Introduction

Historical precedent for the results in this book can be found in the theory of Riemann surfaces. Every compact Riemann surface of genus $g \geq 2$ has representations both as a plane algebraic curve, and so as a branched covering of the complex projective line, and as a quotient of the complex 1-ball, or unit disk, by a freely acting cocompact discrete subgroup of the automorphisms of the 1-ball. The latter result is a direct consequence of the uniformization theorem, which states that the simply-connected Riemann surfaces are the Riemann sphere, the complex plane, and the complex 1-ball.

In complex dimension 2, Hirzebruch (1956) and Yau (1978) showed that the smooth compact connected complex algebraic surfaces, representable as quotients of the complex 2-ball B_2 by freely acting cocompact discrete subgroups of the automorphisms of the 2-ball, are precisely the surfaces of general type whose Chern numbers satisfy $c_1^2 = 3c_2$. Here, c_2 is the Euler characteristic and c_1^2 is the self-intersection number of the canonical divisor. Accordingly, the *proportionality deviation* of a complex surface is defined by the expression $3c_2 - c_1^2$. This book examines the explicit computation of this proportionality deviation for finite covers of the complex projective plane ramified along certain line arrangements. Candidates for ball quotients among these finite covers arise by choosing weights on the line arrangements such that the proportionality deviation vanishes. We then show that these ball quotients actually exist.

The intention of F. Hirzebruch in the original notes was for the material to be presented in a nontechnical way, assuming a minimum of prerequisites, with definitions and results being introduced only as needed. The desire was for the reader to be exposed to the theory of complex surfaces—and of complex surfaces with an orbifold structure—through the examples provided by weighted line arrangements in the projective plane, with an emphasis on their finite covers which are ball quotients. There was no desire to develop a complete theory of surfaces and orbifolds although relevant references are provided, or to treat the latest developments, or to also survey results in higher dimensions. Instead, the goal was that, on reading the book, the student or researcher should be better equipped to go into more technical or more modern territory if interested. It was felt that the topic of the notes was important enough historically to warrant their appearance. F. Hirzebruch wanted the style of the original notes to be like that of his series of lectures, and for them to remain readable and conversational; and, even though a lot of material has been added since by me, I have aimed to retain those qualities.

The plan of the book is as follows. In Chapter 1, we collect in one place the main prerequisites from topology and differential geometry needed for subsequent chapters, although, for convenience, pieces of this material are at times repeated or expanded upon later in the book. Even if the reader is familiar with these topics, it is worth his or her while to leaf through Chapter 1 to see what notions and notations are used later.

In Chapter 2, we apply some of the material from Chapter 1 to Riemann surfaces—which are of real dimension 2—by way of a historical and conceptual motivation for the material on complex surfaces. We also discuss in this chapter the classical Gauss hypergeometric functions of one complex variable and the triangle groups that serve as their monodromy groups. When these triangle groups are hyperbolic, they act on the unit disk, that is, the complex 1-ball. Infinitely many such groups act discontinuously, and their torsion-free normal subgroups of finite index define quotients of the complex 1-ball that are finite coverings of the projective line branched over three points.

In Chapter 3, we study complex surfaces and their coverings branched along divisors, that is, subvarieties of codimension 1. In particular, we discuss coverings branched over transversally intersecting divisors. Applying this to line arrangements in the complex projective plane, we first blow up the projective plane at non-transverse intersection points, that is, at those points of the arrangement where more than two lines intersect. These points are called singular points of the arrangement. This results in a complex surface and transversely intersecting divisors that contain the proper transforms of the original lines. Next, we introduce the group of divisor classes, their intersection numbers, and the canonical divisor class. The Chern numbers c_1^2 and c_2, as well as the proportionality deviation, $\text{Prop} := 3c_2 - c_1^2$, are then defined.

In Chapter 4, we give an overview of the rough classification of (smooth complex connected compact algebraic) surfaces. We present two approaches that, in dimension 2, give the Miyaoka-Yau inequality; $c_1^2 \leq 3c_2$, for surfaces of general type. We only make a few remarks about the first of these, due to Miyaoka, which uses algebraic geometry. The second, due to Aubin and Yau, uses analysis and differential geometry. Here we give more details, since we use the analogous approach to the Miyaoka-Yau inequality for surface orbifolds in Chapter 6. We also discuss why equality in the Miyaoka-Yau inequality characterizes surfaces of general type that are free quotients of the complex 2-ball B_2, that is, orbit spaces for the action of discrete cocompact subgroups of the automorphisms of B_2 that have no fixed points.

In Chapter 5 we arrive at the main topic of the book: the free 2-ball quotients arising as finite covers of the projective plane branched along line arrangements. The material of this chapter is self-contained and largely combinatorial. Let X be the surface obtained by blowing up the singular intersection points of a line arrangement in the complex projective plane. Let Y be a smooth compact complex surface given by a finite cover of X branched along the divisors on X defined by the lines of the arrangement, that is, by their proper transforms and by the blown-up points, also known as exceptional divisors. If Y is of general type with vanishing proportionality deviation Prop, then it is a free 2-ball quotient. In order to find such a Y, we use a formula for Prop due to T. Höfer. This formula expresses Prop as the sum of, first, a quadratic form evaluated at numbers related only to the ramification indices along the proper transforms of the lines, and, second, nonnegative contributions from each blown-up point. The quadratic form itself depends only on the original line arrangement. The contributions from the blown-up points vanish when we impose certain diophantine conditions on the choice of ramification indices. Next, we seek line arrangements and ramification indices that make the quadratic form vanish. Initially, we ask of the arrangements that they have equal ramification indices along each of the proper transforms of the original lines. For these arrangements, the number of intersection points on each line is $(k+3)/3$, where k is the number of lines. We then list all known line arrangements with this property and restrict our attention to them. Next, we replace the ramification indices by weights that are positive or negative integers or infinity, and we list all possible weights that satisfy the diophantine conditions at the blown-up points and also annihilate the quadratic form. Finally, we enumerate all possibilities for the assigned weights of the arrangements, under the assumption that divisors of negative or infinite weight on the blown-up line arrangements do not intersect. When the weight of a divisor on a blown-up line arrangement is negative, the curves above it on Y are exceptional and can be blown down. Blowing down all curves on Y arising from such negative weights, we obtain a smooth surface Y'. When the weight of a divisor on a blown-up line arrangement is infinite, the curves above it on Y are elliptic. Letting C be the union on Y' of all elliptic curves arising in this way, we derive the appropriate expression for the Prop of the possibly non-compact smooth surface $Y' \setminus C$. It is known that we cannot have rational or elliptic curves on a free 2-ball quotient, so this construction is quite natural. For the line arrangements of Chapter 5, we give the complete list of weights such that the Prop of $Y' \setminus C$ vanishes, meaning that $Y' \setminus C$ is a free 2-ball quotient. These weights are all presented in a series of tables in §5.7 at the end of Chapter 5, except for a few extra cases that are listed in §5.6.1. Throughout this chapter, we assume that the finite covers of the line arrangements with vanishing proportionality actually exist. The expression for Prop of $Y' \setminus C$ depends only on the original weighted line arrangement, so we can work with this assumption; but we still need to show that there are such covers.

In Chapter 6 we justify the existence assumption of Chapter 5. Let X be the blow-up of the projective plane at the non-transverse intersection points of a line arrangement. Assign weights—allowed to be positive or negative integers or infinity, denoted by n_i for the proper transforms D_i of the lines, and denoted by m_j for the exceptional divisors E_j given by the blown-up points—in such a way that the divisors with negative or infinite weights are distinct. Let X' be the possibly singular surface obtained by blowing down the D_i and E_j with negative weight, and X'' the possibly singular surface obtained by contracting the images of the D_i and E_j on X' with infinite weight. Let D_i'' be the image in X'' of D_i with weight $n_i > 0$, and E_j'' the image in X'' of E_j with weight $m_j > 0$. The central question of Chapter 6 is the following. When is X'' a (possibly compactified by points) ball quotient $X'' = \overline{\Gamma \backslash B_2}$ for a discrete subgroup Γ of the automorphisms of the ball with natural map $B_2 \to X''$ ramified of order n_i along D_i'' and m_j along E_j''? Any normal subgroup Γ' of Γ of finite index N in Γ, acting on B_2 without fixed points, gives rise to a free ball quotient $\Gamma' \backslash B_2 = Y' \setminus C$, as described in Chapter 5, that is a finite cover of $X_0'' = \Gamma \backslash B_2$ of degree N, ramified of order $n_i > 0$ along $D_i'' \cap X_0''$ and $m_j > 0$ along $E_j'' \cap X_0''$. Such normal subgroups exist due to the work of Borel and Selberg.

Therefore, we approach the proof of the existence of $Y' \setminus C$ by first showing the existence of Γ. The group Γ acts with fixed points, so, in answering this central question, the language of orbifolds is appropriate. Given the simplicity and explicit nature of the orbifolds we study, we use instead the related notion of b-space due to Kato. For the reader seeking a more sophisticated and general treatment, it is worth learning the basics of complex orbifolds that we exclude. Roughly speaking, we show that if all the diophantine conditions on the weights, derived in Chapter 5 to ensure that Prop vanishes, are satisfied, then there is equality in an orbifold, or b-space, version of the Miyaoka-Yau inequality, and, by arguments analogous to those of Chapter 4, the space X_0'' is of the form $\Gamma \backslash B_2$ and the map $B_2 \to X_0''$ has the desired ramification properties described above. To do this, we use the work of R. Kobayashi, S. Nakamura, and F. Sakai [83], which generalizes the work of Aubin-Yau on the Miyaoka-Yau inequality to surfaces with an orbifold structure.

There are both related and alternative approaches dating from about the same time, for example, the approach in the independent work of S. Y. Cheng and S. T. Yau [27], as well as more modern and more general approaches, such as in the work of A. Langer [91], [92], to mention just a few important instances. We choose to base our discussion on the work of R. Kobayashi, S. Nakamura, and F. Sakai, as it was part of the original notes made with F. Hirzebruch and it fits well with the presentation of the work of Aubin-Yau in Chapter 4 as well as with our approach to the material of Chapter 5. The main point of Chapter 6 is to flesh out the details of the orbifold version of the Miyaoka-Yau inequality for our particular situation of line arrangements in the projective plane.

Chapter 7 focuses on the complete quadrilateral line arrangement, and especially on its relationship with the space of regular points of the system of partial differential equations defining the Appell hypergeometric function. The latter is the two-variable analogue of the Gauss hypergeometric function discussed in Chapter 2. Building on work of E. Picard and of T. Terada, Deligne and Mostow established criteria for the monodromy group of this Appell function to act discontinuously on the complex 2-ball. This leads to examples of complex 2-ball quotients, determined by freely acting subgroups of finite index in these monodromy groups that are branched along the blown-up complete quadrilateral. Some of these monodromy groups provide rare examples of non-arithmetic groups acting discontinuously on irreducible bounded complex symmetric domains of dimension greater than 1.

The book concludes with two appendices. The first is by H.-C. Im Hof[1] and supplies a proof of Fenchel's Conjecture about the existence of torsion-free subgroups of finite index in finitely generated Fuchsian groups. The second concerns Kummer coverings of line arrangements in the projective plane.

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