x) = (x+2)(x-3) = -6(1 + x/2)(1 - x/3)$. In general, a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can be written either as $a_n(x - x_1) \cdot \ldots \cdot (x - x_n)$, where a_n is the leading coefficient or as $a_0(1 - x/x_1) \cdot \ldots \cdot (1 - x/x_n)$, where a_0 is the constant term.

7. Actually Euler discarded the root $x = 0$ and therefore obtained the infinite product for $(\sin x)/x$. See Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), vol. 2, pp. 448–449.

8. See *A Source Book in Mathematics*, 1200–1800, ed. D. J. Struik (Cambridge, Mass.: Harvard University Press, 1969), pp. 244–253. For a rigorous proof of equations (6) and (7), see Richard Courant, *Differential and Integral Calculus* (London: Blackie & Son, 1956), vol. 1, pp. 444–445 and 223–224. Other infinite products for π can also be derived from equation (6); for example, by putting $x = \pi/6$ we get

$$\frac{\pi}{3} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \ldots$$
$$= \prod_{n=1}^{\infty} (6n)^2/[(6n)^2 - 1],$$

which actually converges faster than Wallis's product (it takes 55 terms of this product to approximate π to two decimal places, compared to 493 terms of Wallis's product).

9. However, the resulting numerical products become more complicated as i increases. Fortunately there is an easier way to find the coefficients: the left side of equation (11) is $\sin x$, while each term on the right side is equal to $\cos x$ divided by the missing denominator of that term. Thus, to find A_2 we put $x = 3\pi/2$ in equation (11); this will "annihilate" all terms except that of A_2, and for the surviving term we have

$$(\sin x)_{x=3\pi/2} = A_2 \left[\frac{\cos x}{1 - 2x/3\pi} \right]_{x=3\pi/2}.$$

The left side equals -1, but on the right side we get the indeterminate expression 0/0. To evaluate it, we use L'Hospital's rule and transform it into $A_2[(-\sin x)/(-2/3\pi)]_{x=3\pi/2} = -(3\pi/2)A_2$. We thus get $A_2 = 2/(3\pi)$. The other coefficients can be found in the same way.

10. See Petr Beckmann, *A History of* π (Boulder, Colo.: Golem Press, 1977), pp. 132–133; for Leibniz's proof, see George F. Simmons, *Calculus with Analytic Geometry* (New York: McGraw-Hill, 1985), pp. 720–721. The series for $\tan^{-1} x$ can be obtained by writing the expression $1/(1 + x^2)$ as a power series $1 - x^2 + x^4 - + \cdots$ (a geometric series with the quotient $-x^2$) and integrating term by term.

11. That the series $1/5 + 1/21 + 1/45 + \cdots$ converges to 1/3 can be confirmed by noting that each term has the form $1/[(2n+1)^2 - 4] = 1/(2n-1)(2n+3) = (1/4)[1/(2n-1) - 1/(2n+3)]$; thus the series becomes

$$\tfrac{1}{4}[(1 - \tfrac{1}{5}) + (\tfrac{1}{3} - \tfrac{1}{7}) + (\tfrac{1}{5} - \tfrac{1}{9}) + (\tfrac{1}{7} - \tfrac{1}{11}) + \cdots].$$

This is a "telescopic" series in which all terms except the first and third cancel out, resulting in the sum $(1/4)(1 + 1/3) = 1/3$.

12. Assuming, of course, that the series converges. It is proved in calculus texts that the series $\Sigma_{n=1}^{\infty} 1/n^k$, where k is a real number, converges for all $k > 1$, and diverges for $k \leq 1$. In our case $k = 2$, hence S converges.

13. See Simmons, *Calculus*, pp. 722–723 (Euler's proof) and pp. 723–725 (a rigorous proof). One would expect that the series $\Sigma_{n=1}^{\infty} 1/n^2$ converges much faster than the Gregory-Leibniz series because all its terms are positive and involve the squares of the integers. Surprisingly, this is not so: it takes 600 terms to approximate π to two decimal places, compared to 628 terms of the Gregory-Leibniz series.

14. $\lim_{n\to\infty}[\cot(x/2^n)]/2^n = (1/x)\lim_{n\to\infty}[(x/2^n)\cot(x/2^n)] = 1/x$, the last result following from $\lim_{t\to\infty}(1/t)\cot(1/t) = \lim_{u\to 0} u \cot u = \lim_{u\to 0} u/\tan u = 1$, where $u = 1/t$.

15. One may raise the objection that equation (17) expresses π in terms of itself, since the angles in the tangent terms are in radians. However, the trigonometric functions are "immune" to the choice of the angular unit; using degrees instead of radians, equation (17) becomes $1/\pi = (1/4)\tan 45^\circ + (1/8)\tan 45^\circ/2 + \cdots$.

16. On this subject see Kline, *Mathematical Thought*, vol. 2, pp. 442–454 and 460–467. See also my book, *To Infinity and Beyond: A Cultural History of the Infinite* (Princeton, N.J.: Princeton University Press, 1991), pp. 32–33 and 36–39.

17. Simmons, *Calculus*, p. 723.

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