
Contents

Preface	ix
Notation and Conventions	xi
Introduction: An Overview of Some Problems of Unlikely Intersections	1
1 Unlikely Intersections in Multiplicative Groups and the Zilber Conjecture	15
1.1 Torsion points on subvarieties of \mathbb{G}_m^n	16
1.2 Higher multiplicative rank	22
1.3 Remarks on Theorem 1.3 and its developments	29
1.3.1 Fields other than $\overline{\mathbb{Q}}$	29
1.3.2 Weakened assumptions	29
1.3.3 Unlikely intersections of positive dimension and height bounds	31
1.3.4 Unlikely intersections of positive dimension and Zilber’s conjecture	33
1.3.5 Unlikely intersections and reducibility of lacunary polynomials (Schinzel’s conjecture)	35
1.3.6 Zhang’s notion of dependence	36
1.3.7 Abelian varieties (and other algebraic groups)	36
1.3.8 Uniformity of bounds	37
<i>Notes to Chapter 1</i>	39
Sparseness of multiplicatively dependent points	39
Other unlikely intersections	39
A generalization of Theorem 1.3	40
An application of the methods to zeros of linear recurrences	40
Comments on the Methods	41
2 An Arithmetical Analogue	43
2.1 Some unlikely intersections in number fields	43
2.2 Some applications of Theorem 2.1	48
2.3 An analogue of Theorem 2.1 for function fields	50
2.4 Some applications of Theorem 2.2	52
2.5 A proof of Theorem 2.2	54
<i>Notes to Chapter 2</i>	58
Simplifying the proof of Theorem 1.3	58
Rational points on curves over \mathbb{F}_p	58
Unlikely Intersections and Holomorphic GCD in Nevanlinna Theory	60

3 Unlikely Intersections in Elliptic Surfaces and Problems of Masser	62
3.1 A method for the Manin-Mumford conjecture	62
3.2 Masser's questions on elliptic pencils	66
3.3 A finiteness proof	70
3.4 Related problems, conjectures, and developments	77
3.4.1 Pink's and related conjectures	77
3.4.2 Extending Theorem 3.3 from $\overline{\mathbb{Q}}$ to \mathbb{C}	80
3.4.3 Effectivity	83
3.4.4 Extending Theorem 3.3 to arbitrary pairs of points on families of elliptic curves	84
3.4.5 Simple abelian surfaces and Pell's equations over function fields	85
3.4.6 Further extensions and analogues	87
3.4.7 Dynamical analogues	89
<i>Notes to Chapter 3</i>	92
Torsion values for a single point: other arguments	92
A variation on the Manin-Mumford conjecture	93
Comments on the Methods	94
4 About the André-Oort Conjecture	96
4.1 Generalities about the André-Oort Conjecture	96
4.2 Modular curves and complex multiplication	99
4.3 The theorem of André	105
4.3.1 An effective variation	111
4.4 Pila's proof of André's theorem	112
4.5 Shimura varieties	118
<i>Notes to Chapter 4</i>	123
Remarks on Edixhoven's approach to André's theorem	123
Some unlikely intersections beyond André-Oort	124
Definability and o -minimal structures	125
Appendix A Distribution of Rational Points on Subanalytic Surfaces	128
by Umberto Zannier	
Appendix B Uniformity in Unlikely Intersections: An Example for Lines in Three Dimensions	136
by David Masser	
Appendix C Silverman's Bounded Height Theorem for Elliptic Curves: A Direct Proof	138
by David Masser	
Appendix D Lower Bounds for Degrees of Torsion Points: The Transcendence Approach	140
by David Masser	
Appendix E A Transcendence Measure for a Quotient of Periods	143
by David Masser	

CONTENTS

vii

Appendix F Counting Rational Points on Analytic Curves:	
A Transcendence Approach	145
by David Masser	
Appendix G Mixed Problems: Another Approach	147
by David Masser	
Bibliography	149
Index	159