

Contents

1	Introduction	1
2	Berry Phase	6
	2.1 General Formalism	6
	2.2 Gauge-Independent Computation of the Berry Phase	8
	2.3 Degeneracies and Level Crossing	10
	2.3.1 Two-Level System Using the Berry Curvature	10
	2.3.2 Two-Level System Using the Hamiltonian Approach	11
	2.4 Spin in a Magnetic Field	13
	2.5 Can the Berry Phase Be Measured?	14
	2.6 Problems	14
3	Hall Conductance and Chern Numbers	15
	3.1 Current Operators	15
	3.1.1 Current Operators from the Continuity Equation	16
	3.1.2 Current Operators from Peierls Substitution	17
	3.2 Linear Response to an Applied External Electric Field	18
	3.2.1 The Fluctuation Dissipation Theorem	20
	3.2.2 Finite-Temperature Green's Function	22
	3.3 Current-Current Correlation Function and Electrical Conductivity	23
	3.4 Computing the Hall Conductance	24
	3.4.1 Diagonalizing the Hamiltonian and the Flat-Band Basis	25
	3.5 Alternative Form of the Hall Response	29
	3.6 Chern Number as an Obstruction to Stokes' Theorem over the Whole BZ	30
	3.7 Problems	32
4	Time-Reversal Symmetry	33
	4.1 Time Reversal for Spinless Particles	33
	4.1.1 Time Reversal in Crystals for Spinless Particles	34
	4.1.2 Vanishing of Hall Conductance for T -Invariant Spinless Fermions	35
	4.2 Time Reversal for Spinful Particles	35
	4.3 Kramers' Theorem	36
	4.4 Time-Reversal Symmetry in Crystals for Half-Integer Spin Particles	37
	4.5 Vanishing of Hall Conductance for T -Invariant Half-Integer Spin Particles	39
	4.6 Problems	40
5	Magnetic Field on the Square Lattice	41
	5.1 Hamiltonian and Lattice Translations	41
	5.2 Diagonalization of the Hamiltonian of a 2-D Lattice in a Magnetic Field	44
	5.2.1 Dependence on k_y	46
	5.2.2 Dirac Fermions in the Magnetic Field on the Lattice	47
	5.3 Hall Conductance	49
	5.3.1 Diophantine Equation and Streda Formula Method	49

5.4	Explicit Calculation of the Hall Conductance	51
5.5	Problems	59
6	Hall Conductance and Edge Modes: The Bulk-Edge Correspondence	60
6.1	Laughlin's Gauge Argument	60
6.2	The Transfer Matrix Method	62
6.3	Edge Modes	65
6.4	Bulk Bands	65
6.5	Problems	69
7	Graphene	70
7.1	Hexagonal Lattices	70
7.2	Dirac Fermions	72
7.3	Symmetries of a Graphene Sheet	72
7.3.1	Time Reversal	73
7.3.2	Inversion Symmetry	73
7.3.3	Local Stability of Dirac Points with Inversion and Time Reversal	74
7.4	Global Stability of Dirac Points	76
7.4.1	C_3 Symmetry and the Position of the Dirac Nodes	76
7.4.2	Breaking of C_3 Symmetry	79
7.5	Edge Modes of the Graphene Layer	80
7.5.1	Chains with Even Number of Sites	82
7.5.2	Chains with Odd Number of Sites	85
7.5.3	Influence of Different Mass Terms on the Graphene Edge Modes	89
7.6	Problems	90
8	Simple Models for the Chern Insulator	91
8.1	Dirac Fermions and the Breaking of Time-Reversal Symmetry	91
8.1.1	When the Matrices σ Correspond to Real Spin	91
8.1.2	When the Matrices σ Correspond to Isospin	92
8.2	Explicit Berry Potential of a Two-Level System	92
8.2.1	Berry Phase of a Continuum Dirac Hamiltonian	92
8.2.2	The Berry Phase for a Generic Dirac Hamiltonian in Two Dimensions	93
8.2.3	Hall Conductivity of a Dirac Fermion in the Continuum	94
8.3	Skyrmion Number and the Lattice Chern Insulator	95
8.3.1	$M > 0$ Phase and $M < -4$ Phase	96
8.3.2	The $-2 < M < 0$ Phase	96
8.3.3	The $-4 < M < -2$ Phase	98
8.3.4	Back to the Trivial State for $M < -4$	98
8.4	Determinant Formula for the Hall Conductance of a Generic Dirac Hamiltonian	99
8.5	Behavior of the Vector Potential on the Lattice	99
8.6	The Problem of Choosing a Consistent Gauge in the Chern Insulator	100
8.7	Chern Insulator in a Magnetic Field	102
8.8	Edge Modes and the Dirac Equation	103
8.9	Haldane's Graphene Model	104
8.9.1	Symmetry Properties of the Haldane Hamiltonian	106
8.9.2	Phase Diagram of the Haldane Hamiltonian	106
8.10	Problems	107

9	Time-Reversal-Invariant Topological Insulators	109
9.1	The Kane and Mele Model: Continuum Version	109
9.1.1	Adding Spin	110
9.1.2	Spin \uparrow and Spin \downarrow	112
9.1.3	Rashba Term	112
9.2	The Kane and Mele Model: Lattice Version	113
9.3	First Topological Insulator: Mercury Telluride Quantum Wells	117
9.3.1	Inverted Quantum Wells	117
9.4	Experimental Detection of the Quantum Spin Hall State	120
9.5	Problems	121
10	Z_2 Invariants	123
10.1	Z_2 Invariant as Zeros of the Pfaffian	123
10.1.1	Pfaffian in the Even Subspace	124
10.1.2	The Odd Subspace	125
10.1.3	Example of an Odd Subspace: $d_a = 0$ Subspace	125
10.1.4	Zeros of the Pfaffian	126
10.1.5	Explicit Example for the Kane and Mele Model	127
10.2	Theory of Charge Polarization in One Dimension	128
10.3	Time-Reversal Polarization	130
10.3.1	Non-Abelian Berry Potentials at k , $-k$	133
10.3.2	Proof of the Unitarity of the Sewing Matrix B	134
10.3.3	A New Pfaffian Z_2 Index	134
10.4	Z_2 Index for 3-D Topological Insulators	138
10.5	Z_2 Number as an Obstruction	141
10.6	Equivalence between Topological Insulator Descriptions	144
10.7	Problems	145
11	Crossings in Different Dimensions	147
11.1	Inversion-Asymmetric Systems	148
11.1.1	Two Dimensions	149
11.1.2	Three Dimensions	149
11.2	Inversion-Symmetric Systems	151
11.2.1	$\eta_a = \eta_b$	151
11.2.2	$\eta_a = -\eta_b$	152
11.3	Mercury Telluride Hamiltonian	154
11.4	Problems	156
12	Time-Reversal Topological Insulators with Inversion Symmetry	158
12.1	Both Inversion and Time-Reversal Invariance	159
12.2	Role of Spin-Orbit Coupling	162
12.3	Problems	163
13	Quantum Hall Effect and Chern Insulators in Higher Dimensions	164
13.1	Chern Insulator in Four Dimensions	164
13.2	Proof That the Second Chern Number Is Topological	166
13.3	Evaluation of the Second Chern Number: From a Green's Function Expression to the Non-Abelian Berry Curvature	167

13.4	Physical Consequences of the Transport Law of the 4-D Chern Insulator	169
13.5	Simple Example of Time-Reversal-Invariant Topological Insulators with Time-Reversal and Inversion Symmetry Based on Lattice Dirac Models	172
13.6	Problems	175
14	Dimensional Reduction of 4-D Chern Insulators to 3-D Time-Reversal Insulators	177
14.1	Low-Energy Effective Action of (3 + 1)-D Insulators and the Magnetoelectric Polarization	177
14.2	Magnetoelectric Polarization for a 3-D Insulator with Time-Reversal Symmetry	181
14.3	Magnetoelectric Polarization for a 3-D Insulator with Inversion Symmetry	182
14.4	3-D Hamiltonians with Time-Reversal Symmetry and/or Inversion Symmetry as Dimensional Reductions of 4-D Time-Reversal-Invariant Chern Insulators	184
14.5	Problems	185
15	Experimental Consequences of the Z_2 Topological Invariant	186
15.1	Quantum Hall Effect on the Surface of a Topological Insulator	186
15.2	Physical Properties of Time-Reversal Z_2 -Nontrivial Insulators	187
15.3	Half-Quantized Hall Conductance at the Surface of Topological Insulators with Ferromagnetic Hard Boundary	188
15.4	Experimental Setup for Indirect Measurement of the Half-Quantized Hall Conductance on the Surface of a Topological Insulator	189
15.5	Topological Magnetoelectric Effect	189
15.6	Problems	191
16	Topological Superconductors in One and Two Dimensions by Taylor L. Hughes	193
16.1	Introducing the Bogoliubov-de-Gennes (BdG) Formalism for <i>s</i> -Wave Superconductors	193
16.2	<i>p</i> -Wave Superconductors in One Dimension	196
16.2.1	1-D <i>p</i> -Wave Wire	196
16.2.2	Lattice <i>p</i> -Wave Wire and Majorana Fermions	199
16.3	2-D Chiral <i>p</i> -Wave Superconductor	201
16.3.1	Bound States on Vortices in 2-D Chiral <i>p</i> -wave Superconductors	206
16.4	Problems	211
17	Time-Reversal-Invariant Topological Superconductors by Taylor L. Hughes	214
17.1	Superconducting Pairing with Spin	214
17.2	Time-Reversal-Invariant Superconductors in Two Dimensions	215
17.2.1	Vortices in 2-D Time-Reversal-Invariant Superconductors	218
17.3	Time-Reversal-Invariant Superconductors in Three Dimensions	219
17.4	Finishing the Classification of Time-Reversal-Invariant Superconductors	222
17.5	Problems	224

18	Superconductivity and Magnetism in Proximity to Topological Insulator Surfaces by Taylor L. Hughes	226
18.1	Generating 1-D Topological Insulators and Superconductors on the Edge of the Quantum-Spin Hall Effect	226
18.2	Constructing Topological States from Interfaces on the Boundary of Topological Insulators	228
18.3	Problems	234
	APPENDIX: 3-D Topological Insulator in a Magnetic Field	237
	References	241
	Index	245