## Contents

**Preface**

xii

**Acknowledgments**

xv

**INTRODUCTION: WHAT THIS BOOK IS ABOUT**

1

1. Plus 1

2. Sums of Interest 3

**PART ONE. FINITE SUMS**

### CHAPTER 1. PROEM

1. Greatest Common Divisors 11
2. Congruences 14
3. Wilson's Theorem 15
4. Quadratic Residues and Nonresidues 17
5. The Legendre Symbol 19

### CHAPTER 2. SUMS OF TWO SQUARES

1. The Answer 22
2. The Proof Is Not in the Pudding 26
3. The “If” Parts of Theorems 2.1 and 2.3 28
4. The Details 29

### CHAPTER 3. SUMS OF THREE AND FOUR SQUARES

1. Three Squares 32
2. Interlude 33
3. Four Squares 34
4. Sums of More Than Four Squares 35

CHAPTER 4. SUMS OF HIGHER POWERS: WARING’S PROBLEM 37
1. \(g(k)\) and \(G(k)\) 37
2. Sums of Biquadrates 39
3. Higher Powers 40

CHAPTER 5. SIMPLE SUMS 42
1. Return to First Grade 42
2. Adding Small Powers 43

CHAPTER 6. SUMS OF POWERS, USING LOTS OF ALGEBRA 50
1. History 50
2. Squares 52
3. Divertimento: Double Sums 55
4. Telescoping Sums 57
5. Telescoping Sums Redux 59
6. Digression: Euler–Maclaurin Summation 66

PART TWO. INFINITE SUMS

CHAPTER 7. INFINITE SERIES 73
1. Finite Geometric Series 73
2. Infinite Geometric Series 75
3. The Binomial Series 76
4. Complex Numbers and Functions 79
5. Infinite Geometric Series Again 81
6. Examples of Infinite Sums 83
7. \(e, e^x,\) and \(e^z\) 85
8. Power Series 87
9. Analytic Continuation 91
CHAPTER 13. HOW MANY MODULAR FORMS ARE THERE?
1. How to Count Infinite Sets 160
2. How Big Are \( M_k \) and \( S_k \)? 164
3. The \( q \)-expansion 169
4. Multiplying Modular Forms 171
5. Dimensions of \( M_k \) and \( S_k \) 175

CHAPTER 14. CONGRUENCE GROUPS 179
1. Other Weights 179
2. Modular Forms of Integral Weight and Higher Level 182
3. Fundamental Domains and Cusps 182
4. Modular Forms of Half-Integral Weight 184

CHAPTER 15. PARTITIONS AND SUMS OF SQUARES REVISITED 186
1. Partitions 186
2. Sums of Squares 190
3. Numerical Example and Philosophical Reflection 196

CHAPTER 16. MORE THEORY OF MODULAR FORMS 201
1. Hecke Operators 201
2. New Clothes, Old Clothes 208
3. \( L \)-functions 210

CHAPTER 17. MORE THINGS TO DO WITH MODULAR FORMS: APPLICATIONS 213
1. Galois Representations 214
2. Elliptic Curves 217
3. Moonshine 219
4. Bigger Groups (Sato–Tate) 221
5. Envoy 223

Bibliography 225
Index 227