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Trigonometry Becomes Analytic

Thus the analysis of angular sections involves geometric and arithmetic secrets which hitherto have been penetrated by no one.

—François Viète

With the great French mathematician François Viète (also known by his Latin name Franciscus Vieta, 1540–1603), trigonometry began to assume its modern, analytic character. Two developments made this process possible: the rise of symbolic algebra—to which Viète was a major contributor—and the invention of analytic geometry by Fermat and Descartes in the first half of the seventeenth century. The gradual replacement of the cumbersome verbal algebra of medieval mathematics with concise, symbolic statements—a literal algebra—greatly facilitated the writing and reading of mathematical texts. Even more important, it enabled mathematicians to apply algebraic methods to problems that until then had been treated in a purely geometric way.

With Viète, trigonometry underwent a second important change: it admitted infinite processes into its ranks. In 1593 he discovered the famous infinite product

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots$$

(Viète used the abbreviation *etc.* instead of the three dots). It was the first time an infinite process was explicitly written as a mathematical formula, and it heralded the beginning of modern analysis.[1] (We will prove Viète's product in chapter 11.)

In England three individuals made substantial contributions to trigonometry in the first half of the seventeenth century. John Napier's (1550–1617) invention of logarithms in 1614 enormously aided numerical calculations, particularly in trigonometry.[2] William Oughtred (1574–1660) was the first to attempt a systematic use of trigonometric symbols: in his work *Trigonometrie, or, The manner of Calculating the Sides and Angles of Triangles, by the Mathematical Canon, demonstrated* (London, 1657), he used the abbreviations *s*, *t*, *se*, *s co*, *t co*, and *se co* for sine, tangent, secant, cosine ("sine complement"), cotangent, and cosecant, respectively.[3] (In contrast to its long title, the work itself contains, besides the tables, only thirty-six pages of text.) And John Wallis's (1616–1703) work on infinite series was an immediate forerunner to Newton's discoveries in the same field. Wallis, more than anyone else up to his time, realized that synthetic methods in mathematics should give way to analytic ones: he was the first to treat conic sections as quadratic equations rather than geometric objects, as the Greeks had done. (Wallis was also the first major mathematician to write on the history of mathematics, and he introduced the symbol ∞ for infinity.) His most famous formula is the infinite prduct

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots,$$

which together with Viète's product ranks among the most beautiful formulas in mathematics. Wallis arrived at this result by daring intuition and a complex interpolation process that would stretch the patience of a modern reader to the limit;[4] in chapter 12 we will derive his product in a shorter and more elegant way.

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There was yet another reason for the rise of analytic trigonometry in the first half of the seventeenth century: the ever increasing role of mathematics in describing the physical world around us. Whereas the inventors of classical trigonometry were mainly interested in applying it to the heavens (hence the initial predominance of spherical over plane trigonometry), the new era had its feet firmly planted in the mechanical world of daily life. Galileo's discovery that every motion can be resolved into two components along perpendicular lines—and that these components can be treated independently of each other—at once made trigonometry indispensable in the study of motion. The science of artillery—and in the seventeenth century it *was* regarded as a science—was chiefly concerned with finding the range of a projectile fired from a cannon. This range, in the absence of air resistance, is given by the formula $R = (v_0^2 \sin 2\alpha)/g$, where v_0 is the velocity of the projectile as it leaves the cannon (the muzzle velocity), α the angle of firing relative to the ground, and g

the acceleration due to gravity (about 9.81 m/sec^2). This formula shows that for a given velocity, the range depends solely on α: it reaches its maximum when $\alpha = 45^\circ$ and falls off symmetrically on either side of 45°. These facts, of course, had been known empirically for a long time, but their theoretical basis was new in Galileo's time.

Another branch of mechanics vigorously studied in the seventeenth and eighteenth centuries dealt with oscillations. The great sea voyages of the era demanded ever more accurate navigational techniques, and these in turn depended on the availability of clocks of ever greater precision. This led scientists to study the oscillations of pendulums and springs of various kinds. Some of the greatest names of the day were involved in these studies, among them Christiaan Huygens (1629–1695) and Robert Hooke (1635–1703). Huygens discovered the cycloidal pendulum, whose period of oscillation is independent of the amplitude, while Hooke's work on the coiled spring laid the basis for the modern spring-driven watch. On another level, the increased skill and sophistication in building musical instruments—from woodwinds and brass to keyboard instruments and organs—motivated scientists to study the vibrations of sound-producing bodies such as strings, membranes, bells, and air pipes. All this underscored the role of trigonometry in describing *periodic phenomena* and resulted in a shift of emphasis from computational trigonometry (the compilation of tables) to the *relations* among trigonometric functions—the essence of analytic trigonometry.

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In his work *Harmonia mensurarum* (Harmony of mensuration), published posthumously in 1722, the English mathematician Roger Cotes (1682–1716) gave the equivalent of the formula (in modern notation)

$$\phi i = \log(\cos\phi + i\sin\phi),$$

where $i = \sqrt{-1}$ and "log" stands for natural logarithm (logarithm to the base $e = 2.718\ldots$). This, of course, is equivalent to Leonhard Euler's famous formula $e^{i\phi} = \cos\phi + i\sin\phi$, published in 1748 in his great work, *Introductio in analysin infinitorum*. Also in 1722 Abraham De Moivre (1667–1754) derived—though in implicit form—the formula,

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi,$$

which is the basis for finding the *n*th root of a number, real or complex. It took, however, the authority of Euler and his *Introductio* to fully incorporate complex numbers into trigonometry: with him the subject became truly analytic. (We will return to the role of complex numbers in trigonometry in chapter 14.)

These developments moved trigonometry ever farther from its original connection with a triangle. The first to define the trigonometric functions as pure numbers, rather than ratios in a triangle, was Abraham Gotthelf Kästner (1719–1800) of Germany; in 1759 he wrote: “If x denotes the angle expressed in degrees, then the expressions $\sin x$; $\cos x$; $\operatorname{tang} x$ etc. are numbers, which correspond to every angle.”[5] Today, of course, we go a step farther and define the independent variable itself as a real number rather than an angle.

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Almost from its inception the differential and integral calculus was applied to numerous problems in mechanics, first to discrete mechanics (the motion of a single particle or a system of particles), and later to continuous mechanics. Among the latter, the outstanding problem in the second half of the eighteenth century was the vibrating string. This problem has excited mathematicians since the earliest time because of its close association with music. Already Pythagoras, in the sixth century *b.c.*, discovered some of the laws governing the vibrations of a string; this led him to construct a musical scale based on mathematical principles. A full investigation of the problem, however, required methods that were not even available to Newton and Leibniz, namely partial differential equations (equations in which the unknown function and its derivatives depend on two or more independent variables). For the vibrating string the relevant equation is $\partial^2 u/\partial x^2 = (1/c^2)(\partial^2 u/\partial t^2)$, where $u = u(x, t)$ is the displacement from equilibrium of a point at a distance x from one endpoint of the string at time t, and c is a constant that depends on the physical parameters of the string (its tension and linear density).

The attempts to solve this famous equation, known as the one-dimensional wave equation, involved the best mathematical minds of the time, among them the Bernoulli family, Euler, D’Alembert, and Lagrange. Euler and D’Alembert expressed their solutions in terms of arbitrary functions representing two waves, one moving along the string to the right, the other to the left, with a velocity equal to the constant c. Daniel Bernoulli, on the other hand, found a solution involving an infinite series of trigonometric functions. Since these two types of solution to the same problem looked so different, the question arose whether they could be reconciled, and if not, to find out which was the more general. This question was answered by the French mathematician Jean Baptiste Joseph Fourier (1768–1830). In his most important work, *Théorie analytique de la chaleur* (Analytic theory of heat, 1822), Fourier showed that almost any function, when regarded as a periodic function over a given interval, can be represented by a trigonometric series of the form

$$f(x) = a_o + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots,$$

where the coefficients a_i and b_i can be found from $f(x)$ by computing certain integrals. This *Fourier series* is in some respects more general than the familiar Taylor expansion of a function in a power series; for example, while the Taylor series can be applied only to functions $f(x)$ that are continuous and have continuous derivatives, the Fourier series may exist even if $f(x)$ is discontinuous. We will return to these series in chapter 15.

Fourier's theorem marks one of the great achievements of nineteenth-century analysis. It shows that the sine and cosine functions are essential to the study of *all* periodic phenomena, simple or complex; they are, in fact, the building blocks of all such phenomena, in much the same way that the prime numbers are the building blocks of all integers. Fourier's theorem was later generalized to nonperiodic functions (in which case the infinite series becomes an integral), as well as to series involving nontrigonometric functions. These developments proved to be of crucial importance to numerous branches of science, from optics and acoustics to information theory and quantum mechanics.

NOTES AND SOURCES

1. Recent findings indicate that the Hindus may have known several infinite series involving π before Viète; see George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics* (Harmondsworth, U.K.: Penguin Books, 1991), pp. 286–294.

2. See my book, *e: The Story of a Number* (Princeton, N.J.: Princeton University Press, 1994), chaps. 1 and 2.

3. See, however, pp. 37–38 as to the priority of using these symbols. See also Florian Cajori, *William Oughtred: A Great Seventeenth-Century Teacher of Mathematics* (Chicago: Open Court, 1916), pp. 35–39. Cajori notes that the tables in Oughtred's book use a centesimal division of the

degree (i.e., into one hundred parts), a practice that has been revived in our time with the advent of the hand-held calculator.

4. See *A Source Book in Mathematics, 1200–1800* ed. D. J. Struik (Cambridge, Mass.: Harvard University Press, 1969), pp. 244–253.

5. David Eugene Smith, *History of Mathematics* (1925; rpt. New York: Dover, 1958), vol. 2, p. 613. Kästner was the first mathematician to write a work entirely devoted to the history of mathematics (in 4 vols.; Göttingen, 1796–1800).

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