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Six Functions Come of Age

It is quite difficult to describe with certainty the beginning of trigonometry. . . . In general, one may say that the emphasis was placed first on astronomy, then shifted to spherical trigonometry, and finally moved on to plane trigonometry.

—Barnabas Hughes, Introduction to Regiomontanus' *On Triangles*

An early Hindu work on astronomy, the *Surya Siddhanta* (ca. 400 *a.d.*), gives a table of half-chords based on Ptolemy's table (fig. 15). But the first work to refer explicitly to the sine as a function of an angle is the *Aryabhatiya* of Aryabhata (ca. 510), considered the earliest Hindu treatise on pure mathematics.[1] In this work Aryabhata (also known as Aryabhata the elder; born 475 or 476, died ca. 550)[2] uses the word *ardha-jya* for the half-chord, which he sometimes turns around to *jya-ardha* ("chord-half"); in due time he shortens it to *jya* or *jiva*.

Now begins an interesting etymological evolution that would finally lead to our modern word "sine." When the Arabs translated the *Aryabhatiya* into their own language, they retained the word *jiva* without translating its meaning. In Arabic—as also in Hebrew—words consist mostly of consonants, the pronunciation of the missing vowels being understood through common usage. Thus *jiva* could also be pronounced as *jiba* or *jaib*, and *jaib* in Arabic means bosom, fold, or bay. When the Arabic version was translated into Latin, *jaib* was translated into *sinus*, which means bosom, bay, or curve (on lunar maps regions resembling bays are still described as sinus). We find the word *sinus* in the writings of Gherardo of Cremona (ca. 1114–1187), who translated many of the old Greek works, including the *Almagest*, from Arabic into Latin. Other writers followed, and soon the word *sinus*—or *sine* in its English version—became common in mathematical texts throughout Europe. The abbreviated notation *sin* was first used

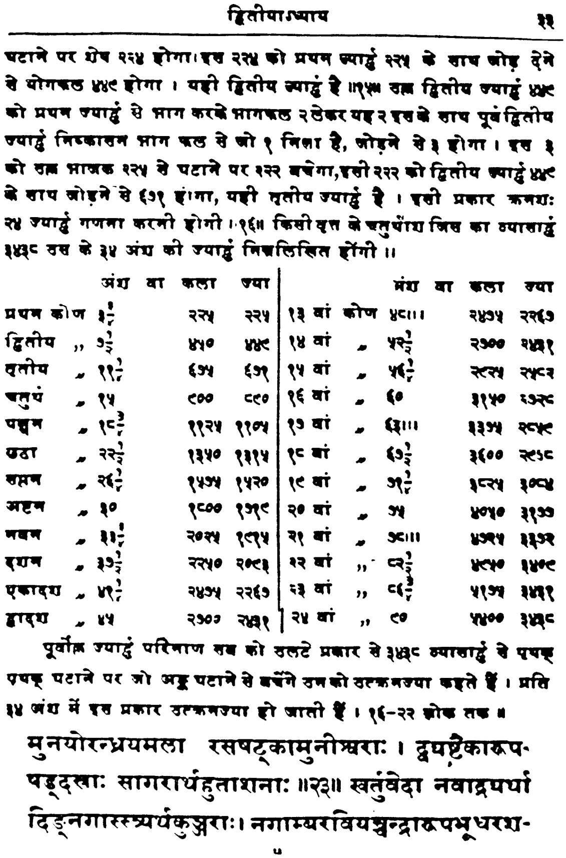

FIG. 15. A page from the Surya Siddhanta.

by Edmund Gunter (1581–1626), an English minister who later became professor of astronomy at Gresham College in London. In 1624 he invented a mechanical device, the "Gunter scale," for computing with logarithms—a forerunner of the familiar slide rule—and the notation *sin* (as well as *tan*) first appeared in a drawing describing his invention.[3]

Mathematical notation often takes unexpected turns. Just as Leibniz objected to William Ougthred's use of the symbol "×" for multiplication (on account of its similarity to the letter x), so did Carl Friedrich Gauss (1777–1855) object to the notation $sin^2 \phi$ for the square of $\sin \phi$:

> $\sin^2 \phi$ is odious to me, even though Laplace made use of it; should it be feared that $\sin^2 \phi$ might become ambiguous, which would perhaps never occur . . . well then, let us write $(\sin \phi)^2$, but not $\sin^2 \phi$, which by analogy should signify $\sin(\sin \phi)$."[4]

Notwithstanding Gauss's objection, the notation $\sin^2 \phi$ has survived, but his concern for confusing it with $\sin(\sin\phi)$ was not without reason: today the repeated application of a function to different initial values is the subject of active research, and expressions like $\sin(\sin(\sin\ldots(\sin\phi)\ldots))$ appear routinely in the mathematical literature.

The remaining five trigonometric functions have a more recent history. The cosine function, which we regard today as equal in importance to the sine, first arose from the need to compute the sine of the complementary angle. Aryabhata called it *kotijya* and used it in much the same way as trigonometric tables of modern vintage did (until the hand-held calculator made them obsolete), by tabulating in the same column the sines of angles from 0° to 45° and the cosines of the complementary angles. The name *cosinus* originated with Edmund Gunter: he wrote *co.sinus*, which was modified to *cosinus* by John Newton (1622–1678), a teacher and author of mathematics textbooks (he is unrelated to Isaac Newton) in 1658. The abbreviated notation *Cos* was first used in 1674 by Sir Jonas Moore (1617–1679), an English mathematician and surveyor.

The functions secant and cosecant came into being even later. They were first mentioned, without special names, in the works of the Arab scholar Abul-Wefa (940–998), who was also one of the first to construct a table of tangents; but they were of little use until navigational tables were computed in the fifteenth century. The first printed table of secants appeared in the work *Canon doctrinae triangulorum* (Leipzig, 1551) by Georg Joachim Rhæticus (1514–1576), who studied with Copernicus and became his first disciple; in this work all six trigonometric functions appear for the first time. The notation *sec* was suggested in 1626 by the French-born mathematician Albert Girard (1595–1632), who spent most of his life in Holland. (Girard was the first to understand the meaning of negative roots in geometric problems; he also guessed that a polynomial has as many roots as its degree, and was an early advocate of the use of parentheses in algebra.) For sec A he wrote $\overset{\text{sec}}{\text{A}}$, with a similar notation for tan A, but for sin A and cos A he wrote A and a, respectively.

The tangent and cotangent ratios, as we have seen, originated with the gnomon and shadow reckoning. But the treatment of these ratios as functions of an angle began with the Arabs. The first table of tangents and cotangents was constructed around 860 by Ahmed ibn Abdallah al-Mervazi, commonly known as Habash al-Hasib ("the computer"), who wrote on astronomy and astronomical instruments.[5] The astronomer al-Battani (known in Europe as Albategnius; born in Battan, Mesopotamia, ca. 858, died 929) gave a rule for finding the elevation of the sun above the horizon in terms of the length s of the shadow cast by a vertical gnomon of height h; his rule (ca. 920),

$$s = \frac{h \sin (90^\circ - \phi)}{\sin \phi},$$

is equivalent to the formula $s = h \cot \phi$. Note that in expressing it he used only the sine function, the other functions having not yet been known by name. (It was through al-Battani's work that the Hindu half-chord function—our modern sine—became known in Europe.) Based on this rule, he constructed a "table of shadows"—essentially a table of cotangents—for each degree from 1° to 90°.

The modern name "tangent" did not make its debut until 1583, when Thomas Fincke (1561–1646), a Danish mathematician, used it in his *Geometria Rotundi*; up until then, most European writers still used terms taken from shadow reckoning: *umbra recta* ("straight shadow") for the horizontal shadow cast by a vertical gnomon, and *umbra versa* ("turned shadow") for a vertical shadow cast by a gnomon attached to a wall. The word "cotangens" [sic] was first used by Edmund Gunter in 1620. Various abbreviations were suggested for these functions, among them *t* and *t co* by William Oughtred (1657) and *T* and *t* by John Wallis (1693). But the first to use such abbreviations consistently was Richard Norwood (1590–1665), an English mathematician and surveyor; in a work on trigonometry published in London in 1631 he wrote: "In these examples *s* stands for *sine*: *t* for *tangent*: *sc* for *sine complement* [i.e., cosine]: *tc* for *tangent complement*: *sec* for *secant*." We note that even today there is no universal conformity of notation, and European texts often use "tg" and "ctg" for tangent and cotangent.

The word "tangent" comes from the Latin *tangere*, to touch; its association with the tangent function may have come from the following observation: in a circle with center at O and radius r (fig. 16), let AB be the chord subtended by the central angle 2α, and OQ the bisector of this angle. Draw a line parallel to AB and tangent to the circle at Q, and extend OA and OB until they meet this line at C and D, respectively. We have

$$AB = 2r \sin \alpha, CD = 2r \tan \alpha,$$

showing that the tangent function is related to the tangent line in the same way as the sine function is to the chord. Indeed, this construction forms the basis of the modern definition of the six trigonometric functions on the unit circle.

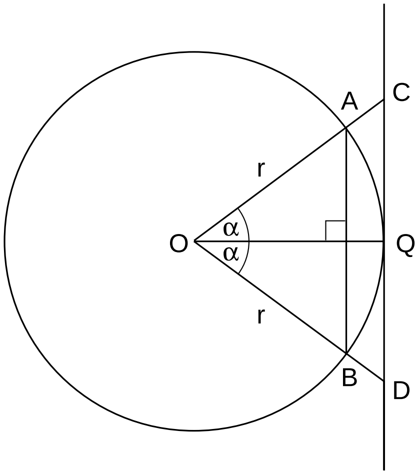

Fig. 16. $AB = 2r \sin \alpha$, $CD = 2r \tan \alpha$.

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Through the Arabic translations of the classic Greek and Hindu texts, knowledge of algebra and trigonometry gradually spread to Europe. In the eighth century, Europe was introduced to the Hindu numerals—our modern decimal numeration system—through the writings of Mohammed ibn Musa al-Khowarizmi (ca. 780–ca. 840). The title of his great work, *'ilm al-jabr wa'l muqabalah* ("the science of reduction and cancelation") was transliterated into our modern word "algebra," and his own name evolved into the word "algorithm." The Hindu-Arabic system was not immediately accepted by the public, who preferred to cling to the old Roman numerals. Scholars, however, saw the advantages of the new system and advocated it enthusiastically, and contests between "abacists," who computed with the good old abacus, and "algorists," who did the same symbolically with paper and pen, became a common feature of medieval Europe.

It was mainly through Leonardo Fibonacci's exposition of the Hindu-Arabic numerals in his *Liber Abaci* (1202) that the decimal system finally took general hold in Europe. The first trigonometric tables using the new system were computed around 1460 by Georg von Peurbach (1423–1461). But it was his pupil Johann Müller (1436–1476), known as Regiomontanus (because he was born in Königsberg, which in German means "the royal mountain") who wrote the first comprehensive treatise on trigonometry up to date. In his *De triangulis omnimodis libri quinque* ("of triangles of every kind in five books," ca. 1464)[6] he developed the subject starting from basic geometric concepts and leading to the definition of the sine function; he then showed how to solve any triangle—plane or spherical—using either the sine of an angle or the sine of its complement (the cosine). The Law of Sines is stated here in verbal form, and so is the rule for finding the area of a triangle, $A = (ab \sin \gamma)/2$. Curiously the tangent function is absent, possibly because the main thrust of the work was spherical trigonometry where the sine function is dominant.

De triangulis was the most influential work on trigonometry of its time; a copy of it reached Copernicus, who studied it thoroughly (see p. 45). However, another century would pass before the word "trigonometry" appeared in a title of a book. This honor goes to Bartholomäus Pitiscus (1561–1613), a German clergyman whose main interest was mathematics. His book, *Trigonometriae sive de dimensione triangulorum libri quinque* (On trigonometry, or, concerning the properties of triangles, in five books), appeared in Frankfort in 1595. This brings us to the beginning of the seventeenth century, when trigonometry began to take the analytic character that it would retain ever since.

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Notes and Sources

1. *The Aryabhatiya of Aryabhata: An Ancient Indian Work on Mathematics and Astronomy*, translated with notes by Walter Eugene Clark (Chicago: University of Chicago Press, 1930). In this work (p. 28) the value of π is given as 3.1416; this is stated in verbal form as a series of mathematical instructions, a common feature of Hindu mathematics. See also David Eugene Smith, *History of Mathematics*, (1925; rpt. New York: Dover, 1958), vol. 1, pp. 153–156, and George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics* (Harmondsworth, U.K.: Penguin Books, 1992), pp. 265–266.

2. In 1975 India named its first satellite after him.

3. For a detailed history of trigonometric notation, see Florian Cajori, *A History of Mathematical Notations* (1929; rpt. Chicago: Open Court, 1952), vol. 2, pp. 142–179; see also Smith, *History of Mathematics*, vol. 2, pp. 618–619 and 621-623. A list of trigonometric symbols, with their authors and dates, can be found in Vera Sanford, *A Short History of Mathematics* (1930; rpt. Cambridge, Mass.: Houghton Mifflin, 1958), p. 298.

4. Gauss-Schumacher correspondence, as quoted in Robert Edouard Moritz, *On Mathematics and Mathematicians (Memorabilia Mathematica)* (1914; rpt. New York: Dover, 1942), p. 318.

5. Smith, *History of Mathematics*, vol. 2, p. 620. Cajori, however, credits al-Battani as the first to construct a table of cotangents (*A History of Mathematics*, 1893; 2d ed. New York: Macmillan, 1919, p. 105).

6. English translation with an introduction and notes by Barnabas Hughes (Madison: University of Wisconsin Press, 1967).