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Kari Astala, Tadeusz Iwaniec & Gaven Martin: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane

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