
Contents

Contributors	v
Preface	xv
1 Kähler Manifolds by E. Cattani	1
1.1 Complex Manifolds	2
1.1.1 Definition and Examples	2
1.1.2 Holomorphic Vector Bundles	11
1.2 Differential Forms on Complex Manifolds	15
1.2.1 Almost Complex Manifolds	15
1.2.2 Tangent and Cotangent Space	16
1.2.3 De Rham and Dolbeault Cohomologies	20
1.3 Symplectic, Hermitian, and Kähler Structures	23
1.3.1 Kähler Manifolds	25
1.3.2 The Chern Class of a Holomorphic Line Bundle	28
1.4 Harmonic Forms—Hodge Theorem	30
1.4.1 Compact Real Manifolds	30
1.4.2 The $\bar{\partial}$ -Laplacian	35
1.5 Cohomology of Compact Kähler Manifolds	36
1.5.1 The Kähler Identities	36
1.5.2 The Hodge Decomposition Theorem	37
1.5.3 Lefschetz Theorems and Hodge–Riemann Bilinear Relations	39
A Linear Algebra	45
A.1 Real and Complex Vector Spaces	45
A.2 The Weight Filtration of a Nilpotent Transformation	50
A.3 Representations of $\mathfrak{sl}(2, \mathbb{C})$ and Lefschetz Theorems	51
A.4 Hodge Structures	54
B The Kähler Identities by P. A. Griffiths	58
B.1 Symplectic Linear Algebra	58
B.2 Compatible Inner Products	61
B.3 Symplectic Manifolds	63
B.4 The Kähler Identities	64
Bibliography	67

2	The Algebraic de Rham Theorem by F. El Zein and L. Tu	70
	Introduction	70
	Part I. Sheaf Cohomology, Hypercohomology, and the Projective Case	72
2.1	Sheaves	72
2.1.1	The Étale Space of a Presheaf	72
2.1.2	Exact Sequences of Sheaves	73
2.1.3	Resolutions	74
2.2	Sheaf Cohomology	75
2.2.1	Godement's Canonical Resolution	76
2.2.2	Cohomology with Coefficients in a Sheaf	79
2.2.3	Flasque Sheaves	80
2.2.4	Cohomology Sheaves and Exact Functors	83
2.2.5	Fine Sheaves	85
2.2.6	Cohomology with Coefficients in a Fine Sheaf	87
2.3	Coherent Sheaves and Serre's GAGA Principle	89
2.4	The Hypercohomology of a Complex of Sheaves	92
2.4.1	The Spectral Sequences of Hypercohomology	94
2.4.2	Acyclic Resolutions	96
2.5	The Analytic de Rham Theorem	98
2.5.1	The Holomorphic Poincaré Lemma	98
2.5.2	The Analytic de Rham Theorem	99
2.6	The Algebraic de Rham Theorem for a Projective Variety	100
	Part II. Čech Cohomology and the Algebraic de Rham Theorem in General	101
2.7	Čech Cohomology of a Sheaf	101
2.7.1	Čech Cohomology of an Open Cover	102
2.7.2	Relation Between Čech Cohomology and Sheaf Cohomology	103
2.8	Čech Cohomology of a Complex of Sheaves	105
2.8.1	The Relation Between Čech Cohomology and Hypercohomology	106
2.9	Reduction to the Affine Case	108
2.9.1	Proof that the General Case Implies the Affine Case	109
2.9.2	Proof that the Affine Case Implies the General Case	109
2.10	The Algebraic de Rham Theorem for an Affine Variety	111
2.10.1	The Hypercohomology of the Direct Image of a Sheaf of Smooth Forms	112
2.10.2	The Hypercohomology of Rational and Meromorphic Forms	112
2.10.3	Comparison of Meromorphic and Smooth Forms	116
	Bibliography	121
3	Mixed Hodge Structures by F. El Zein and Lê D. T.	123
3.1	Hodge Structure on a Smooth Compact Complex Variety	128
3.1.1	Hodge Structure (HS)	128
3.1.2	Spectral Sequence of a Filtered Complex	133

3.1.3	Hodge Structure on the Cohomology of Nonsingular Compact Complex Algebraic Varieties	137
3.1.4	Lefschetz Decomposition and Polarized Hodge Structure	141
3.1.5	Examples	144
3.1.6	Cohomology Class of a Subvariety and Hodge Conjecture	147
3.2	Mixed Hodge Structures (MHS)	150
3.2.1	Filtrations	151
3.2.2	Mixed Hodge Structures (MHS)	157
3.2.3	Induced Filtrations on Spectral Sequences	164
3.2.4	MHS of a Normal Crossing Divisor (NCD)	168
3.3	Mixed Hodge Complex	171
3.3.1	Derived Category	172
3.3.2	Derived Functor on a Filtered Complex	179
3.3.3	Mixed Hodge Complex (MHC)	184
3.3.4	Relative Cohomology and the Mixed Cone	188
3.4	MHS on the Cohomology of a Complex Algebraic Variety	191
3.4.1	MHS on the Cohomology of Smooth Algebraic Varieties	192
3.4.2	MHS on Cohomology of Simplicial Varieties	200
3.4.3	MHS on the Cohomology of a Complete Embedded Algebraic Variety	209
Bibliography		214
4 Period Domains by J. Carlson		217
4.1	Period Domains and Monodromy	221
4.2	Elliptic Curves	225
4.3	Period Mappings: An Example	230
4.4	Hodge Structures of Weight 1	234
4.5	Hodge Structures of Weight 2	237
4.6	Poincaré Residues	239
4.7	Properties of the Period Mapping	243
4.8	The Jacobian Ideal and the Local Torelli Theorem	244
4.9	The Horizontal Distribution—Distance-Decreasing Properties	247
4.10	The Horizontal Distribution—Integral Manifolds	250
Bibliography		255
5 Hodge Theory of Maps, Part I by L. Migliorini		257
5.1	Lecture 1: The Smooth Case: E_2 -Degeneration	258
5.2	Lecture 2: Mixed Hodge Structures	261
5.2.1	Mixed Hodge Structures on the Cohomology of Algebraic Varieties	261
5.2.2	The Global Invariant Cycle Theorem	263
5.2.3	Semisimplicity of Monodromy	264

5.3	Lecture 3: Two Classical Theorems on Surfaces and the Local Invariant Cycle Theorem	266
5.3.1	Homological Interpretation of the Contraction Criterion and Zariski’s Lemma	266
5.3.2	The Local Invariant Cycle Theorem, the Limit Mixed Hodge Structure, and the Clemens–Schmid Exact Sequence	270
Bibliography		272
6	Hodge Theory of Maps, Part II by M. A. de Cataldo	273
6.1	Lecture 4	273
6.1.1	Sheaf Cohomology and All That (A Minimalist Approach)	273
6.1.2	The Intersection Cohomology Complex	284
6.1.3	Verdier Duality	286
6.2	Lecture 5	289
6.2.1	The Decomposition Theorem (DT)	289
6.2.2	The Relative Hard Lefschetz and the Hard Lefschetz for Intersection Cohomology Groups	292
Bibliography		295
7	Variations of Hodge Structure by E. Cattani	297
7.1	Local Systems and Flat Connections	298
7.1.1	Local Systems	298
7.1.2	Flat Bundles	300
7.2	Analytic Families	302
7.2.1	The Kodaira–Spencer Map	303
7.3	Variations of Hodge Structure	306
7.3.1	Geometric Variations of Hodge Structure	306
7.3.2	Abstract Variations of Hodge Structure	310
7.4	Classifying Spaces	311
7.5	Mixed Hodge Structures and the Orbit Theorems	315
7.5.1	Nilpotent Orbits	315
7.5.2	Mixed Hodge Structures	318
7.5.3	SL_2 -Orbits	321
7.6	Asymptotic Behavior of a Period Mapping	323
Bibliography		329
8	Variations of Mixed Hodge Structure by P. Brosnan and F. El Zein	333
8.1	Variation of Mixed Hodge Structures	335
8.1.1	Local Systems and Representations of the Fundamental Group	335
8.1.2	Connections and Local Systems	337
8.1.3	Variation of Mixed Hodge Structure of Geometric Origin	340
8.1.4	Singularities of Local Systems	346

8.2	Degeneration of Variations of Mixed Hodge Structures	352
8.2.1	Diagonal Degeneration of Geometric VMHS	352
8.2.2	Filtered Mixed Hodge Complex (FMHC)	354
8.2.3	Diagonal Direct Image of a Simplicial Cohomological FMHC	357
8.2.4	Construction of a Limit MHS on the Unipotent Nearby Cycles	359
8.2.5	Case of a Smooth Morphism	360
8.2.6	Polarized Hodge–Lefschetz Structure	364
8.2.7	Quasi-projective Case	366
8.2.8	Alternative Construction, Existence and Uniqueness	367
8.3	Admissible Variation of Mixed Hodge Structure	369
8.3.1	Definition and Results	370
8.3.2	Local Study of Infinitesimal Mixed Hodge Structures After Kashiwara	372
8.3.3	Deligne–Hodge Theory on the Cohomology of a Smooth Variety	375
8.4	Admissible Normal Functions	391
8.4.1	Reducing Theorem 8.4.6 to a Special Case	394
8.4.2	Examples	394
8.4.3	Classifying Spaces	397
8.4.4	Pure Classifying Spaces	397
8.4.5	Mixed Classifying Spaces	397
8.4.6	Local Normal Form	398
8.4.7	Splittings	399
8.4.8	A Formula for the Zero Locus of a Normal Function	400
8.4.9	Proof of Theorem 8.4.6 for Curves	401
8.4.10	An Example	402

Bibliography **407**

9 Algebraic Cycles and Chow Groups by J. Murre **410**

9.1	Lecture I: Algebraic Cycles. Chow Groups	410
9.1.1	Assumptions and Conventions	410
9.1.2	Algebraic Cycles	411
9.1.3	Adequate Equivalence Relations	413
9.1.4	Rational Equivalence. Chow Groups	414
9.2	Lecture II: Equivalence Relations. Short Survey on the Results for Divisors	418
9.2.1	Algebraic Equivalence (Weil, 1952)	419
9.2.2	Smash-Nilpotent Equivalence	419
9.2.3	Homological Equivalence	420
9.2.4	Numerical Equivalence	421
9.2.5	Final Remarks and Résumé of Relations and Notation	422
9.2.6	Cartier Divisors and the Picard Group	422
9.2.7	Résumé of the Main Facts for Divisors	423
9.2.8	References for Lectures I and II	425
9.3	Lecture III: Cycle Map. Intermediate Jacobian. Deligne Cohomology	425

9.3.1	The Cycle Map	425
9.3.2	Hodge Classes. Hodge Conjecture	427
9.3.3	Intermediate Jacobian and Abel–Jacobi Map	428
9.3.4	Deligne Cohomology. Deligne Cycle Map	431
9.3.5	References for Lecture III	434
9.4	Lecture IV: Algebraic Versus Homological Equivalence. Griffiths Group	434
9.4.1	Lefschetz Theory	434
9.4.2	Return to the Griffiths Theorem	437
9.4.3	References for Lecture IV	440
9.5	Lecture V: The Albanese Kernel. Results of Mumford, Bloch, and Bloch–Srinivas	440
9.5.1	The Result of Mumford	440
9.5.2	Reformulation and Generalization by Bloch	442
9.5.3	A Result on the Diagonal	443
9.5.4	References for Lecture V	445

Bibliography **446**

10 Spreads and Algebraic Cycles by M. L. Green **449**

10.1	Introduction to Spreads	449
10.2	Cycle Class and Spreads	452
10.3	The Conjectural Filtration on Chow Groups from a Spread Perspective	456
10.4	The Case of X Defined over \mathbb{Q}	460
10.5	The Tangent Space to Algebraic Cycles	463

Bibliography **467**

11 Absolute Hodge Classes by F. Charles and C. Schnell **469**

11.1	Algebraic de Rham Cohomology	470
11.1.1	Algebraic de Rham Cohomology	471
11.1.2	Cycle Classes	473
11.2	Absolute Hodge Classes	476
11.2.1	Algebraic Cycles and the Hodge Conjecture	477
11.2.2	Galois Action, Algebraic de Rham Cohomology, and Absolute Hodge Classes	478
11.2.3	Variations on the Definition and Some Functoriality Properties	481
11.2.4	Classes Coming from the Standard Conjectures and Polarizations	484
11.2.5	Absolute Hodge Classes and the Hodge Conjecture	488
11.3	Absolute Hodge Classes in Families	491
11.3.1	The Variational Hodge Conjecture and the Global Invariant Cycle Theorem	491
11.3.2	Deligne’s Principle B	494
11.3.3	The Locus of Hodge Classes	496
11.3.4	Galois Action on Relative de Rham Cohomology	498

11.3.5	The Field of Definition of the Locus of Hodge Classes	500
11.4	The Kuga–Satake Construction	502
11.4.1	Recollection on Spin Groups	502
11.4.2	Spin Representations	503
11.4.3	Hodge Structures and the Deligne Torus	504
11.4.4	From Weight 2 to Weight 1	504
11.4.5	The Kuga–Satake Correspondence Is Absolute	506
11.5	Deligne’s Theorem on Hodge Classes on Abelian Varieties	508
11.5.1	Overview	508
11.5.2	Hodge Structures of CM-Type	510
11.5.3	Reduction to Abelian Varieties of CM-Type	514
11.5.4	Background on Hermitian Forms	516
11.5.5	Construction of Split Weil Classes	519
11.5.6	André’s Theorem and Reduction to Split Weil Classes	521
11.5.7	Split Weil Classes are Absolute	523
Bibliography		528
12 Shimura Varieties by M. Kerr		531
12.1	Hermitian Symmetric Domains	532
A.	Algebraic Groups and Their Properties	532
B.	Three Characterizations of Hermitian Symmetric Domains	536
C.	Cartan’s Classification of Irreducible Hermitian Symmetric Domains	538
D.	Hodge-Theoretic Interpretation	540
12.2	Locally Symmetric Varieties	543
12.3	Complex Multiplication	547
A.	CM-Abelian Varieties	547
B.	Class Field Theory	551
C.	Main Theorem of CM	555
12.4	Shimura Varieties	556
A.	Three Key Adélic Lemmas	556
B.	Shimura Data	558
C.	The Adélic Reformulation	561
D.	Examples	563
12.5	Fields of Definition	567
A.	Reflex Field of a Shimura Datum	567
B.	Canonical Models	568
C.	Connected Components and VHS	571
Bibliography		574
Index		577