

Introduction

A Natural System of Units, the Cube of Physics, Being Overweight, and Hawking Radiation

Planck gave us natural units

Max Planck* is properly revered for his profound contribution to quantum mechanics. But he is also much loved for his second greatest contribution to physics: in a far-reaching and insightful paper, he gave us a natural system of units.

Once upon a time, we used some English king's feet to measure lengths.[†] Einstein recognized that with the universal speed of light c , we no longer need separate units for length and time. Even the proverbial guy and gal in the street understand that henceforth, we could measure length in lightyears.

We and another civilization, be they in some other galaxy, would now be able to agree on a unit of distance, if we could only communicate to them what we mean by one year or one day. Therein lies the rub: our unit for measuring time derives from how fast our home planet spins and revolves around its star. Only homeboys would know. How could we possibly communicate to a distant civilization this period of rotation we call a day, which is merely an accident of how some interstellar debris came together to form the rock we call home?

* In his personal life, Planck suffered terribly. He lost his first wife, then his son in action in World War I, then both daughters in childbirth. In World War II, bombs totally demolished his house, while the Gestapo tortured his other son to death for trying to assassinate Hitler.

[†] Notions we take for granted today still had to be thought up by someone. Maxwell, in his magnum opus on electromagnetism, proposed that the meter be tied to the wavelength of light emitted by some particular substance, adding that such a standard "would be independent of any changes in the dimensions of the earth, and should be adopted by those who expect their writings to be more permanent than that body." The various eminences of our subject could be quite sarcastic.

Newton's discovery of the universal law of gravity brought another constant G into physics. Comparing the kinetic energy $\frac{1}{2}mv^2$ of a particle of mass m in a gravitational potential with its potential energy $-GMm/r$ and canceling off m , we see that the combination* GM/c^2 has dimensions of length. In other words, having two universal constants c and G at hand allows us to measure masses in terms of our unit for length (or equivalently time), or lengths in terms of our unit for mass.

Planck with his constant \hbar made a monumental contribution to physics by noting that the quantum world gives us for free a fundamental set of units that physicists call natural units.

Three big names, three basic principles, three natural units

To see how, note that Heisenberg's uncertainty principle tells us that \hbar divided by the momentum Mc is a length. Equating the two lengths GM/c^2 and \hbar/Mc , we see that the combination $\hbar c/G$ has dimensions of mass squared. In other words, the three fundamental constants G , c , and \hbar allow us to define a mass,¹ known rightfully as the Planck mass

$$M_{\text{P}} = \sqrt{\frac{\hbar c}{G}} \quad (1)$$

We can immediately define, with Heisenberg's help, a Planck length

$$l_{\text{P}} = \frac{\hbar}{M_{\text{P}}c} = \sqrt{\frac{\hbar G}{c^3}} \quad (2)$$

and, with Einstein's help, a Planck time

$$t_{\text{P}} = \frac{l_{\text{P}}}{c} = \sqrt{\frac{\hbar G}{c^5}} \quad (3)$$

Einstein, Newton, and Heisenberg—three big names, three basic principles, three natural units to measure space, time, and energy by. We have reduced the MLT system to nothing! We no longer have to invent or find some unit, such as the good king's foot, to measure the universe with. We measure mass in units of M_{P} , length in units of l_{P} , and time in units of t_{P} . Another way of saying this is that in these natural units, $c = 1$, $G = 1$, and $\hbar = 1$. The natural system of units is understood no matter where your travels might take you, within this galaxy or far beyond.

Newton small, so Planck huge, and the Mother of All Headaches

The Planck mass works out to be 10^{19} times the proton mass M_{p} . That humongous number 10^{19} , as we will see, is responsible for the Mother of All Headaches plaguing fundamental

* You will learn shortly what this combination means physically.

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physics today.² That M_P is so gigantic compared to the known particles can be traced back to the extreme feebleness of gravity: G is tiny, so M_P is enormous.

As the Planck mass is huge, the Planck length and time are teeny. If you insist on contaminating the purity of natural units by manmade ones, t_P comes out to be $\sim 5.4 \times 10^{-44}$ second, the Planck length $l_P \sim 1.6 \times 10^{-33}$ centimeter, and the Planck mass $M_P \sim 2.2 \times 10^{-5}$ gram!

It is important to realize how profound Planck's insight was. Nature herself, far transcending any silly English king or some self-important French revolutionary committee, gives us a set of units to measure her by. We have managed to get rid of all manmade units. We needed three fundamental constants, each associated with a fundamental principle, and we have precisely three!

This suggests that we have discovered all* the fundamental principles that there are. Had we not known about the quantum, then we would have to use one manmade unit to describe the universe, which would be weird. From that fact alone, we would have to go looking for quantum physics.

The cube of physics

Here is a nifty summary of all of physics as a cube (see figure 1). Physics started with Newtonian mechanics at one corner of the cube, and is now desperately trying to get to the opposite corner, where sits the alleged Holy Grail. The three fundamental constants, c^{-1} , \hbar , and G , characterizing Einstein, Planck or Heisenberg, and Newton, label the three axes. As we turned on one or the other of three constants (in other words, as each of these constants came into physics), we took off from the home base of Newtonian mechanics.[†] Much of 20th century physics consisted of getting from one corner of the cube to another. Consider the bottom face³ of the cube. When we turned on c^{-1} we went from Newtonian mechanics to special relativity. When we turned on \hbar , we went from Newtonian mechanics to quantum mechanics. When we turned on both c^{-1} and \hbar , we arrived at quantum field theory, in my opinion the greatest monument of 20th century physics.

Newton himself had already moved up the vertical axis from Newtonian mechanics to Newtonian gravity by turning on G . Turning on c^{-1} , Einstein took us from that corner to Einstein gravity, the main subject of this book.[‡] All the *Stürm und Drang* of the past few decades is the attempt to cross from that corner to the Holy Grail of quantum gravity, when (glory glory hallelujah!) all three fundamental constants are turned on.[§]

* These days, fundamental principles are posted on the physics archive with abandon. There might be hundreds by now.

[†] By this I mean the three laws, $F = ma$ and so on, not including the law of universal gravitation.

[‡] The corner with $c^{-1} = 0$ but $\hbar \neq 0$ and $G \neq 0$ is relatively unpublicized and generally neglected. It covers phenomena described adequately by nonrelativistic quantum mechanics in the presence of a gravitational field. Two fascinating experiments in this area are: (1) dribbling neutrons like basketballs, and (2) interfering a neutron beam with itself in a gravitational field.⁴

[§] This statement carries a slight caveat, which we will come to in chapter VII.3.

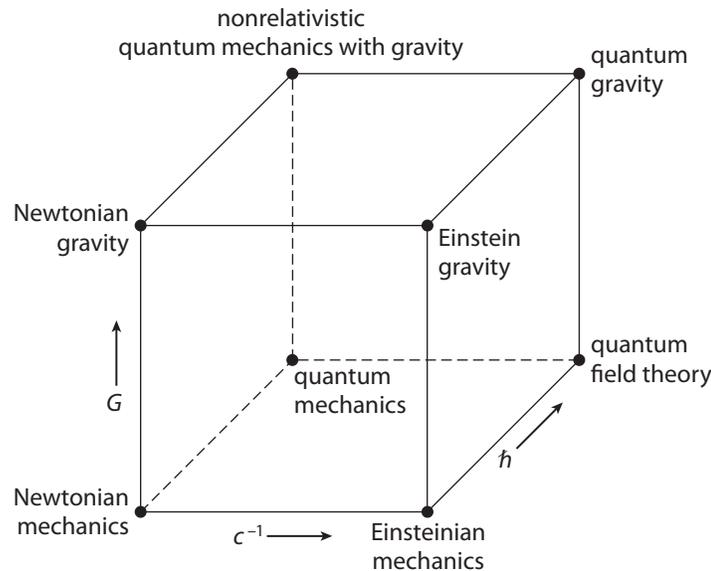


Figure 1 The cube of physics.

In our everyday existence, we are aware of only two corners of this cube, because these three fundamental constants are either absurdly small or absurdly large compared to what humans experience.

The universe's obesity index

As the obesity epidemic sweeps over the developed countries, one government after another has issued some kind of obesity index, basically dividing body weight by size. As we have seen, for an object of mass M , the combination GM/c^2 is a length that can be compared to the characteristic size of the object. So, Nature has her own obesity index for any object, from electron to galaxy. Indeed, as is well known, John Michell in 1783 and the Marquis Pierre-Simon Laplace in 1796 pointed out that even light cannot escape from an object excessively massive for its size.

More precisely, consider an object of mass M and radius R . A particle of mass m at the surface of this object has a gravitational potential energy $-GMm/R$ and kinetic energy $\frac{1}{2}mv^2$. Equating these two energies gives the escape velocity $v_{\text{escape}} = \sqrt{2GM/R}$. Setting v_{escape} to c tells us that if $2GM > Rc^2$, not even light can escape, and the object is a black hole.⁵ Remarkably, even though the physics behind the argument* is not correct in detail (as we now know, we should not treat light as a Newtonian corpuscle with a tiny mass), this

* This often cited Newtonian argument actually does not establish the existence of black hole defined as an object from which nothing could escape. The escape velocity refers to the initial speed with which we attempt to fling something into outer space. In the Newtonian world, we could certainly escape from any massive planet in a rocket with a powerful enough engine.

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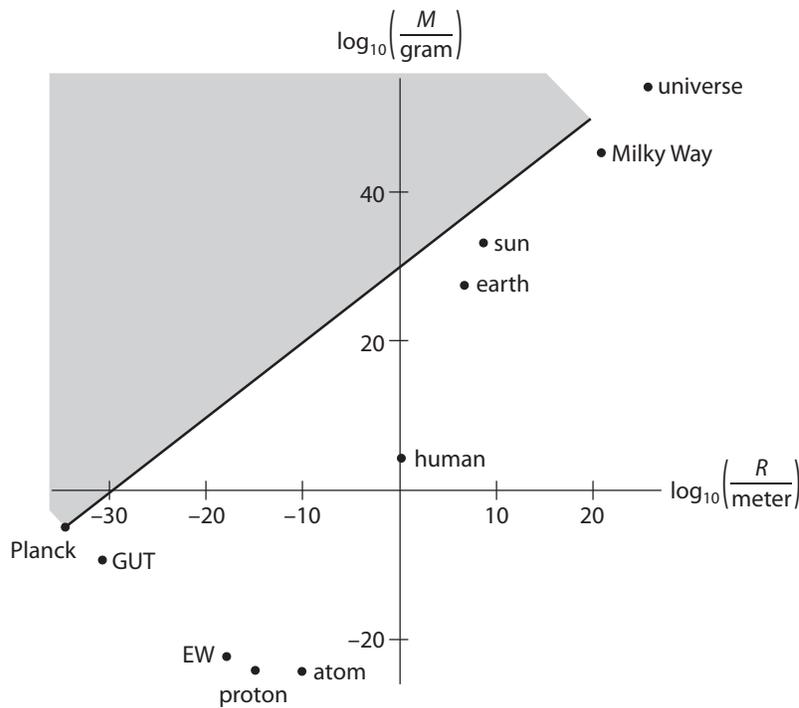


Figure 2 A plot of M versus R for various objects in the universe. EW stands for electroweak and GUT for grand unified theory. The shaded area represents the “black hole” regime with $2GM > R$.

criterion, including the factor of 2, turns out to hold in Einstein’s theory. Figure 2 shows a plot of M versus R for various objects in the universe.

Hawking radiation

Unless you have been hiding out in the jungles of New Guinea, you would have heard that in an extremely influential paper, Stephen Hawking, building on the earlier work of Jacob Bekenstein and others, and working in collaboration with Gary Gibbons, pointed out this purely classical argument needs to be amended when quantum effects are included: black holes evaporate and radiate particles.

In fact, the temperature of the radiation, known as the Hawking temperature T_H of the black hole, can be estimated by using dimensional analysis. You may be puzzled,* since there are two masses in the problem, the mass M of the black hole and the Planck mass M_P . With two masses, any function of M/M_P is admissible, and so dimensional analysis appears to be inapplicable. Indeed, we need one more piece of information. The key is that

* I was talking to a distinguished condensed matter physicist just the other day, and he was puzzled about precisely this point. So your unspoken question may be widespread.

Newton's constant G is a multiplicative measure of the strength of gravity. In Einstein's theory as well as in Newton's, the gravitational field around an object of mass M can only depend on the combination of GM . Let us now set c and \hbar (but not G) to 1. The combination GM is a length and hence an inverse mass. On the other hand, Boltzmann and the founding fathers of statistical mechanics had long ago revealed to us that temperature, a highly mysterious concept at one time, is merely the average energy⁶ of the microscopic constituents of macroscopic matter. Hence temperature has the dimensions of energy, that is, of a mass in units with $c = 1$.

It follows immediately that $T_H \sim \frac{1}{GM}$. This "sophisticated" dimensional analysis captures an essential piece of physics: the radiation is explosive! As the black hole radiates energy, M goes down and T_H goes up, and thus the black hole radiates faster. The radiative mass loss accelerates. Certainly not something you want to see in the kitchen: an object that gets hotter as it loses energy.

In chapter VII.3, we will see that the overall numerical constant can be determined in a couple of lines of algebra, so that

$$T_H = \frac{\hbar c^3}{8\pi GM} \tag{4}$$

We have restored c and \hbar by high school dimensional analysis using everyday unnatural units. It is gratifying to see that indeed, with $\hbar = 0$ and quantum effects turned off, $T_H = 0$, and the black hole does not radiate.

Thermodynamics states that entropy S is given by $dE = TdS$. Here E is just the mass of the black hole. Integrating $\frac{dS}{dM} = \frac{1}{T_H} \sim GM$, we obtain

$$S \sim GM^2 \sim \left(\frac{M}{M_p}\right)^2 \tag{5}$$

Note that, as expected, S is dimensionless.

Using the fact that the black hole has radius $R \sim GM$ and hence surface area $A \sim R^2$, we conclude that

$$S \sim \frac{R^2}{G} \sim \frac{A}{l_p^2} \tag{6}$$

You should be shocked, shocked, shocked. Most theoretical physicists were, and are.

Not shocked?

Normally, the entropy of a system is extensive, that is, proportional to its volume. Somehow, a black hole has an entropy proportional to its surface area rather than to its volume. This fact has led to the so-called holographic principle. Many fundamental physicists believe that this mysterious property of black holes holds the key to quantum gravity.

All of this merely from dimensional analysis!

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Notes

1. Some readers might wonder why we do not use the mass of the electron m_e . In modern particle physics, the electron may not always have had the mass it has now, and in fact it might have been massless in the early universe. The masses of elementary particles depend on quantum field theoretic notions known as spontaneous symmetry breaking and the Higgs mechanism. We should express m_e in terms of M_p , not M_p in terms of m_e . In different areas of physics, different units are used: for example, the size of the hydrogen atom might be used as a length unit.
2. I return to this problem in due course, in chapter X.8, for example.
3. This face, regarded as a square, was discussed in the very first section of the first chapter in *QFT Nut*.
4. See appendix 5 to chapter X.8; for more details, see J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, pp. 110 and 133.
5. Named by John Wheeler almost 200 years later.
6. The Boltzmann constant k , which is merely a conversion factor between energy units and the markings on some tubes containing mercury known as degrees, has been set to 1.