Abraham De Moivre

Abraham De Moivre was born in Vitry in the province of Champagne, France, on May 26, 1667, to a Protestant family. He showed an early interest in mathematics and studied it—secretly—at the various religious schools he was attending. In 1685 Louis XIV revoked the Edict of Nantes—a decree issued in 1598 granting religious freedom to French Protestants—and a period of repression followed. By one account De Moivre was imprisoned for two years before leaving for London, where he would spend the rest of his life. He studied mathematics on his own and became very proficient in it. By sheer luck he happened to be at the house of the Earl of Devonshire, where he worked as a tutor, at the very moment when Isaac Newton stepped out with a copy of the *Principia*, his great work on the theory of gravitation. De Moivre took up the book, studied it on his own, and found it far more demanding than he had expected (it is a difficult text even for a modern reader). But by assiduous study—he used to tear pages out of the huge volume so he could study them between his tutoring sessions—he not only mastered the work but became an expert on it, so much so that Newton, in later years, would refer to De Moivre questions addressed to himself, saying, "Go to Mr. De Moivre; he knows these things better than I do."

In 1692 he met Edmond Halley (of comet fame), who was so impressed by his mathematical ability that he communicated to the Royal Society De Moivre's first paper, on Newton's "method of fluxions" (i.e., the differential calculus). Through Halley, De Moivre became a member of Newton's circle of friends that also included John Wallis and Roger Cotes. In 1697 he was elected to the Royal Society and in 1712 was appointed as member of the Society's commission to settle the bitter priority dispute between Newton and Leibniz over the invention of the calculus. He was also elected to the academies of Paris and Berlin.

Despite these successes, De Moivre was unable to secure himself a university position—his French origin was one reason—and even Leibniz's attempts on his behalf were unsuccessful. He made a meager living as a tutor of mathematics, and for the rest of his life would lament having to waste his time walking between the homes of his students. His free time was spent in the coffeehouses and taverns on St. Martin's Lane in London, where he answered all kinds of mathematical questions addressed to him by rich patrons, especially about their chances of winning in gambling.

When he grew old he became lethargic and needed longer sleeping hours. According to one account, he declared that beginning on a certain day he would need twenty more minutes of sleep on each subsequent day. On the seventy-third day—November 27, 1754—when the additional sleeping time accumulated to 24 hours, he died; the official cause was recorded as "somnolence" (sleepiness). He was eighty-seven years old, joining a long line of distinguished English mathematicians who lived well past their eighties: William Oughtred, who died in 1660 at the age of 86, John Wallis (d. 1703 at 87), Isaac Newton (d. 1727 at 85), Edmond Halley (d. 1742 at 86), and in our time, Alfred North Whitehead (d. 1947 at 86) and Bertrand Russell, who died in 1970 at 98. The poet Alexander Pope paid him tribute in *An Essay on Man*:

Who made the spider parallels design,
Sure as Demoivre, without rule or line?

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De Moivre's mathematical work covered mainly two areas: the theory of probability, and algebra and trigonometry (considered as a unified field). In probability he extended the work of his predecessors, particularly Christiaan Huygens and several members of the Bernoulli family. A generalization of a problem first posed by Huygens is known as *De Moivre's problem*: Given n dice, each having f faces, find the probability of throwing any given number of points.[1] His many investigations in this field appeared in his work *The Doctrine of Chances: or, a Method of Calculating the Probability of Events in Play* (London, 1718, with expanded editions in 1738 and 1756); it contains numerous problems about throwing dice, drawing balls of different colors from a bag, and questions related to life annuities. Here also is stated (though he was not the first to discover it) the law for finding the probability of a compound event. A second work, *A Treatise of Annuities upon Lives* (London, 1725 and 1743), deals with the analysis of mortality statistics (which Halley had begun some years earlier), the division of annuities among several heirs, and other questions of interest to financial institutes and insurance companies.

In the theory of probability one constantly encounters the expression $n!$ (read "n factorial"), defined as $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. The value of $n!$ grows very rapidly with increasing n; for example, $10! = 3{,}628{,}800$ while $20! = 2{,}432{,}902{,}008{,}176{,}640{,}000$. To find $n!$ one must first find $(n-1)!$, which in turn requires finding $(n-2)!$ and so on, making a direct calculation of $n!$ for large n extremely time consuming. It is therefore desirable to have an approximation formula that would estimate $n!$ for large n by a single calculation. In a paper written in 1733 and presented privately to some friends, De Moivre developed the formula

$$n! \approx c n^{n+1/2} e^{-n},$$

where c is a constant and e the base of natural logarithms.[2] He was unable, however, to determine the numerical value of this constant; this task befell a Scot, James Stirling (1692–1770), who found that $c = \sqrt{2\pi}$. Stirling's formula, as it is known today, is thus as much due to De Moivre; it is usually written in the form

$$n! \approx \sqrt{2\pi n}\, n^n e^{-n}.$$

As an example, for $n = 20$ the formula gives $2.422,786,847 \times 10^{18}$, compared to the correct, rounded value $2.432,902,008 \times 10^{18}$.

De Moivre's third major work, *Miscellanea Analytica* (London, 1730), deals, in addition to probability, with algebra and analytic trigonometry. A major problem at the time was how to factor a polynomial such as $x^{2n} + px^n + 1$ into quadratic factors. This problem arose in connection with Cotes' work on the decomposition of rational expressions into partial fractions (then known as "recurring series"). De Moivre completed Cotes's work, left unfinished by the latter's early death (see p. 182). Among his many results we find the following formula, sometimes known as "Cotes' property of the circle":

$$x^{2n} + 1 = [x^2 - 2x\cos \pi/2n + 1][x^2 - 2x\cos 3\pi/2n + 1] \cdots [x^2 - 2x\cos (2n-1)\pi/2n + 1].$$

To obtain this factorization, we only need to find (using De Moivre's theorem) the $2n$ different roots of the equation $x^{2n} + 1 = 0$, that is, the $2n$ complex values of $\sqrt[2n]{-1}$, and then multiply the corresponding linear factors in conjugate pairs. The fact that trigonometric expressions appear in the factorization of a purely algebraic expression such as $x^{2n} + 1$ amazes any student who encounters such a formula for the first time; in De Moivre's time it amazed even professional mathematicians.

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De Moivre's famous theorem,

$$(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi,$$

was suggested by him in 1722 but was never explicitly stated in his work; that he knew it, however, is clear from the related formula

$$\cos \phi = \tfrac{1}{2}(\cos n\phi + i \sin n\phi)^{1/n} + \tfrac{1}{2}(\cos n\phi - i \sin n\phi)^{1/n},$$

which he had already found in 1707 (De Moivre derived it for positive integral values of n; Euler in 1749 proved it for any real n).[3] He frequently used it in *Miscellanea Analytica* and in numerous papers he published in the *Philosophical Transactions*, the official journal of the Royal Society. For example, in a paper published in 1739 he showed how to extract the roots of any binomial of the form $a + \sqrt{b}$ or $a + \sqrt{-b}$ (he calls the latter an "impossible binomial"). As a specific example he shows how to find the three cube roots of $81 + \sqrt{-2{,}700}$ (in modern notation, $81 + (30\sqrt{3})i$). The discussion is verbal rather than symbolic, but it is precisely the method we find today in any trigonometry textbook: write $81 + (30\sqrt{3})i$ in polar form as $r(\cos\theta + i\sin\theta)$, where $r = \sqrt{[81^2 + (30\sqrt{3})^2]} = \sqrt{9{,}261} = 21\sqrt{21}$ and $\theta = \tan^{-1}(30\sqrt{3})/81 = \tan^{-1}(10\sqrt{3})/27 = 32.68°$. Then compute the expression $(\sqrt[3]{r})[\cos(\theta + 360°k)/3 + i\sin(\theta + 360°k)/3]$ for $k = 0, 1,$ and 2. We have $\sqrt[3]{(21\sqrt{21})} = (21^{3/2})^{1/3} = 21^{1/2} = \sqrt{21}$ and $\theta/3 = 10.89°$, so the roots are $\sqrt{21}$ cis $(10.89° + 120°k)$, where "cis" stands for cos + i sin. Using a table or a calculator to find the sines and cosines, we get the three required roots: $(9 + (\sqrt{3})i)/2$, $-3 + (2\sqrt{3})i$, and $(-3 - (5\sqrt{3})i)/2$. De Moivre adds:

> There have been several authors, and among them the eminent Wallis, who have thought that those cubic equations which are referred to the circle, may be solved by the extraction of the cube root of an imaginary quantity, and of $81 + \sqrt{-2{,}700}$, without regard to the table of sines, but this is a mere fiction and a begging of the question. For on attempting it, the result always recurs back again to the same question as that first proposed. And the thing cannot be done directly, without the help of the table of sines, especially when the roots are irrational, as has been observed by many others.[4]

De Moivre must have surely wondered why the three roots come out as "nice" irrational complex numbers, even though the angle θ is not any "special" angle like $15^\circ, 30^\circ$, or 45°. He says that it is a "fiction" (i.e., impossible) to find the cubic root of a complex number without a table of sines; and to avoid any misunderstanding, he repeats this statement again toward the end: "And the thing cannot be done directly, without the help of the table of sines, especially when the roots are irrational." He is of course right in the general case: to find the three cubic roots of a complex number $z = x + iy$, we must express it in polar form, $z = r \text{ cis } \theta$, where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1} y/x$; next we compute $\sqrt[3]{r}$ and $\theta/3$, then—using a table of sines—we find $\cos\theta/3$ and $\sin\theta/3$, and finally $(\sqrt[3]{r}) \text{ cis } (\theta/3 + 120^\circ k)$ for $k = 0, 1$, and 2. Ironically, however, the very example De Moivre brings to illustrate the procedure could be solved without recourse to a table! Let us see how.

We wish to find the three cubic roots of $z = x + iy = 81 + (30\sqrt{3})i = r \text{ cis } \theta$, where $r = 21\sqrt{21}$ and $\tan\theta = (10\sqrt{3})/27$. From this last equation (or by directly computing x/r) we find that $\cos\theta = 81/(21\sqrt{21}) = (9\sqrt{21})/49$. We now use the identity $\cos\theta = 4\cos^3\theta/3 - 3\cos\theta/3$ to find the value of $\cos\theta/3$; letting $x = \cos\theta/3$, we have

$$(9\sqrt{21})/49 = 4x^3 - 3x \tag{1}$$

or

$$196x^3 - 147x - 9\sqrt{21} = 0. \tag{2}$$

The substitution $y = x/(9\sqrt{21})$ reduces this equation to

$$333{,}396\,y^3 - 147y - 1 = 0. \tag{3}$$

This new equation is radical-free, but its leading coefficient looks hopelessly large. It so happens, however, that 147 is divisible by 21 and 333,396 is divisible by 21^3. Writing $z = 21y$, the equation becomes

$$36z^3 - 7z - 1 = 0, \tag{4}$$

a pretty simple equation whose three roots are $1/2$, $-1/3$ and $-1/6$—all rational numbers! Substituting back, we have $y = z/21 = 1/42, -1/63$ and $-1/126$, and finally $x = \cos\theta/3 = (9\sqrt{21})y = (3\sqrt{21})/14, -(\sqrt{21})/7$, and $-(\sqrt{21})/14$. For each of these values we now find $\sin\theta/3$ from the identity $\sin\theta/3 = \pm\sqrt{(1 - \cos^2\theta/3)}$; we get $\sin\theta/3 = (\sqrt{7})/14, (2\sqrt{7})/7$, and $-(5\sqrt{7})/14$ (the last is negative because the corresponding point is in Quadrant III of the complex plane). We also have $\sqrt[3]{r} = (21\sqrt{21})^{1/3} = \sqrt{21}$. The three required roots are thus

$$\sqrt{21}\left[3\sqrt{21}/14 + (\sqrt{7}/14)i\right] = \tfrac{1}{2}(9 + \sqrt{3}i),$$

$$\sqrt{21}\left[-\sqrt{21}/7 + (2\sqrt{7}/7)i\right] = -3 + \sqrt{3}i,$$

and

$$\sqrt{21}\left[-\sqrt{21}/14 - (5\sqrt{7}/14)i\right] = \tfrac{1}{2}(-3 - 5\sqrt{3}i);$$

they are shown in figure 27.

Of course, the "natural" way of handling this problem would be to solve equation (1) directly, using a formula named after the Italian Girolamo Cardano (also known as Jerome Cardan, 1501–1576) but actually developed independently by two other Italians, Scipione del Ferro (ca. 1465–1526) and Nicolo Tartaglia (ca. 1506–1557).[5] Cardano's formula is analogous to the familiar quadratic formula for solving equations of the second degree but is considerably more complicated; it is based on the fact that any cubic equation in normal form, $y^3 + ay^2 + by + c = 0$ (where the leading coefficient $= 1$) can be brought to the *reduced* form $x^3 + px + q = 0$ (with no quadratic term) by the substitution $y = x - a/3$, where $p = b - a^2/3$ and $q = 2a^3/27 - ab/3 + c$. Since equation (1) is already free of a quadratic term, we only need to divide it by its leading coefficient, getting $x^3 + px + q = 0$, where $p = -3/4$ and $q = -(9\sqrt{21})/196$. Cardano's formula now requires one to compute the quantities $P = \sqrt[3]{-q/2 + \sqrt{q^2/4 + p^3/27}}$ and $Q = \sqrt[3]{-q/2 - \sqrt{q^2/4 + p^3/27}}$. Substituting the values of p

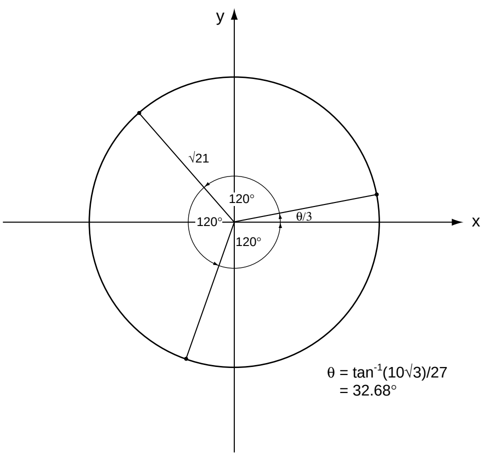

FIG. 27. The three cube roots of $81 + \sqrt{-2700}$.

and q into these expressions we get, after considerable simplification, $P, Q = \frac{1}{14}\sqrt[3]{-63\sqrt{21} \pm (70\sqrt{7})i}$. So we must now find the cubic root of the complex numbers $-63\sqrt{21} \pm (70\sqrt{7})i$, and to do so we must express them in polar form R cis ϕ. We have $R = \sqrt{[(-63\sqrt{21})^2 + (70\sqrt{7})^2]} = 343$ and $\phi = \pm\tan^{-1}(70\sqrt{7})/(63\sqrt{21}) = \pm\tan^{-1}(10\sqrt{3})/27$—the very same angle we had in the first place! This is what De Moivre meant in his enigmatic statement, "For on attempting it, the result always recurs back again to the same question as that first proposed."

Could a mathematician of De Moivre's caliber have overlooked the fact that his own example could be solved without using a table of sines? Apparently he did. Even Einstein once ignored the possibility that a denominator in one of his equations might be zero. This was in 1917, when he applied his general theory of relativity to cosmological questions. A young Russian astronomer, Aleksandr Friedmann, noticed this seemingly benign oversight and concluded that the particular case Einstein had ignored implied, no less, that the universe might be expanding![6]

Go to Chapter 6

Notes and Sources

1. Florian Cajori, *A History of Mathematics* (1893; 2d ed. New York: Macmillan, 1919), p. 230.

2. This paper also gives the first statement of the formula for the normal distribution. See David Eugene Smith, *A Source Book in Mathematics* (1929; rpt. New York: Dover, 1959), pp. 566–568.

3. From this relation and its companion,

$$i \sin\phi = \tfrac{1}{2}(\cos n\phi + i \sin n\phi)^{1/n} - \tfrac{1}{2}(\cos n\phi - i \sin n\phi)^{1/n}$$

we get, upon adding, $\cos\phi + i\sin\phi = (\cos n\phi + i \sin n\phi)^{1/n}$, from which De Moivre's theorem follows immediately. For Euler's proof that the formula is valid for any real n, see Smith, pp. 452–454.

4. Ibid., pp. 447–450. Two of the roots appearing there, $-3/2 + (5\sqrt{3})i/2$ and $-3 + (\sqrt{3})i/2$, are clearly wrong, probably as a result of a misprint.

5. The history of the cubic equation is a long one, replete with controversy and intrigue. See David Eugene Smith, *History of Mathematics* (1925; rpt. New York: Dover, 1958), vol. 2, pp. 454–466; Victor J. Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 328–337; and David M. Burton, *History of Mathematics: An Introduction* (Dubuque, Iowa: Wm. C. Brown, 1995), pp. 288–299.

6. Ronald W. Clark, *Einstein: The Life and Times* (1971; rpt. New York: Avon Books, 1972), p. 270.