Introduction

What Philosophy of Mathematics Is Today

One of the starting points for this book project was a job talk that I presented at the philosophy department of some research university. I discussed how mathematical signs shift their meanings and described mathematical processes of sense making, some of which are covered in this book. One of the department professors commented succinctly that "this is not philosophy of mathematics." He explained that a philosopher of mathematics should take the notions and terms that we or our historical sources use when discussing mathematics and provide them with some sort of rational reconstruction. That was not what I was doing.

In a sense, this professor was right. What I did was not philosophy of mathematics as usually practiced today. This can be easily verified. I went through the top 150 entries in the *Philosopher's Index* with the words "mathematics" or "mathematical" in their abstracts that were posted over the last couple of years. This showed that the most popular debate in contemporary philosophy of mathematics, representing almost 40 percent of research production, is how to describe the mode of existence of mathematical entities, especially in the context of their application to natural sciences (the *PhilPapers* bibliographies suggest a similar proportion).

So the main problem that bothers mainstream philosophy of mathematics today has to do with the kind of reality attributed to mathematical objects and statements. Indeed, in most situations, one speaks of mathematical objects and statements like one speaks of other scientific objects and statements. But there are obvious problems with this way of speaking, because mathematical objects are hard to tie down to spatio-temporal phenomena, and the claims that involve mathematical objects are therefore hard to conceive as truly referential, as realists require. At the same time, many mathematical claims are highly applicable to empirical phenomena, which makes it difficult to think of them as contingent constructs of reason, as nominalists tend to do.

So the main contemporary philosophical task in mainstream philosophy of mathematics is to trace a rational account of the terms "true" and "exist" so as to allow their consistent use in both science and mathematics, and, at the same time, respect the common usage of these terms and common mathematical habits. But this latter pair of constraints is in conflict: we may either redefine "true" and "exist" creatively to generate consistency over their use in science and mathematics, but lose contact with common usage, or we can hold on to common usage, but face obstacles when applying these terms to real, existing mathematical practice. The philosophical debate is thus about stretching the terms "true" and "exist" in ways that cover the most important aspects of common usage and conceptual consistency. Since deciding what is "most important" involves nonconsensual prioritizing, the debate continues to spin.

This analytic approach refers to an established canon of philosophies of mathematics. The main references are the following: Plato's transcendent and ideal mathematical forms recollected through empirical experience and dialectical reason; Kant's view of mathematics as a science of the forms *through* which we organize time and space (a middle ground between empiricist accounts of mathematics in terms of observations *in* time and space and rationalist accounts in terms of pure nonempirical reason); the logicist attempt to reduce mathematics to logic as the science of pure reason; the intuitionist attempt to confine mathematics to what can be actually constructed in our minds (or, more exactly, in a Kantian-like form of temporal intuition); the formalist articulation of mathematics as a system of meaningless signs subject to purely syntactic rules whose consistency is to be analyzed by means of a constructive and finitary logic; the logical positivist articulation of mathematics as a system of syntactic logical truths used to tie together empirical observations; and the empiricist-holist view of mathematics and empirical science as an inseparable continuum. This canon provides the backdrop for the contemporary search for a satisfactory articulation of mathematical objectivity and truth.

This entire canon revolves around a foundational question: "To what kind of ontological ground can we *reduce* mathematics?" It is tied together by a search for a unified ontological substrate and for a unified language to discuss it. So from the point of view of this tradition, the professor who criticized my presentation was right. I was not doing philosophy of mathematics, and I continue not to do it in this book. (Obviously, I didn't get the job.)

What Else Philosophy of Mathematics Can Be

Before trying to articulate a different kind of philosophy of mathematics, I should explain who this philosophy is meant for. As I see it, there are three main target groups for the philosophy of mathematics: philosophers, mathematicians, and people who engage with mathematics less intensively in their professional and daily lives.

Philosophers are usually interested in mathematics as a test case for some general philosophical system. Since it is not just any test case, but one that has some extreme characteristics (for instance, it is seen as an extremely rigorous branch of knowledge), mathematics is considered a very important test case. Since the interest in mathematics is usually entangled with science and logic, and since these domains are favorites of the analytical tradition, philosophers' philosophy of mathematics is usually analytic—hence the current focus on analytic versions of realism versus nominalism and questions concerning math-science relations.

Mathematicians, as far as I can see, are not terribly interested in the philosophy of mathematics. They often have philosophical views, but they are usually not very keen on challenging or developing them—they don't usually consider this as worthy of too much effort. They're also very suspicious of philosophers. Indeed, mathematicians know better than anyone else what it is that they're doing. The idea of having a philosopher lecture them about it feels kind of silly, or even intrusive.

So we turn to people who have something to do with mathematics in their professional or daily lives, but are not focused on mathematics. Such people often have some sort of vague, sometimes naïve, conceptions of mathematics. One of the most striking manifestations of these folk views is the following: If I say something philosophical that people don't understand, the default assumption is that I use big pretentious words to cover small ideas. If I say something mathematical that people don't understand, the default assumption is that I'm saying something so smart and deep that they just can't get it.

There's an overwhelming respect for mathematics in academia and in wider circles. So much so that bad, trivial, and pointless forms of mathematization are often mistaken for important achievements in the social sciences, and sometimes in the humanities as well. It is often assumed that all ambiguities in our vague verbal communication disappear once we switch to mathematics, which is supposed to be purely univocal and absolutely true. But a mirror image of this approach is also common. According to this view, mathematics is a purely mechanical, inhuman, and irrelevantly abstract form of knowledge.

I believe that the philosophy of mathematics should try to confront such naïve views. To do that, one doesn't need to reconstruct a rational scheme underlying the way we speak of mathematics, but rather paint a richer picture of mathematics, which tries to affirm, rather than dispel, its ambiguities, humanity, and historicity.

This approach represents a minority strand in the philosophy of mathematics—but a growing minority. There's a whole tradition that's been coming together since the 1980s (if not earlier), which, today, is referred to under the title "philosophy of mathematical practice" (Azzouni 1994; Ernest 1998; Hersh 1997; Van Kerkhove 2009; Van Kerkhove and Van Bendegem 2007; Mancosu 1996, 2011; Rotman 2000; Tymoczko 1998). This tradition tries to explain what it is that mathematicians do when they do mathematics, and to shift the focus from "what it is" to "how it works." This shift does not quite exclude the question of "what mathematics is"; rather, it asks "what it is in motion," as it is being produced, understood, interpreted, and applied, rather than "what it is at rest," when we try look at it as a complete, given object or stratum of reality/discourse.

When proponents of this school of philosophy of mathematics look at some historical or contemporary practices that are problematic according to contemporary formal standards (for example, the use of infinitesimals or argument by diagrams), they do not try to reconstruct them in more rigorous terms; rather they look at whatever it is that mathematicians do, even when what they do is not formally rigorous (I, personally, tend to focus on semiosis—how mathematical signs accumulate and change meanings).

This latter branch of the philosophy of mathematics is highly descriptive and deeply entangled with the history and sociology of mathematics. Indeed, when we look at what mathematicians do, we find that their practice changes historically and is irreducibly embedded in social institutions. Due to the descriptive and concrete flavor of this kind of work, many see it as unworthy of the title "philosophical." It often has to call itself "science studies" to survive outside the traditional disciplinary boundaries of philosophy, history, and sociology. In many ways, what I do with mathematical texts is more similar to what some researchers in literature departments do with literary texts than to what people in philosophy departments do (this obviously has to do with the fact that some branches of continental philosophy were exiled to literature departments).

Nevertheless, I believe that this form of dealing with mathematics, regardless of whether we choose to call it philosophy or not, is genuinely important today. The approach that I'm promoting here can help a general academic readership reform some folk views of mathematics, and reposition mathematics as a humanely accessible endeavor, enjoying many unique characteristics, but still comparable to other branches of knowledge.

The approach advocated here is gaining more and more interest from philosophers, and is less alienating to mathematicians than mainstream philosophical accounts. Hopefully, it can generate a discourse that will draw together philosophers, mathematicians, and nonspecialists, so as to reintegrate the scattered sectarian debates on the present and future of mathematics into a lively and more pertinent cross-cultural conversation.

The uncritical idolizing of mathematics as *the best* model of knowledge, just like the opposite trend of disparaging mathematics as mindless drudgery, are both detrimental to the organization and evaluation of contemporary academic knowledge. Instead, mathematics should be appreciated and judged as one among many practices of shaping knowledge. Understanding what this practice consists of would allow the academic community to give mathematics its due credit and place within and outside the academic system.

But before we go any further, in order to give a more concrete sense of what I am trying to deal with, let's consider the following vignette.

A Vignette: Option Pricing and the Black-Scholes Formula

The point of the following vignette is to give a concrete example of how mathematics relates to its wider scientific and practical context. It will show that mathematics has force, and that its force applies even when actual mathematical claims do not quite work as descriptions of reality. The rest of this book will then try to philosophize about this force: where it comes from, how it works, and how it interacts with other forces.

The context of this vignette is option pricing. An "option" is the right (but not the obligation) to make a certain transaction at a certain cost at a certain time. For example, I could own the option to buy 100 British pounds for 150 US dollars three months from today. If I own this option, and three months from today 100 pounds are worth more than 150 dollars, I'd be likely to use the option. If 100 pounds turn out to be worth less than 150 dollars, I will most probably simply discard it.

Such options could be used as insurance. The preceding option, for example, would insure me against a drop in the dollar-pound exchange rate, if I needed such insurance. It could also serve as a simple bet for financial gamblers. But what price should one put on this kind of insurance or bet?

There are two narratives to answer this question. The first says that until 1973, no one really knew how to price such options, and prices were determined by supply, demand, and guesswork. More precisely, there existed some reasoned means to price options, but they all involved putting a price on the risk one was willing to take, which is a rather subjective issue.

In two papers published in 1973, Fischer Black and Myron Scholes, followed by Robert Merton, came up with a reasoned formula for pricing options that did not require putting a price on risk. This feat was deemed so important that in 1997 Scholes and Merton were awarded the Nobel Prize in economics for their formula (Black had died two years earlier). Indeed, "Black, Merton and Scholes thus laid the foundation for the rapid growth of markets for derivatives in the last ten years"—at least according to the Royal Swedish Academy press release (1997).

But there's another way to tell the story. This other way claims that options go back as far as antiquity, and option pricing has been studied as early as the seventeenth century. Option pricing formulas were established well before Black and Scholes, and so were various means to factor out putting a price on risk (based on something called put-call parity rather than the Nobel-winning method of dynamic hedging, but we can't go into details here). Moreover, according to this narrative, the Black-Scholes formula simply doesn't work and isn't used (Derman and Taleb 2005; Haug and Taleb 2011).

If we wanted to strike a compromise between the two narratives, we could say that the Black-Scholes model was a new and original addition to existing models, and that it works under suitable ideal conditions, which are not always approximated by reality. But let's try to be more specific.

The idea behind the Black-Scholes model is to reconstruct the option by a dynamic process of buying and selling the underlying assets (in our preceding example, pounds and dollars). It provides an initial cost and a recipe that tells you how to continuously buy and sell these dollars and pounds as their exchange rate fluctuates over time in order to guarantee that by the time of the transaction, the money one has accumulated together with the 150 dollars dictated by the option would be enough to buy 100 pounds. This recipe depends on some clever, deep, and elegant mathematics.

This recipe is also risk free and will necessarily work, provided some conditions hold. These conditions include, among others, the capacity to always instantaneously buy and sell as many pounds/dollars as I want and a specific probabilistic model for the behavior of the exchange rate (Brownian motion with a fixed and known future volatility, where volatility is a measure of the fluctuations of the exchange rate).

The preceding two conditions do not hold in reality. First, buying and selling is never really unlimited and instantaneous. Second, exchange rates do not adhere precisely to the specific probabilistic model.

But if we can buy and sell fast enough, and the Brownian model is a good enough approximation, the pricing formula should work well enough. Unfortunately, prices sometimes follow other probabilistic models (with some infinite moments), where the Black and Scholes formula may fail to be even approximately true. The latter flaw is sometimes cited as an explanation for some of the recent market crashes—but this is a highly debated interpretation.

Another problem is that the future volatility (a measure of cost fluctuations from now until the option expires) of whatever the option buys and sells has to be known for the model to work. One could rely on past volatility, but when comparing actual option prices and the Black-Scholes formula, this doesn't quite work. The volatility rate that is required to fit the Black-Scholes formula to actual market option pricing is not simply the past volatility.

In fact, if one compares actual option prices to the Black-Scholes formula, and tries to calculate the volatility that would make them fit, it turns out that there's no single volatility for a given commodity at a given time. The cost of wilder options (for selling or buying at a price far removed from the present price) reflects higher volatility than the more tame options. So something is clearly empirically wrong with the Black-Scholes model, which assumes a fixed (rather than stochastic) future volatility for whatever the option deals with, regardless of the terms of the option.

So the Black-Scholes formula is nice in theory, but needn't work in practice. Haug and Taleb (2011) even argue that practitioners simply don't use it, and have simpler practical alternatives. They go as far as to say that the Black-Scholes formula is like "scientists lecturing birds on how to fly, and taking credit for their subsequent performance—except that here it would be lecturing them the wrong way" (101, n. 13). So why did the formula deserve a Nobel prize?

Looking at some informal exchanges between practitioners, one can find some interesting answers. The discussion I quote from the online forum *Quora* was headed by the question "Is the Black-Scholes Formula Just Plain Wrong?" (2014). All practitioners agree that the formula is not used as such. Many of them don't quite see it as an approximation either. But this does not mean that they think it is useless. One practitioner (John Hwang) writes:

> Where Black-Scholes really shines, however, is as a common language between options traders. It's the oldest, simplest, and the most intuitive option pricing model around. Every option trader understands it, and it is easy to calculate, so it makes sense to communicate implied volatility [the volatility that would make the formula fit the actual price] in terms of Black-Scholes.... As a proof, the exchanges disseminate [Black-Scholes] implied volatility in addition to price data.

Another practitioner (Rohit Gupta) adds that this "is done because traders have better intuition in terms of volatilities instead of quoting various prices." In the same vein, yet another practitioner (Joseph Wang) added:

> One other way of looking at this is that Black-Scholes provides something of a baseline that lets you compare the real world to a nonexistent ideal world.... Since we don't live in an ideal world, the numbers are different, but the Black-Scholes framework tells us *how different* the real world is from the idealized world.

So the model earned its renown by providing a common language that practitioners understand well, and allowing them to understand actual contingent circumstances in relation to a sturdy ideal.

Now recall that practitioners extrapolate the implied volatility by comparing the Black-Scholes formula to actual prices, rather than plug a given volatility into the formula to get a price. This may sound like data fitting. Indeed, one practitioner (Ron Ginn) states that "if the common denominator of the crowd's opinion is more or less Black-Scholes ... smells like a self fulfilling prophecy could materialize," or, put in a more elaborate manner (Luca Parlamento):

> I just want to add that CBOE [Chicago Board Options Exchange] in early '70 was looking to market a new product: something called "options." Their issue was that how you can market something that no one can evaluate? You can't! You need a model that helps people exchange stuff, turn[s] out that the BS formula ... did the job. You have a way to make people easily agree on prices, create a liquid market and ... "why not" generate commissions.

The tone here is more sinister: the formula is useful because it's there, because it's a reference point that allows a market to grow around it.

But why did *this* specific formula attract the market, and become a common reference point, possibly even a self-fulfilling prophecy? Why not any of the other older or contemporary pricing practices, which are no worse? Why was this specific pricing model deemed Nobel worthy?

The answer, I believe, lies in the mathematics. The formula depends on a sound and elegant argument. The mathematics it uses is sophisticated, and enjoys a record of good service in physics, which imparts a halo of scientific prestige. Moreover, it is expressed in the language of an expressive mathematical domain that makes sense to practitioners (and, of course, it also came at the right time).

This is the force of mathematics. It's a language that the practitioners of the relevant niches understand and value. It feels well founded and at least ideally true. If it is sophisticated and comes with a good track record in other scientific contexts, it is assumed to be deep and somehow true. All this helps build rich practical networks around mathematical ideas, even when these ideas do not reflect empirical reality very well.

This book is about the force of mathematics, its origins and its unfolding. We all have some basic ideas about how and why mathematics works, and to what extent it is true or useful. But if we want to understand the surprising force of mathematics demonstrated in this vignette, we need to engage in a more careful analysis of mathematical practice.

Outline of This Book

The purpose of this book is to investigate what this force of mathematics builds on, and how it works in practice. To do that, I will discuss mathematics not only from the point of view of applications but also from the point of view of its production.

Chapter 1 introduces some histories of canonized philosophies of mathematics. These historical narratives are structured so as to highlight *not* the specific philosophical questions of the canon, but some overarching concerns that philosophical debates reflect. This serves to creatively rearrange the canon of the philosophy of mathematics and introduce some of the problems that this book will engage.

This articulation of overarching problems is reflected in chapter 2 by a real historical case study. This chapter attempts to flesh out the preceding chapter's problems by describing economical-mathematical practice with algebraic signs and subtracted numbers in the *abbaco* tradition of the Italian late Middle Ages and Renaissance. This chapter follows the vein of Wagner (2010b, 2010c), but it is thoroughly reorganized, and includes new material.

This leads us to chapter 3: a general outline of a philosophy of mathematical practice that forms the theoretical core of this book (a philosophically inclined reader who wants to read just one chapter of this book should probably go directly there). This chapter reflects on the function of mathematical statements (following Wittgenstein), their epistemological position, mathematical consensus, and mathematical interpretation and semiosis. The various positions expressed in the philosophical survey of the first chapter and the historical case study of the second are rearticulated as real constraints that apply to mathematical practice. Different mathematical cultures negotiate these constraints in different ways, and no single constraint serves as a final foundation. The chapter then proceeds to engage more mainstream notions of reality and truth of mathematical entities and statements (following Grosholz and Maddy), and suggests how a takeoff on Putnam's notion of relevance might relativize them. The fourth section of this chapter includes material revised from Wagner (2010a).

Chapter 4 attempts to reflect some of the ideas of the previous chapter with concrete case studies, focusing on problems of mathematical semiosis: how mathematical signs obtain and change their senses. The case studies are abridged and simplified versions of Wagner (2009b, 2009c), dealing with generating functions and the stable marriage problem in combinatorics. This opens up questions of how meaning is transferred within and across mathematical contexts—a question that belongs to the study of mathematical cognition.

Chapter 5 will articulate the preceding cognitive concerns in a more systematic manner. In order to introduce the notion of embodied mathematical cognition, I will first review the neuro-cognitive debate on the mental representation of numbers (focusing on Dehaene and Walsh). I will then present the cognitive theory of mathematical metaphor and suggest a rearticulation, based on Wagner (2013). This theory will be further enriched by an engagement with Walter Freeman's theory of meaning. We will conclude with an appropriation of Deleuze's *Logic of Sensation* to the context of mathematical practice, drawing on material from Wagner (2009a).

Chapter 6 will flesh out the cognitive problematic from the previous chapter with case studies of medieval and early modern geometric algebra adapted from Wagner (2013) and of the history of notions of infinity adapted from Wagner (2012). These case studies will demonstrate the limitations of the cognitive theory of mathematical metaphor in accounting for the formation of actual historical mathematical life worlds.

Chapter 7 will complement the discussion by thinking of mathematics not only as subject to constraints but also as feeding back into the reality that shapes it. A brief narrative will follow Fichte, Schelling, and Hermann Cohen to derive inspiration for rethinking the reality of ideas, and suggest how mathematics reforms the world where it lives. These philosophical approaches will lead us to offer a solution to Mark Steiner's formulation of Wigner's problem of the "unreasonable" applicability of mathematics to the natural sciences (or at least a reduction of the problem to a more containable intra-mathematical setting), elaborating an argument briefly outlined in Wagner (2012).

The book should be accessible to readers with a general interest in philosophy and mathematics. Some of the case studies and examples may require the equivalent of basic undergraduate calculus or linear algebra courses. Only a few scattered examples require higher mathematical training, and they can be skipped. The modular structure of the book should help readers avoid sections that are too theoretical or too technical for their taste.

I include various historic and contemporary case studies, some of which may appear rather strange to contemporary readers. My purpose is to recall that mathematics was and can be different from the mathematics that we are used to today. This helps us gain a wider view of the possibilities and contingencies of mathematical practice that contemporary imagination tends to suppress. In turn, this will provide us with a better, more complete understanding of the landscape that the title "mathematics" subsumes.