Foreword

In 1965, generalizing and clarifying a result of Weil, Narasimhan and Seshadri [**56**] established a bijective correspondence between the set of equivalence classes of unitary irreducible representations of the fundamental group of a compact Riemann surface X of genus ≥ 2 and the set of isomorphism classes of stable vector bundles of degree zero on X. The correspondence was then extended to all projective and smooth complex varieties by Donaldson [**21**]. The analogue for arbitrary linear representations is due to Simpson; to obtain a correspondence of the same type as Narasimhan and Seshadri, we need an additional structure on the vector bundle. This led to the notion of a Higgs bundle that was first introduced by Hitchin for algebraic curves. If X is a smooth scheme over a field K, a *Higgs bundle* on X is a pair (M, θ) consisting of a locally free $\mathscr{O}_X$-module of finite type M and an $\mathscr{O}_X$-linear morphism $\theta \colon M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/K}$ such that $\theta \wedge \theta = 0$. Simpson's main result [**67, 68, 69, 70**] establishes an equivalence between the category of (complex) finite-dimensional linear representations of the fundamental group of a smooth and projective complex variety and the category of semi-stable Higgs bundles with vanishing Chern classes (cf. [**54**]).

Simpson's results and the important developments they inspired have led, in recent years, to the search for a p-adic analogue. Looking back, the first examples of such a construction (which did not yet use the terminology of Higgs bundles) can be found in the work of Hyodo [**43**], who had treated the conceptually important case of p-adic variations of Hodge structures, called *Hodge–Tate local systems*. At present, the most advanced approach to such a correspondence is due to Faltings [**27, 28**]. It aims to describe all p-adic representations of the geometric fundamental group of a smooth algebraic variety over a p-adic field in terms of Higgs bundles. The constructions use several tools that he developed to establish the existence of Hodge–Tate decompositions [**24**], in particular his theory of almost étale extensions [**26**]. Once completed, this *p-adic Simpson correspondence* should thus naturally provide the best Hodge–Tate type statements in p-adic Hodge theory. But at present, Faltings' construction seems satisfying only for curves, and even in that case, many fundamental questions remain open.

In this volume, we undertake a systematic development of the *p-adic Simpson correspondence* started by Faltings following two new approaches, one by the first two authors (A.A. and M.G.), the other by the third author (T.T.). The need to resume and develop Faltings' construction was felt given the number of results sketched in a rather short and extremely dense article [**27**]. This correspondence applies to objects more general than p-adic representations of the geometric fundamental group, introduced by Faltings and called *generalized representations*. We focus mainly on those p-adically close to the trivial representation, qualified by him as *small*. Though some of our constructions extend beyond this setting, let us make clear right away that we do not, however, discuss the descent techniques that enabled Faltings, in the case of curves, to get rid of this smallness condition.

Independently of the work of Faltings, Deninger and Werner [**19, 20, 18**] have developed a partial analogue to the theory of Narasimhan and Seshadri for p-adic curves, which should correspond to Higgs bundles with vanishing Higgs field in the p-adic Simpson correspondence. On the other hand, also inspired by the complex case, Ogus and Vologodsky [**59**] have introduced a correspondence between modules with integrable connection and Higgs modules for varieties in characteristic p. That work, in turn, inspired the first approach developed in this volume for the p-adic Simpson correspondence. Let us, however, note that unlike the complex case where a link between the two variants of the Simpson correspondence is established (by definition) through the Riemann–Hilbert correspondence, the link between the p-adic and the modulo p Simpson correspondences is not, at present, known. Several works exploring this direction are in progress [**52, 61, 66**].

Let us now give some indications on the structure of this volume. The first approach is presented in Chapters I, II, and III. Chapter I provides an overview of this approach and can also serve as an introduction to the general theme of this volume. In Chapter II, we study the case of an affine scheme of a particular type, qualified also as *small* by Faltings. We introduce the notion of *Dolbeault generalized representation* and the companion notion of *solvable Higgs module*, and then construct a natural equivalence between these two categories. We prove that this approach generalizes simultaneously Faltings' construction for small generalized representations and Hyodo's theory of p-adic variations of Hodge–Tate structures. In Chapter III, we address the global aspects of the theory. We introduce the *Higgs–Tate algebra*, which is the main novelty of this approach compared to that of Faltings, the notion of *Dolbeault module* that globalizes that of Dolbeault generalized representation, and the companion notion of *solvable Higgs bundle*. The main result is the equivalence between the category of Dolbeault modules and that of solvable Higgs bundles. We also prove the compatibility of this equivalence with the natural cohomologies. The general construction is obtained from the affine case by a gluing technique relying on the *Faltings topos*, developed in a more general context in Chapter VI.

This first approach, like Faltings' original one, requires the datum of a deformation of the scheme over a p-adic infinitesimal thickening of order 1 introduced by Fontaine. The second approach, developed in Chapter IV of this volume, avoids this additional datum. For this purpose, we introduce a crystalline type topos, and replace the notion of Higgs bundles by that of *Higgs (iso)crystals*. The link between these two notions uses *Higgs envelopes* and calls to mind the link between classical crystals and modules with integrable connections. The main result is the construction of a fully faithful functor from the category of Higgs (iso)crystals satisfying an *overconvergence* condition to that of *small* generalized representations. We also prove the compatibility of this functor with the natural cohomologies. Finally, we compare the period rings used in the two approaches developed in this volume, showing the compatibility of the two constructions.

The last part of the volume, consisting of Chapters V and VI, contains results of wider interest in p-adic Hodge theory. Chapter V provides a concise introduction to Faltings' theory of *almost étale extensions*, a tool that has become essential in many questions in arithmetic geometry, even beyond p-adic Hodge theory. The point of view adopted here is closer to Faltings' original one than the more systematic development given later by Gabber and Ramero. Chapter VI is devoted to the *Faltings topos*. Though it is the general framework for Faltings' approach in p-adic Hodge theory, this topos remains relatively unexplored. We present a new approach to it based on a generalization of Deligne's *covanishing topos*. Along the way, we correct the original definition of Faltings.

The reader will find at the end of the volume the facsimile of Faltings' article, which is reprinted from *Advances in Mathematics* 198(2), Faltings, Gerd, "A p-adic Simpson Correspondence," pp. 847–862, Copyright 2005, with permission from Elsevier. We thank the author warmly for having allowed us to reproduce it.

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