Introduction

The main subject of this book is the construction of good compactifications of PEL-type Shimura varieties in mixed characteristics. In this introduction, we will summarize the history of such a construction, clarify its aims, and describe what has been achieved (by others and by us). After this, we will also give an outline of the topics in each chapter of the book.

To avoid introducing a heavy load of notation and definitions (which is perhaps typical for this subject), we shall not attempt to give any precise mathematical statement here. (We shall try not to introduce general non-PEL-type Shimura varieties either.) Nevertheless, we hope that the information we provide here will help the reader locate more precise details in the main body of the text.

COMPLEX ANALYTIC THEORY

It is classically known, especially since the work of Shimura, that complex abelian varieties with so-called PEL structures (polarizations, endomorphisms, and level structures) can be parameterized by unions of arithmetic quotients of Hermitian symmetric spaces. Simple examples include modular curves as quotients of the Poincaré upper-half plane, Hilbert modular spaces as quotients of products of the Poincaré upper-half plane, and Siegel moduli spaces as quotients of Siegel upper-half spaces. These analytically defined spaces carry objects such as modular forms and their generalizations that possess mysterious arithmetic properties suggesting strong links between analysis, geometry, and number theory.

Thanks to Baily and Borel [18], each such arithmetic quotient can be given an algebro-geometric structure because it can be embedded as a Zariski open subvariety of a canonically associated complex normal projective variety called the *Satake–Baily–Borel* or *minimal compactification*. Thus the above-mentioned parameter spaces can be viewed as unions of complex quasi-projective varieties. These parameter spaces are called *PEL-type Shimura varieties*. They admit canonical models over number fields, as investigated by Shimura and many others (see, in particular, [36] and [37]). (For simplicity, in this introduction, we shall henceforth allow varieties to have more than one irreducible component.)

Although the minimal compactifications mentioned above are normal and canonical, Igusa [72] and others have discovered that, in general, minimal compactifications are highly singular. In [17], Mumford and his coworkers constructed a large class of (noncanonical) compactifications in the category of complex algebraic spaces, called *toroidal compactifications*, labeled by certain combinatorial data called cone decompositions. Within this class, there are plenty of nonsingular compactifications, many among them are projective, hence providing a theory of smooth compactifications (in the category of schemes) for

PEL-type Shimura varieties over the complex numbers. Based on the work of many people since Shimura, it is known that both minimal compactifications and toroidal compactifications admit canonical models over the same number fields (the so-called *reflex fields*) over which the Shimura varieties are defined (see Pink's thesis [110], and also [66]).

GOOD COMPACTIFICATIONS

To better clarify the aim of the theory, let us collect (without being precise, yet already a very long list) certain important properties of these compactifications over $\mathbb{C}$ (or over any number field where they are defined):

1. PEL-type Shimura varieties (or their canonical models), being quasi-projective varieties, can be realized as coarse moduli spaces of certain smooth algebraic stacks (defined by moduli problems of abelian schemes with PEL structures of prescribed types). By abuse of notation, let us denote Shimura varieties by Sh_H, where H is the so-called *level*, suppressing other data involved in the definition.

2. Each Sh_H has a canonical minimal compactification $\mathsf{Sh}_H^{\min}$, which is a normal projective scheme containing Sh_H as an open dense subvariety. Moreover,

    a) The minimal compactification $\mathsf{Sh}_H^{\min}$ has a stratification by locally closed subvarieties, including Sh_H as the dense open stratum, consisting otherwise of "Galois orbits" (in a suitable sense due to Pink [110]) of smaller PEL-type Shimura varieties. These strata appearing in the boundary of $\mathsf{Sh}_H^{\min}$ will be called *cusps*. (In general, there can be cusps of different *ranks*, and the closures of cusps in $\mathsf{Sh}_H^{\min}$ can overlap with each other.)

    b) Hecke operators can be defined geometrically on Sh_H and $\mathsf{Sh}_H^{\min}$ by algebraic correspondences given by finite morphisms (from some $\mathsf{Sh}_{H'}$ at a higher level H').

    c) A warning: As long as Sh_H is not compact, there always exist zero-dimensional cusps of $\mathsf{Sh}_H^{\min}$. Therefore, although [37] excludes zero-dimensional Shimura varieties for technical reasons, they have to be included in every theory of compactifications. This is an issue already addressed by Pink [110].

    d) The construction of $\mathsf{Sh}_H^{\min}$ in [18] comprises two steps: Firstly, one constructs a compact topological space. Secondly, one shows that the graded algebra of sections of powers of an extension of the canonical line bundle of Sh_H is finitely generated, which allows one to define an embedding of Sh_H into a projective space. Both steps involve methods which are transcendental in nature.

3. Each Sh_H has a collection $\{\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}\}_\Sigma$ of toroidal compactifications, which are algebraic spaces labeled by the combinatorial data Σ called *cone decompositions*. Moreover,

   a) Each toroidal compactification $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$ has a stratification by locally closed subvarieties, depending on the choice of Σ, including Sh_H as the dense open stratum, together with a canonical morphism $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma} \to \mathsf{Sh}^{\mathrm{min}}_H$ mapping each stratum on the source to a stratum on the target.

   b) The local charts of each $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$ have a nice description given by finite quotients of neighborhoods of torus bundles over abelian schemes over the cusps of $\mathsf{Sh}^{\mathrm{min}}_H$. (Thus, the toroidal compactifications can be viewed as explicit desingularizations of the minimal compactifications along their cusps.)

   c) The directed partial order on the collection of Σ's given by *refinements* induces a directed partial order on the collection $\{\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}\}_\Sigma$ of toroidal compactifications given by *proper morphisms*, and locally these morphisms are given explicitly by equivariant morphisms between toroidal embeddings.

   d) Assume that H is neat for simplicity. There exist combinatorial conditions on Σ ensuring that the corresponding toroidal compactifications $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$ are smooth or projective, or both, which can always be achieved by replacing Σ with a refinement.

      In the smooth case, one can arrange that the boundary divisor $\mathsf{D}_{\infty,H} := (\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma} - \mathsf{Sh}_H)_{\mathrm{red}}$ has *normal crossings*, in which case each connected component of a (locally closed) boundary stratum of $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$ is smooth and open dense in an intersection of the irreducible components of $\mathsf{D}_{\infty,H}$. By refining the cone decompositions, one can arrange that the normal crossings of $\mathsf{D}_{\infty,H}$ are all *simple*.

      In the projective case, $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$ is a *projective variety* because it is the normalization of the blowup of $\mathsf{Sh}^{\mathrm{min}}_H$ along a coherent sheaf of ideals $\mathcal{J}$ supported on the boundary of the projective variety $\mathsf{Sh}^{\mathrm{min}}_H$. (Nevertheless, one cannot construct $\mathcal{J}$ without first constructing the algebraic space $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$.)

   e) There is a long list of nice properties enjoyed by the (collection of) toroidal compactifications, making possible a comprehensive theory of the (coherent sheaf) cohomology of automorphic bundles (see [66], [67], and [101]).

   f) The finite morphisms $\mathsf{Sh}_{H'} \to \mathsf{Sh}_H$ defining Hecke correspondences extend to proper morphisms of the form $\mathsf{Sh}^{\mathrm{tor}}_{H',\Sigma'} \to \mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$, which are not finite in general, but nevertheless induce the correct actions on the cohomology of automorphic bundles mentioned above.

g) Given any smooth toroidal compactification $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$, the analytic construction of $\mathsf{Sh}^{\mathrm{min}}_{H}$ can be replaced with a simpler algebraic construction: firstly, by properness of $\mathsf{Sh}^{\mathrm{tor}}_{H,\Sigma}$, the algebra of sections of powers of a canonical extension of the canonical line bundle of Sh_H is finitely generated; secondly, one takes the minimal compactification to be the projective spectrum of this graded algebra. (Here the canonical line bundle of Sh_H can be replaced with the so-called Hodge line bundle.)

Certainly, in characteristic zero, the existence of (collections of) good compactifications can be handled by the (embedded) resolution of singularities due to Hironaka [69, 70]. However, the theory of toroidal compactifications is appealing not only because it is more explicit, but also because—as we shall see below—it can be generalized in *mixed characteristics*, where resolution of singularities is not known in general.

ALGEBRAIC THEORY IN MIXED CHARACTERISTICS

Since abelian varieties (with PEL structures) make sense over rings of algebraic integers localized at some precise sets of *good primes*, we obtain *integral models* of PEL-type Shimura varieties by defining suitable moduli problems of such abelian varieties. Moreover, the precise sets of good primes can be determined such that the moduli problems are *smooth* algebraic stacks over base schemes with good residue characteristics. (See, for example, [89], [124], and [83]. Up to some well-known defect called the *failure of Hasse's principle*, PEL-type Shimura varieties over $\mathbb{C}$ are the coarse moduli spaces of the complex fibers of such smooth algebraic stacks.)

One naturally (or naively) asks whether we have analogues in *good* mixed characteristics of the above-mentioned good compactifications over $\mathbb{C}$, enjoying an equally long (or even longer) list of nice properties. For example, we would like these compactifications to have stratifications by locally closed *smooth* subschemes. Moreover, we would like to describe their local structures by certain universal objects they carry. (For example, we would like to know whether universal abelian schemes over the moduli problems extend to any reasonable objects over the compactifications, and if such extensions enjoy any kind of universal properties. Moreover, if this is the case, we would like to study whether and how the Kodaira–Spencer morphisms of abelian schemes extend.)

It is not surprising that the existence of such compactifications in mixed characteristics is desirable for arithmetic applications of Shimura varieties. Certainly, we hope the answers to all of these questions are in the affirmative.

In this book, we will show that, under reasonable assumptions excluding no single PEL-type Shimura varieties, this is indeed the case. We will call the good (toroidal or minimal) compactifications in mixed characteristics *arithmetic* (*toroidal or minimal*) *compactifications*.

WHAT IS KNOWN?

Certainly there are the modular and Hilbert modular cases, where the cusps (in the minimal compactifications) are zero-dimensional and never overlap with each other. Apart from these special cases, the first construction of arithmetic compactifications involving cusps of arbitrarily large ranks (and with nontrivially overlapping closures) is given by Faltings and Chai.

Based on the work of Mumford [105], Faltings and Chai developed a theory of degeneration for polarized abelian varieties over complete adic rings satisfying certain reasonable normality conditions, and (based on this theory of degeneration) constructed smooth toroidal compactifications of the integral models of Siegel modular varieties (parameterizing principally polarized abelian schemes with principal level structures, over base schemes over which the primes dividing the levels are invertible). (See [45] , [26], and especially [46].)

The key point in their construction is the *gluing process* in the étale topology. Such a process is feasible because there exist local charts over which the sheaves of relative log differentials can be explicitly calculated and compared. The above-mentioned theory of degeneration and the theory of toroidal embeddings over arbitrary bases play a major role in the construction of these local charts.

As a by-product, as mentioned above, they obtained the minimal compactifications of the integral models of Siegel modular varieties by taking the projective spectra of certain graded algebras. (Unlike in the theory over $\mathbb{C}$, we do not know of any direct construction of minimal compactifications.)

In Larsen's thesis [90] (see also [91]), he applied the techniques of Faltings and Chai and constructed arithmetic compactifications of integral models of Picard modular varieties, namely, Shimura varieties associated with unitary groups defined by Hermitian pairings of real signature $(2, 1)$ over imaginary quadratic fields. (This is an example where the cusps are zero-dimensional and nonoverlapping, but they nevertheless parameterize more data than in the modular and Hilbert modular cases.) Larsen's compactification theory played a foundational role in the Montréal volume [88].

Before moving on, let us mention that there is also the unpublished revision of Fujiwara's master's thesis [50] on the arithmetic compactifications of PEL-type Shimura varieties involving simple components of only types A and C. The main difference between his work and Faltings–Chai's is his rigid-analytic methods in the gluing process. However, as far as we can understand, his boundary construction (*before* the gluing step) is incomplete. Moreover, he did not verify (or even describe in sufficient detail) most of the expected good properties. (His project is in some sense unfinished, as suggested by the title of [50].)

WHAT IS NEW?

As mentioned above, in this book our goal is to construct (good) arithmetic compactifications of (smooth integral models of) PEL-type Shimura varieties (as defined in Kottwitz's paper [83]).

Our construction is based on a generalization of Faltings and Chai's in [46]. It is a very close imitation from the perspective of algebraic geometry. Thus our work can be viewed as a long student exercise justifying the claims in [46, pp. 95–96 and 137] that their method works for general PEL-type Shimura varieties.

However, there do exist some difficulties if one tries to imitate Faltings and Chai's theory naively.

The first main issue, perhaps surprisingly to some readers, concerns the definition (or rather our understanding) of the moduli problems. In the Siegel case, it is enough to consider symplectic isomorphisms between finite étale group schemes; but in general, this is not enough. With the presence of endomorphisms by nontrivial (and possibly nonmaximal) orders, and with the presence of nonperfect pairings, the correct definition of level structures involves not only a symplectic condition, but also a liftability condition. For example, a moduli problem defined with *no* level structure is in general not the same as a moduli problem defined with *level one* structures. (A moduli problem with no level structure can be nondisjoint unions of several moduli problems with level one structures.)

Thus, our first main objective is to formulate certain liftability and pairing conditions on the degeneration data, so that the combination of these two conditions can predict the existence of level structures of a prescribed type on the generic fiber of the corresponding degenerating families. This involves a Weil-pairing calculation that we believe is *new*. (Perhaps also surprisingly to some readers, the assumption that we are working in the good reduction case does not mean we only need smooth group schemes. The necessity of considering the liftability condition is exactly because we have to deal with certain nonsmooth fibers of the group scheme defined by the endomorphism and the pairing. The Siegel case, somehow, is the only case where this group scheme is smooth everywhere!)

WHAT IS NOT COVERED?

As the title of the book suggests, we have only considered integral models of PEL-type Shimura varieties. It is possible to consider smooth integral models of more general Shimura varieties, such as Hodge-type or abelian-type Shimura varieties, as in the works of Vasiu, Kisin, and others, which require fundamentally different considerations very much beyond the scope of this book. (Nevertheless, all known constructions for Hodge-type or abelian-type Shimura varieties depend logically on the theory for Siegel moduli in Faltings–Chai, and the construction of Hecke actions away from the residue characteristics—which are desirable for obvious reasons—requires some minor generalizations in this book.)

Within the context of PEL-type Shimura varieties, there have always been people working on compactifications of *nonsmooth* integral models. Let us explain why we do not consider this further generality in our work. The main reason is not to do with the theory of degeneration data or the techniques of constructing local boundary charts. It is rather to do with the definition of Shimura varieties themselves, and the expectations of the results to be arrived at. In some cases, there seems to be more than one reasonable way of stating them. This certainly does not mean that it is impossible to compactify a particular nonsmooth integral model. However, it would probably be more sensible if we know for what purpose we compactify it, and if we know for what reason it should be compactified in a particular way. This is beyond this (already lengthy) work.

Also, we are not working along the lines of the *canonical compactifications* constructed by Alexeev and Nakamura [2, 1], or by Olsson [108], because it is much less clear how one should define Hecke actions on their canonical compactifications. (The main component in their compactifications can be related to the toroidal compactifications constructed by some particular choices of cone decompositions. However, such cone decompositions are not preserved by most Hecke actions.) It is possible that their construction could be useful for the construction of minimal compactifications (with Hecke actions), but we believe that the argument will be somewhat indirect. Nevertheless, we would like to emphasize that their compactifications are described by moduli problems allowing deformation-theoretic considerations along the boundary. Hence their compactifications might be more useful, at least, for applications to algebraic geometry.

We warn the reader that we have not explained here why the algebro-geometric construction in this book (and in [46]) and the analytic construction in [17] are consistent with each other. (It seems a folklore belief that it suffices to compare the descriptions of these toroidal compactifications, as commented after the statement of [46, Ch. IV, Prop. 5.15] and in some other works. But we do not know a logical justification for such a belief.) This consistency is certainly desirable for practical reasons, but we shall defer its proof to the article [87] (which is written after, and depends on a revision of, [86], or rather this book).

STRUCTURE OF THE EXPOSITION

In Chapter 1, we lay down the foundations and give the definition of the moduli problems we consider. For the purposes of proving representability and constructing compactifications, we shall use the definition by *isomorphism classes* of abelian schemes with additional structures. On the other hand, there is the definition by *isogeny classes* of abelian schemes with addition structures, which agrees with the definition in [83] when specialized to the same bases. We shall explain that there is a canonical isomorphism from each of the moduli problems defined by *isomorphism classes* to a canonically associated moduli problem defined by *isogeny classes*. Consequently, as explained in [83], the complex fibers of these moduli problems contain (complex) Shimura varieties associated with the reductive groups mentioned above as open and closed subalgebraic stacks.

In Chapter 2, we elaborate on the representability of the moduli problems defined in Chapter 1. Our treatment is biased towards the prorepresentability of local moduli and Artin's criterion of algebraic stacks. We do not need geometric invariant theory or the theory of Barsotti–Tate groups. The argument is very elementary and might be considered outdated by the experts in this area. (Indeed, it may not be enough for the study of bad reductions.) Although readers might want to skip this chapter as they might be willing to believe the representability of moduli problems, there are still some reasons to include this chapter. For example, the Kodaira–Spencer morphisms of abelian schemes with PEL structures are of fundamental importance in our argument for the gluing of boundary charts (in Chapter 6), and they are best understood via the study of deformation theory. Furthermore, the proof of the formal smoothness of local moduli functors illustrates how the linear algebraic assumptions are used. Some of the linear algebraic facts are used again in the construction of boundary components, and it is an interesting question whether one can propose a satisfactory intuitive explanation of this coincidence.

In Chapter 3, we explain well-known notions important for the study of semi-abelian schemes, such as groups of multiplicative type and torsors under them, biextensions, cubical structures, semi-abelian schemes, Raynaud extensions, and certain *dual objects* for the last two notions extending the notion of dual abelian varieties. Our main references for these are [43], [61], and in particular [103].

In Chapter 4, we reproduce the theory of degeneration data for polarized abelian varieties, as elaborated on in the first three chapters of [46]. In the main theorems (of Faltings and Chai) that we present, we have made some modifications to the statements according to our own understanding of the proofs. Examples of this sort include, in particular, Definitions 4.2.1.1 and 4.5.1.2, Theorem 4.2.1.14, and Remarks 4.2.1.2, 4.2.1.3, and 4.5.1.4.

In Chapter 5, we supply a theory of degeneration data for endomorphism structures, Lie algebra conditions, and level structures, based on the theory of degeneration in Chapter 4. People often claim that the degeneration theory for general PEL-type structures is just a straightforward consequence of the functoriality of the merely polarized case. However, the Weil-pairing calculation carried out in this chapter may suggest that this is not true. As far as we can see, functoriality does not seem to imply properties about pairings in an explicit way. There are conceptual details to be understood beyond simple implications of functoriality. However, we are able to present in this chapter a theory of degeneration data for abelian varieties with PEL structures, together with the notion of cusp labels.

In Chapter 6, we explain the algebraic construction of toroidal compactifications. For this purpose we need one more basic tool, namely, the theory of toroidal embeddings for torsors under groups of multiplicative type. Based on this theory, we begin the general construction of local charts on which degeneration data for PEL structures are tautologically associated. The key ingredient in these constructions is the construction of the tautological PEL structures, including particularly the level structures. The construction depends heavily on the way we classify the degeneration data and cusp labels developed in Chapter 5. As explained above, there are complications that are not seen in special cases such as Faltings and Chai's work. The next important step is the description of good formal models, and good algebraic models approximating them. The correct formulation of necessary properties and the actual construction of these good algebraic models are the key to the gluing process in the étale topology. In particular, this includes the comparison of local structures using the Kodaira–Spencer morphisms mentioned above. As a result of gluing, we obtain the arithmetic toroidal compactifications in the category of algebraic stacks. The chapter is concluded by a study of Hecke actions on towers of arithmetic toroidal compactifications.

In Chapter 7, we first study the automorphic forms that are defined as global sections of certain invertible sheaves on the toroidal compactifications. The local structures of toroidal compactifications lead naturally to the theory of Fourier–Jacobi expansions and the Fourier–Jacobi expansion principle. As in the case of Siegel modular schemes, we also obtain the algebraic construction of arithmetic minimal compactifications (of the coarse moduli associated with moduli problems), which are projective normal schemes defined over the same integral bases as the moduli problems are. As a by-product of codimension counting, we obtain Koecher's principle for arithmetic automorphic forms (of naive parallel weights). Furthermore, following the generalization in [26, Ch. IV] and [46, Ch. V, §5] of Tai's result in [17, Ch. IV, §2] to Siegel moduli schemes in mixed characteristics, we can show the projectivity of a large class of arithmetic toroidal compactifications by realizing them as normalizations of blowups of the corresponding minimal compactifications.

We have included two appendices containing basic information about algebraic stacks and Artin's criterion for them. There is also an index of notation and terminology at the end of the book.

Our overall treatment might seem unreasonably lengthy, and some of the details might have made the arguments more clumsy than they should be. Even so, we have tried to provide sufficient information, so that readers should have no trouble correcting our foolish mistakes, or improving our unnecessarily inefficient arguments. It is our belief that it is the right of the reader, but not the author, to skip details. At least, we hope that readers will not have to repeat some of the elementary but tedious tasks we have gone through.

NOTATION AND CONVENTIONS

All rings, commutative or not, will have an identity element. All left or right module structures, or algebra structures, will preserve the identity elements. Unless the violation is clear from the context, or unless otherwise specified, every ring homomorphism will send the identity to the identity element. Unless otherwise specified, all modules will be assumed to be left modules by default. An exception is ideals in noncommutative rings, in which case we shall always describe precisely whether it is a left ideal, a right ideal, or a two-sided ideal. All involutions in this work are anti-automorphisms of order two. The dual of a left module is naturally equipped with a left module structure over the opposite ring, and hence over the same ring if the ring admits an involution (which is an anti-isomorphism from the ring to its opposite ring).

We shall use the notation $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{A}$, $\mathbb{A}^\infty$, and $\hat{\mathbb{Z}}$ to denote respectively the ring of rational integers, rational numbers, real numbers, complex numbers, adeles, finite adeles, and integral adeles, without any further explanation.

More generally, for each set $\Box$ of rational primes, which can be either finite or infinite in cardinality, or even empty, we denote by $\mathbb{Z}_{(\Box)}$ the unique localization of $\mathbb{Z}$ (at the multiplicative subset of $\mathbb{Z}$ generated by nonzero integers prime-to-$\Box$) having $\Box$ as its set of nonzero height-one primes, and denote by $\hat{\mathbb{Z}}^\Box$ (resp. $\mathbb{A}^{\infty,\Box}$, resp. $\mathbb{A}^\Box$) the integral adeles (resp. finite adeles, resp. adeles) away from $\Box$. (When $\Box$ is empty, we have $\mathbb{Z}_{(\Box)} = \mathbb{Q}$, $\mathbb{A}^{\infty,\Box} = \mathbb{A}^\infty$, and $\mathbb{A}^\Box = \mathbb{A}$.)

We say that an integer m is prime-to-$\Box$ if m is not divisible by any prime number in $\Box$. In this case, we write $\Box \nmid m$.

These conventions and notation are designed so that results would be compatible if $\Box$ were literally just a prime number.

The notation $[A : B]$ will mean either the index of B in A as a subgroup when we work in the category of lattices, or the degree of A over B when we work in the category of finite-dimensional algebras over a field. We allow this ambiguity because there is no interesting overlap of these two usages.

The notation δ_{ij}, when i and j are indices, means the Kronecker delta, which is 1 when $i = j$ and 0 when $i \neq j$, as usual.

In our exposition, *schemes* will almost always mean *quasi-separated preschemes*, unless otherwise specified (see Remark A.2.5 and Lemma A.2.6). All *algebraic stacks* that we will encounter are *Deligne–Mumford stacks* (cf. [39], [46, Ch. I, §4], [92], and Appendix A).

The notion of *relative schemes* over a ringed topos can be found in [65], which, in particular, is necessary when we talk about relative schemes over formal schemes. We shall generalize this notion tacitly to relative schemes over formal algebraic stacks.

By a *normal* scheme we mean a scheme whose local rings are all integral and integrally closed in its fraction field. A ring R is normal if $\mathrm{Spec}(R)$ is normal. We do not need R to be integral and/or noetherian in such a statement.

We shall almost always interpret *points* as *functorial points*, and hence fibers as fibers over functorial points. By a *geometric point* of a point we mean a morphism from an algebraically closed field to the scheme we consider. We will often use the relative notion of various scheme-theoretic concepts without explicitly stating the convention.

We will use the notation $\mathbf{G}_{\mathrm{m}}$, $\mathbf{G}_{\mathrm{a}}$, and $\boldsymbol{\mu}_n$ to denote, respectively, the *multiplicative group*, the *additive group*, and the group scheme kernel of $[n] : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ over $\mathrm{Spec}(\mathbb{Z})$. Their base change to other base schemes S will often be denoted by $\mathbf{G}_{\mathrm{m},S}$, $\mathbf{G}_{\mathrm{a},S}$, and $\boldsymbol{\mu}_{n,S}$, respectively.

For each scheme S and each set X, we denote by X_S the sheaf of locally constant functions over S valued in X. When X carries additional structures such as being an algebra or a module over some ring, then X_S is a sheaf also carrying such additional structures. (We can also interpret X_S as a scheme over S defined by the disjoint union of copies of S indexed by elements in X.)

For each scheme S and each $\mathscr{O}_S$-algebra (resp. graded $\mathscr{O}_S$-algebra) $\mathcal{A}$, we denote by $\underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathcal{A})$ (resp. $\underline{\mathrm{Proj}}_{\mathscr{O}_S}(\mathcal{A})$) the spectrum (resp. homogeneous spectrum) of $\mathcal{A}$ over S. This is often denoted by $\mathbf{Spec}(\mathcal{A})$ (resp. $\mathbf{Proj}(\mathcal{A})$), or by $\mathrm{Spec}(\mathcal{A})$ (resp. $\mathrm{Proj}(\mathcal{A})$) as in [63, II, 1.3.1, resp. 3.1.3]. Our underlined notation is compatible with our other notation $\underline{\mathrm{Hom}}$, $\underline{\mathrm{Isom}}$, $\underline{\mathrm{Pic}}$, etc. for sheafified objects.

Throughout the text, there will be relative objects such as sheaves, group schemes, torsors, extensions, biextensions, cubical structures, etc. These objects are equipped with their base schemes or algebraic stacks (or their formal analogues) by definition. Unless otherwise specified, morphisms between objects with these structures will be given by morphisms respecting the bases, unless otherwise specified. (We will make this clear when there is room for ambiguity.) We will be more explicit when the structures are defined or studied, but will tacitly maintain this convention afterwards.

The typesetting of this work will be sensitive to small differences in notation. Although no difficult simultaneous comparison between similar symbols will be required, the differences should not be overlooked when looking for references. More concretely, we have used all the following fonts: A (normal), A (Roman), $\mathbf{A}$ (boldface), $\mathbb{A}$ (blackboard boldface), A (sans serif), $\mathtt{A}$ (typewriter), $\mathcal{A}$ (calligraphic), $\mathfrak{A}$ (Fraktur), and $\mathscr{A}$ (Ralph Smith's formal script). The tiny difference between A (normal) and *A* (italic) in width, which does exist, seems to be extremely difficult to see. So we shall never use both of them. We distinguish between A and $\underline{A}$, where the latter almost always means the relative version of A (as a sheaf or functor, etc.). We distinguish between Greek letters in each of the pairs ϵ and ε, ρ and ϱ, σ and ς, ϕ and φ, and π and ϖ. The musical symbols $^\flat$ (flat), $^\natural$ (natural), and $^\sharp$ (sharp) will be used following Grothendieck (cf., for example, [61, IX]) and some other authors. The difference in each of the pairs $^\flat$ and ^{b}, and $^\sharp$ and $^\#$, should not lead to any confusion. The notation $^\heartsuit$ and $^\diamondsuit$ are used, respectively, for Mumford families and good formal models, where the convention for the former follows from [46]. We distinguish between the two star signs * and *. The two dagger forms † and ‡ are used as superscripts. The differences between v, ν, υ, and the dual sign $^\vee$ should not be confusing because they are never used for similar purposes. The same is true for i, ι, $\imath$, and $\jmath$. Since we will never need calculus in this work, the symbols ∂, $\int$, and $\oint$ are used as variants of d or S.

Finally, unless it comes with "resp.", the content of each set of parentheses in text descriptions is *not an option*, but rather a reminder, a remark, or a supplement of information.