François Viète

It is unfortunate that the names of so many of those who helped shape mathematics into its present form have largely vanished from today's curriculum. Among them we may mention Regiomontanus, Napier, and Viète, all of whom made substantial contributions to algebra and trigonometry.

François Viète was born in Fontenay le Comte, a small town in western France, in 1540 (the exact day is unknown). He first practiced law and later became involved in politics, serving as member of the parliament of Brittany. As was the practice of many learned men at the time, he latinized his name to Franciscus Vieta; but unlike others—among them Regiomontanus—the Latin version was not adopted universally. We shall use the French Viète.

Throughout his life Viète practiced mathematics only in his leisure time, regarding it as an intellectual recreation rather than a profession. He was not alone in this attitude: Pierre Fermat, Blaise Pascal, and René Descartes all made great contributions to mathematics at their leisure while officially occupying a variety of political, diplomatic, and, in Descartes' case, military positions. Viète began his scientific career as a tutor to Catherine of Phartenay, the daughter of a prominent military figure, for whom he wrote several textbooks. As his reputation grew, he was called on to serve the monarch Henry IV in his war against Spain. Viète proved himself an expert in breaking the enemy code: a secret message to the Spanish monarch Philip II from his liaison officer was intercepted by the French and given to Viète, who succeded in deciphering it. The Spaniards, amazed that their code had been broken, accused the French of using sorcery, "contrary to the practice of the Christian faith."[1]

Viète's most important work is his *In artem analyticam isagoge* (Introduction to the analytical art; Tours, 1591), considered the earliest work on symbolic algebra. In this work he introduced a system of notation that came close to the one we use today: he denoted known quantities by consonants and unknowns by vowels. (The present custom of using a, b, c, etc. for constants and x, y, z for unknowns was introduced by Descartes in 1637.) He defined an equation as "a comparison of an unknown magnitude with a determinate one" and gave the basic rules for solving equations—moving a term from one side of the equation to the other, dividing an equation by a common factor, and so on. He called his method *ars analytice* and the new algebra *logistica speciosa* (literally, the art of calculating with species, i.e., general quantities), to distinguish it from the old *logistica numerosa*. This transition from verbal to symbolic algebra is considered one of the most important developments in the history of mathematics.

Viète applied the rules of algebra to *any* quantity, arithmetic or geometric, thereby putting to rest the age-old distinction between pure numbers and geometric entities. In other respects, however, he was rather conservative. For example, he always insisted on making an equation dimensionally homogeneous: instead of the modern equation $mx = b$ he would write "M in A aequatur B quadratus", meaning that a given quantity M (represented by a consonant) multiplied by an unknown quantity A (a vowel) is equal to the square of a given number B. This shows that he was still clinging to the old Greek view that regarded operations among numbers as geometric in nature; since a product of two numbers represents the area of a rectangle having these numbers as sides, one must equate such a product to the area of a square whose side is given. (Today, of course, we regard algebraic quantities as pure, dimensionless numbers.) It is also interesting to note that Viète used the modern symbols $+$ and $-$ for addition and subtraction, but for equality he used the verbal description "aequatur." For A^2 he wrote *A quadratus* and for A^3, *A cubus* (although later he abbreviated these to *Aq* and *Ac*). Clearly Viète could not entirely free himself from the shackles of the old verbal algebra. His work reflects the time in which he lived—a period of transition from the old world to the new.

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Of particular interest to us are Viète's contributions to trigonometry. His first work on this subject appeared in 1571 under the title *Canon mathematicus seu ad triangula cum appendicibus*. Here he gives the first systematic treatment in the Western world of the methods for solving plane and spherical triangles, using all six trigonometric functions. He develops the three sum-to-product formulas (e.g., $\sin\alpha + \sin\beta = 2\sin(\alpha+\beta)/2 \cdot \cos(\alpha-\beta)/2$, with similar formulas for $\sin\alpha + \cos\beta$ and $\cos\alpha + \cos\beta$), from which Napier may have gotten the idea of logarithms, since they allow one (when used in reverse) to reduce a product of two numbers to the sum of two other numbers. And he was the first to state the Law of Tangents in its modern form: $(a+b)/(a-b) = [\tan(\alpha+\beta)/2]/[\tan(\alpha-\beta)/2)]$, where a and b are two sides of a triangle and α and β the opposite angles (see page 152).

Viète was the first mathematician to systematically apply algebraic methods to trigonometry. For example, by letting $x = 2\cos\alpha$ and $y_n = \cos n\alpha$, he obtained the recurrence formula

$$y_n = xy_{n-1} - y_{n-2},$$

which, when translated back into trigonometry, becomes

$$\cos n\alpha = 2\cos\alpha \cdot \cos(n-1)\alpha - \cos(n-2)\alpha.$$

One can now express $\cos(n-1)\alpha$ and $\cos(n-2)\alpha$ in terms of the cosines of still lower multiples of α; and by continuing the process, one ends up with a formula expressing $\cos n\alpha$ in terms of $\cos\alpha$ and $\sin\alpha$. Viète was able to do this for all integers n up to 10. He was so proud of his achievement that he exclaimed, "Thus the analysis of angular sections involves geometric and arithmetic secrets which hitherto have been penetrated by no one."[2] To appreciate his feat, we mention that the general formulas expressing $\cos n\alpha$ and $\sin n\alpha$ in terms of $\cos\alpha$ and $\sin\alpha$ were found by Jakob Bernoulli in 1702, more than a hundred years after Viète's work.[3]

Viète's adeptness in applying algebraic transformations to trigonometry served him well in a famous encounter between Henry IV and Netherland's ambassador to France. Adriaen van Roomen (1561–1615), professor of mathematics and medicine in Louvain (Belgium), in 1593 published a work entitled *Ideae mathematicae*, which contained a survey of the most prominent living mathematicians of the time.[4] Not a single French mathematician was mentioned, prompting the Dutch ambassador to speak disdainfully about France's scientific achievements. To prove his point, he presented Henry IV with a problem that appeared in the *Ideae*—with a prize offered to whoever will solve it—and boasted that surely no French mathematician will come up with a solution. The problem was to solve the 45th-degree equation

$$x^{45} - 45x^{43} + 945x^{41} - 12{,}300x^{39} + - \cdots$$
$$+ 95{,}634x^5 - 3{,}795x^3 + 45x = c,$$

where c is a constant.

Henry summoned Viète, who immediately found one solution, and on the following day came up with twenty-two more.

What happened is described by Florian Cajori in *A History of Mathematics*:

> Viète, who, having already pursued similar investigations, saw at once that this awe-inspiring problem was simply the equation by which $c = 2\sin\phi$ was expressed in terms of $x = 2\sin(\phi/45)$; that, since $45 = 3 \cdot 3 \cdot 5$, it was necessary only to divide an angle once into five equal parts, and then twice into three—a division which could be effected by corresponding equations of the fifth and third degrees.[5]

To follow Viète's line of thought, let us first look at a simpler problem. Suppose we are asked to solve the equation

$$x^3 - 3x + 1 = 0.$$

We rewrite it as $1 = 3x - x^3$ and make the substitution $x = 2y$:

$$\frac{1}{2} = 3y - 4y^3.$$

If we have a keen eye, we might recognize the similarity between this equation and the identity

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha.$$

In fact, we can make the two equations coincide if we write $1/2 = \sin 3\alpha$ and $y = \sin\alpha$; in this new form, the problem amounts to finding $\sin\alpha$, given that $\sin 3\alpha = 1/2$. But if $\sin 3\alpha = 1/2$, then $3\alpha = 30^\circ + 360^\circ k$, where $k = 0, \pm 1, \pm 2, \ldots$, and so $\alpha = 10^\circ + 120^\circ k$. Hence $y = \sin(10^\circ + 120^\circ k)$, and finally $x = 2y = 2\sin(10^\circ + 120^\circ k)$. However, because the sine function has a period of 360°, it suffices to consider only the values $k = 0, 1, 2$. Our three solutions are then

$$x_0 = 2\sin 10^\circ = 0.347\cdots,$$

$$x_1 = 2\sin 130^\circ = 1.532\cdots,$$

and

$$x_2 = 2\sin 250^\circ = -1.879\cdots.$$

With a calculator we can easily check that these are indeed the three solutions. Thus a trigonometric identity helped us to solve a purely algebraic equation.

Now it is one thing to solve a cubic equation using trigonometry, but quite another to solve an equation of degree 45. How then did Viète find his solutions? In a work entitled *Responsum* (1595) he outlined his method, which we summarize here in modern notation: let

$$c = 2\sin 45\theta,\ y = 2\sin 15\theta,\ z = 2\sin 5\theta,\ x = 2\sin\theta.$$

Our task is to find $x = 2\sin\theta$, given that $c = 2\sin 45\theta$. We will do this in three stages. We again start with the identity $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$. Multiplying by 2 and susbtituting first $\alpha = 15\theta$, we get

$$c = 3y - y^3. \tag{1}$$

Next, we substitute $\alpha = 5\theta$ and get

$$y = 3z - z^3. \tag{2}$$

We now use the identity

$$\sin^5\alpha = \frac{5}{8}\sin\alpha - \frac{5}{16}\sin 3\alpha + \frac{1}{16}\sin 5\alpha.^6$$

Multiplying by 32, replacing α by θ and expressing $2\sin 3\theta$ in terms of $x = 2\sin\theta$, we get

$$x^5 = 10x - 5(3x - x^3) + z,$$

which after simplifying becomes

$$z = 5x - 5x^3 + x^5. \tag{3}$$

If we now back-substitute equation (3) into (2) and then (2) into (1) and expand, we get exactly van Roomen's equation!

Viète thus split the original problem into three simpler ones. But why did he find only twenty-three solutions, when we know that the original equation must have forty-five solutions (all real, as follows from the geometric meaning of the problem: to divide an arbitrary angle into forty-five equal parts)? The reason is that in Viète's time it was still the practice to regard the *chord length* of an angle, rather than its sine, as the basic trigonometric function (see chapter 2); and since length is non-negative, he had to reject all negative solutions as meaningless. The complete set of solutions is given by

$$x_k = 2\sin(\theta + 360^\circ k/45),\ k = 0, 1, 2, \dots, 44;$$

of these (assuming that $45\theta \leq 180^\circ$, for otherwise $\sin 45\theta$ itself would already be negative), only the first twenty-three are positive, corresponding to angles in quadrants I and II.

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Among Viète's many other contributions we should mention his discovery of the relation between the roots of a quadratic equation $ax^2 + bx + c = 0$ and its coefficients ($x_1 + x_2 = -b/a$ and $x_1 x_2 = c/a$), although his rejection of negative roots prevented him from stating this relation as a general rule; the development of a numerical method for approximating the solutions of algebraic equations; and his discovery of the famous infinite product for π that bears his name (see page 50). Most of his works were originally printed for private circulation only; they were collected, edited, and published in 1646, more than forty years after Viète's death, by the Dutch mathematician Frans van Schooten (1615–1660).[7]

During Viète's last years he was embroiled in a bitter controversy with the German mathematician Christopher Clavius (1537–1612) over the reformation of the calendar that had been ordered by Pope Gregory XIII in 1582. Viète's caustic attacks on Clavius, who was the pope's adviser in this matter, made him many enemies and resulted in his adversaries' rejection of his new algebra. It is also worth noting that Viète consistently opposed the Copernican system, attempting instead to improve the old geocentric system of Ptolemy. We see here the inner conflict of a man who was at once an innovator of the first rank and a conservative deeply rooted in the past. Viète died in Paris on December 13, 1603, at the age of sixty-three. With him, algebra and trigonometry began to take the form we know today.[8]

NOTES AND SOURCES

1. W. W. Rouse Ball, *A Short Account of the History of Mathematics* (1908; rpt. New York: Dover, 1960), p. 230.

2. Florian Cajori, *A History of Mathematics* (1893; 2nd ed. New York: MacMillan, 1919), p. 138.

3. These formulas are

$$\cos n\alpha = \cos^n \alpha - \frac{n(n-1)}{2!} \cos^{n-2} \alpha \cdot \sin^2 \alpha + \cdots$$

and

$$\sin n\alpha = \frac{n}{1!} \cos^{n-1} \alpha \cdot \sin \alpha - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} \alpha \cdot \sin^3 \alpha + \cdots .$$

4. In this work the value of π is given to seventeen decimal places, a remarkable achievement at the time.

5. Cajori, *History of Mathematics*, p. 138.

6. This identity can be obtained from the formula $\sin 5\alpha = 5 \sin \alpha - 20 \sin^3 \alpha + 16 \sin^5 \alpha$ by replacing $\sin^3 \alpha$ with $(3 \sin \alpha - \sin 3\alpha)/4$ and solving for $\sin^5 \alpha$.

7. The van Schooten family produced three generations of mathematicians, all of whom were born and lived in Leyden: Frans senior (1581–1646), Frans junior mentioned above, and his half-brother Petrus (1634–1679). Of the three, the most prominent was Frans junior, who edited the Latin edition of Descartes's *La Géométrie*; he also wrote on perspective and advocated the use of three-dimensional coordinates

in space. He was the teacher of the great Dutch scientist Christiaan Huygens.

8. There is no biography of Viète in English. Sketches of his life and work can be found in Ball, *Short Account*, pp. 229–234; Cajori, *History of Mathematics*, pp. 137–139; Joseph Ehrenfried Hofmann, *The History of Mathematics*, translated from the German by Frank Gaynor and Henrietta O. Midonick (New York: Philosophical Library, 1957), pp. 92–101; and in the *DSB*.

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