Introduction

Elasticity imaging is used to determine the characteristics of structures inside an elastic body based on observations of the displacement fields made on a part of the boundary surface or inside the body. The aim is to recover certain material and geometric parameters characteristic to these structures.

The main motivations of elasticity imaging are non-destructive testing of elastic structures for material impurities, exploration geophysics, and medical diagnosis, in particular, for detection of potential tumors of diminishing size.

Elasticity imaging for medical diagnosis aims at providing a quantitative visualization of mechanical properties of human tissues using the relation between the wave propagation velocity and the tissue viscoelastic properties. Different imaging modalities can be used to measure the displacement field in the interior of tissue in response to a time-harmonic or a dynamic excitation. The two major techniques are based on magnetic resonance imaging and on ultrasound. When magnetic resonance imaging is used, the excitation is at a relatively low frequency, and then the time-harmonic elastic response of the tissue is measured. On the other hand, when ultrasound is used, the excitation is dynamic and broadband. However, even in this case, the dynamic displacement field is resolved into its time-harmonic components via a Fourier transform. The time-harmonic displacement in the interior of the tissue offers a wealth of information that may be used to characterize the tissue viscoelastic properties by solving an imaging problem. These properties in turn carry information about the tissue composition, micro-structure, physiology, and pathology. Changes in tissue elasticity are generally correlated with pathological phenomena such as weakening of vessel walls or cirrhosis of the liver. Many cancers appear as extremely hard nodules because of the recruitment of collagen during tumorigenesis. It is therefore very interesting and challenging for diagnostic applications to find ways for generating resolved images that depict tissue elasticity or stiffness.

Elasticity imaging plays also an essential role in a wide range of industrial areas and at almost any stage in the production or life cycle of many components. It is important to determine the integrity of a structure, quantitatively measure some characteristic of an object or ensure a cost effective operation, safety and reliability of a plant, with resultant benefit to the community. As a number of infrastructures currently in service reach the end of their expected serviceable life, elasticity testing methods to evaluate their durability, and thus to ensure their structural integrity, have received gradually increasing attention. Concrete structures are designed, in particular, on the basis of the compression strength of the concrete. Concrete often degrades by the corrosion of the embedded steel-reinforcing bars, which can lead to internal stress. When reinforcing bars in concrete become corroded (*i.e.*, rusted), rust is generated on the surface of the bars, and small cracks begin to grow as the rusting expands. A change in the physical properties of the concrete due to degradation of a concrete structure affects the concrete's compression strength.

Elasticity imaging, which uses the direct relationship between the velocities of the elastic waves and the elastic properties of the material through which they are propagating, is the method of choice for estimating the compression strength of concrete.

In environmental sciences, a major application of elasticity imaging is the monitoring of potentially dangerous structures like active fault zones prone to damaging earthquakes or volcanic edifices. Oil reservoir monitoring is also of primary interest for the oil industry in order to get insight into the depletion dynamics of a reservoir. Quantitative elasticity imaging of elastic properties of the subsurface is essential for oil- and gas-reservoir characterization and for monitoring carbon dioxide sequestration with time-lapse acquisitions. Indeed, fluids and gases have significant effects on the elastic properties of the subsurface in terms of Poisson's ratio anomalies. This quantitative imaging is also required for near-surface imaging in the framework of civil engineering applications because the shear properties of the shallow weathered layers strongly impact the elastic wavefield.

Many challenging mathematical problems arise in elasticity imaging techniques and pose interesting mathematical riddles that often lead to the investigation of fundamental problems in various branches of mathematics.

This book covers recent mathematical, numerical and statistical approaches for elasticity imaging of inclusions and cracks with waves at zero, single or multiple non-zero frequencies. The inclusions and cracks of small size are believed to be the starting point of fatigue failure in elastic materials. An inclusion or a crack is called small when the product of its characteristic size with the operating frequency is less than one while it is called extended when this factor is much larger than one. There are two interesting problems: one is of finding small elastic inclusions and the other is of reconstructing shape deformations of an extended elastic inclusion. In both situations, we are interested in imaging small perturbations with respect to known situations.

Recently, there have been important developments on asymptotic imaging, stochastic modeling, and analysis of both deterministic and stochastic elastic wave propagation phenomena. The aim of this book is to put them together in a coherent way. An emphasis is laid down on deriving the best possible imaging functionals for small inclusions and cracks in the sense of stability and resolution. For imaging extended elastic inclusions, we design accurate optimal control methodologies and evaluate the effect of uncertainties of the geometric or physical parameters on their stability and resolution properties. We also provide an asymptotic framework for vibration testing. Localized damage to a mechanical structure affects its dynamic characteristics. The modification is characterized by changes in the eigenparameters, *i.e.*, eigenvalues and the associated eigenvectors. Considerable effort has been spent in obtaining a relationship between the changes in the eigenparameters, the damage location, characteristics, and size. In this book, we relate the measured eigenparameters to the elastic inclusion or crack location, orientation, and size. We design a method that can be used to identify, locate, and estimate inclusions and cracks in elastic structures by measuring their modal characteristics.

The book is organized as follows. In Chapter 1, after reviewing some well-known results on the solvability and layer potentials for static and time-harmonic elasticity equations, we proceed to prove representation formulas for solutions of the elasticity equations. Then we establish Helmholtz-Kirchhoff identities. These formulas are our main tool in later chapters for analyzing the resolution of elastic wave imaging approaches. Chapter 2 collects some recent results on the elasticity equations with high contrast coefficients. Chapter 3 covers the method of small-volume expansions. It provides the leading-order terms in the asymptotic expansions of the solutions to the static and time-harmonic elasticity equations with respect to the size of a small inclusion. We also introduce the concept of elastic moment tensor associated with an elastic inclusion and present its main properties. The results of Chapter 2 are used in Chapter 3 in order to show that the asymptotic expansion of the solution in the presence of small inclusions holds uniformly with respect to material parameters. The results of this chapter can be extended to anisotropic inclusions. Chapter 4 deals with the perturbations of the displacement (or traction) vector that are due to the presence of a small crack with homogeneous Neumann boundary conditions in an elastic medium. An asymptotic formula for the boundary perturbations of the displacement as the length of the crack tends to zero is derived. It carries information about the location, size, and orientation of the crack. Chapter 5 is devoted to direct imaging of small inclusions and cracks in the static regime. It focuses on MUSIC- and migration-type algorithms for detecting the small defects. Chapter 6 introduces a topological derivative based imaging framework for detecting elastic inclusions in the time-harmonic regime. Based on a weighted Helmholtz decomposition of the topological derivative based imaging functional, we achieve optimal resolution imaging. Its stability properties with respect to both medium and measurement noises are investigated. Chapters 8 and 9 discuss imaging techniques for extended elastic inclusions. We start with inverse source problems and introduce time-reversal techniques. Then we focus on reconstructing shape changes of an extended target. We introduce several algorithms and analyze their resolution and stability for the linearized reconstruction problem. Finally, we describe optimal control approaches for solving the nonlinear problem. Chapter 9 extends time-reversal techniques for imaging in viscoelastic media. Chapter 10 is to propose efficient methods for reconstructing both the shape and the elasticity parameters of an inclusion using internal displacement measurements. Chapter 11 is on vibration testing. Following the asymptotic formalism developed in this book, we derive asymptotic formulas for eigenvalue perturbations due to small inclusions, cracks, and shape deformations. We propose efficient algorithms for detecting small elastic inclusions and cracks or perturbations in the interface of an inclusion from modal measurements. In Appendix A we review useful probabilistic tools for elastic imaging in the presence of noise. In Appendix B we derive, based on the stationary phase theorem, asymptotics of the attenuation operator. In Appendix C we recall the main results of Gohberg and Sigal in [100] concerning the generalization to operator-valued functions of classical results in complex analysis.

The book opens a door for a mathematical and numerical framework for elasticity imaging of nano-particles and cellular structures.

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