Introduction and Overview

A DIFFERENTIAL FIELD WITH NO ESCAPE

Our principal object of interest is the *differential field* $\mathbb{T}$ of *transseries*. Transseries are formal series in an indeterminate $x > \mathbb{R}$, such as

$$\begin{aligned} \varphi(x) \;=\; & -3\,\mathrm{e}^{\mathrm{e}^{x}} + \mathrm{e}^{\frac{\mathrm{e}^{x}}{\log x} + \frac{\mathrm{e}^{x}}{\log^2 x} + \frac{\mathrm{e}^{x}}{\log^3 x} + \cdots} - x^{11} + 7 \\ & + \frac{\pi}{x} + \frac{1}{x\log x} + \frac{1}{x\log^2 x} + \frac{1}{x\log^3 x} + \cdots \\ & + \frac{2}{x^2} + \frac{6}{x^3} + \frac{24}{x^4} + \frac{120}{x^5} + \frac{720}{x^6} + \cdots \\ & + \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x^2} + 3\,\mathrm{e}^{-x^3} + 4\,\mathrm{e}^{-x^4} + \cdots, \end{aligned} \tag{1}$$

where $\log^2 x := (\log x)^2$, etc. As in this example, each transseries is a (possibly transfinite) sum, with terms written from left to right, in asymptotically decreasing order. Each term is the product of a real coefficient and a *transmonomial*. Appendix A contains the inductive construction of $\mathbb{T}$, including the definition of "transmonomial" and other notions about transseries that occur in this introduction. For expositions of $\mathbb{T}$ with proofs, see [112, 122, 194]. In [112], $\mathbb{T}$ is denoted by $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$, and its elements are called *logarithmic-exponential series*. At this point we just mention that transseries can be added and multiplied in the natural way, and that with these operations, $\mathbb{T}$ is a field containing $\mathbb{R}$ as a subfield. Transseries can also be differentiated term by term, subject to $r' = 0$ for each $r \in \mathbb{R}$ and $x' = 1$. In this way $\mathbb{T}$ acquires the structure of a *differential field*.

Why transseries?

Transseries naturally arise in solving differential equations at infinity and studying the asymptotic behavior of their solutions, where ordinary power series, Laurent series, or even Puiseux series in x^{-1} are inadequate. Indeed, functions as simple as e^{x} or $\log x$ cannot be expanded with respect to the asymptotic scale $x^{\mathbb{R}}$ of real powers of x at $+\infty$. For merely solving algebraic equations, no exponentials or logarithms are needed: it is classical that the fields of Puiseux series over $\mathbb{R}$ and $\mathbb{C}$ are real closed and algebraically closed, respectively.

One approach to asymptotics with respect to more general scales was initiated by Hardy [163, 165], inspired by earlier work of du Bois-Reymond [51] in the late 19th century. Hardy considered *logarithmico-exponential functions*: real-valued functions built up from constants and the variable x using addition, multiplication, division, exponentiation and taking logarithms. He showed that such a function, when defined on some interval $(a, +\infty)$, has eventually constant sign (no oscillation!), and so the germs at $+\infty$ of these functions form an ordered field H with derivation $\frac{d}{dx}$. Thus H is what Bourbaki [62] calls a *Hardy field*: a sub*field* K of the ring of germs at $+\infty$ of differentiable functions $f\colon (a, +\infty) \to \mathbb{R}$ with $a \in \mathbb{R}$, closed under taking derivatives; for more precision, see Section 9.1. Each Hardy field is naturally an ordered differential field. The Hardy field H is rather special: every $f \in H$ satisfies an algebraic differential equation over $\mathbb{R}$. But H lacks some closure properties that are desirable for a comprehensive theory. For instance, H has no antiderivative of e^{x^2} (by Liouville; see [361]), and the functional inverse of $(\log x)(\log\log x)$ doesn't lie in H, and is not even asymptotic to any element of H: [111, 190]; see also [333].

With $\mathbb{T}$ and transseries we go beyond H and logarithmico-exponential functions by admitting *infinite* sums. It is important to be aware, however, that by virtue of its inductive construction, $\mathbb{T}$ does not contain, for example, the series

$$x + \log x + \log\log x + \log\log\log x + \cdots,$$

which does make sense in a suitable extension of $\mathbb{T}$. Thus $\mathbb{T}$ allows only certain kinds of infinite sums. Nevertheless, it turns out that the differential field $\mathbb{T}$ enjoys many remarkable closure properties that H lacks. For instance, $\mathbb{T}$ is closed under natural operations of exponentiation, integration, composition, compositional inversion, and the resolution of *feasible* algebraic differential equations (where the meaning of *feasible* can be made explicit). This makes $\mathbb{T}$ of interest for different areas of mathematics:

Analysis

In connection with the Dulac Problem, $\mathbb{T}$ is sufficiently rich for modeling the asymptotic behavior of so-called Poincaré return maps. This analytically deep result is a crucial part of Écalle's solution of the Dulac Problem [119, 120, 121]. (At the end of this introduction we discuss this in more detail.)

Computer algebra

Many transseries are concrete enough to compute with them, in the sense of computer algebra [190, 402]. Moreover, many of the closure properties mentioned above can be made effective. This allows for the automation of an important part of asymptotic calculus for functions of one variable.

Logic

Given an o-minimal expansion of the real field, the germs at $+\infty$ of its definable one-variable functions form a Hardy field, which in many cases can be embedded into $\mathbb{T}$. This gives useful information about the possible asymptotic behavior of these definable functions; see [21, 292] for more about this connection.

Soon after the introduction of $\mathbb{T}$ in the 1980s it was suspected that $\mathbb{T}$ might well be a kind of *universal domain* for the differential algebra of Hardy fields and similar ordered differential fields, analogous to the role of the algebraically closed field $\mathbb{C}$ as a universal domain for algebraic geometry of characteristic 0 (Weil [461, Chapter X, §2]), and of $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{C}(\!(t)\!)$ in related ordered and valued settings. This is corroborated by the strong closure properties enjoyed by $\mathbb{T}$. See in particular p. 148 of Écalle's book [120] for eloquent expressions of this idea. The present volume and the next substantiate the *universal domain* nature of the differential field $\mathbb{T}$, using the language of *model theory.* The model-theoretic properties of the classical fields $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{C}(\!(t)\!)$ are well established thanks to Tarski, Seidenberg, Robinson, Ax & Kochen, Eršov, Cohen, Macintyre, Denef, and others; see [443, 395, 350, 28, 29, 131, 84, 275, 100]. Our goal is to analyze likewise the differential field $\mathbb{T}$, which comes with a definable ordering and valuation, and in this book we achieve this goal.

The ordered and valued differential field $\mathbb{T}$

For what follows, it will be convenient to quickly survey some of the most distinctive features of $\mathbb{T}$. Appendix A contains precise definitions and further details.

Each transseries $f = f(x)$ can be uniquely decomposed as a sum

$$f \;=\; f_{\succ} + f_{\asymp} + f_{\prec},$$

where $f_{\succ}$ is the *infinite part of* f, $f_{\asymp}$ is its constant term (a real number), and $f_{\prec}$ is its *infinitesimal part*. In the example (1) above,

$$\begin{aligned}
\varphi_{\succ} &= -3\,\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^{\frac{\mathrm{e}^x}{\log x} + \frac{\mathrm{e}^x}{\log^2 x} + \frac{\mathrm{e}^x}{\log^3 x} + \cdots} - x^{11},\\
\varphi_{\asymp} &= 7,\\
\varphi_{\prec} &= \frac{\pi}{x} + \frac{1}{x\log x} + \cdots.
\end{aligned}$$

In this example, $\varphi_{\succ}$ happens to be a finite sum, but this is not a necessary feature of transseries: take for example $f := \frac{\mathrm{e}^x}{\log x} + \frac{\mathrm{e}^x}{\log^2 x} + \frac{\mathrm{e}^x}{\log^3 x} + \cdots$, with $f_{\succ} = f$. Declaring a transseries to be positive iff its dominant (= leftmost) coefficient is positive turns $\mathbb{T}$ into an *ordered* field extension of $\mathbb{R}$ with $x > \mathbb{R}$. In our example (1), the dominant transmonomial of $\varphi(x)$ is $\mathrm{e}^{\mathrm{e}^x}$ and its dominant coefficient is -3, whence $\varphi(x)$ is negative; in fact, $\varphi(x) < \mathbb{R}$.

The inductive definition of $\mathbb{T}$ involves constructing a certain exponential operation $\exp\colon \mathbb{T} \to \mathbb{T}^{\times}$, with $\exp(f)$ also written as e^f, and

$$\exp(f) \;=\; \exp(f_{\succ})\cdot\exp(f_{\asymp})\cdot\exp(f_{\prec}) \;=\; \exp(f_{\succ})\cdot \mathrm{e}^{f_{\asymp}}\cdot\sum_{n=0}^{\infty}\frac{f_{\prec}^n}{n!}$$

where the first factor $\exp(f_{\succ})$ is a transmonomial, the second factor $\mathrm{e}^{f_{\asymp}}$ is the real number obtained by exponentiating the real number $f_{\asymp}$ in the usual way, and the third factor $\exp(f_{\prec}) = \sum_{n=0}^{\infty} \frac{f_{\prec}^n}{n!}$ is expanded as a series in the usual way. Conversely, each transmonomial is of the form $\exp(f_{\succ})$ for some transseries f. Viewed as an exponential field, $\mathbb{T}$ is an *elementary* extension of the exponential field of real numbers; see [111]. In particular, $\mathbb{T}$ is real closed, and so its ordering is existentially definable (and universally definable) from its ring operations:

$$f \geqslant 0 \iff f = g^2 \text{ for some } g. \tag{2}$$

However, as emphasized above, our main interest is in $\mathbb{T}$ as a *differential field,* with derivation $f \mapsto f'$ on $\mathbb{T}$ defined termwise, with $r' = 0$ for $r \in \mathbb{R}$, $x' = 1$, $(\mathrm{e}^f)' = f'\,\mathrm{e}^f$, and $(\log f)' = f'/f$ for $f > 0$. Let us fix here some notation and terminology in force throughout this volume: a *differential field* is a field K of characteristic 0 together with a single derivation $\partial\colon K \to K$; if ∂ is clear from the context we often write a' instead of $\partial(a)$, for $a \in K$. The *constant field* of a differential field K is the subfield

$$C_K := \{a \in K : a' = 0\}$$

of K, also denoted by C if K is clear from the context. The constant field of $\mathbb{T}$ turns out to be $\mathbb{R}$, that is,

$$\mathbb{R} = \{f \in \mathbb{T} : f' = 0\}.$$

By an *ordered differential field* we mean a differential field equipped with a total ordering on its underlying set making it an *ordered field* in the usual sense of that expression. So $\mathbb{T}$ is an *ordered differential field.* More important than the ordering is the *valuation* on $\mathbb{T}$ with valuation ring

$$\mathcal{O}_{\mathbb{T}} := \big\{f \in \mathbb{T} : |f| \leqslant r \text{ for some } r \in \mathbb{R}\big\} = \{f \in \mathbb{T} : f_{\succ} = 0\},$$

a convex subring of $\mathbb{T}$. The unique maximal ideal of $\mathcal{O}_{\mathbb{T}}$ is

$$\mathcal{o}_{\mathbb{T}} := \big\{f \in \mathbb{T} : |f| \leqslant r \text{ for all } r > 0 \text{ in } \mathbb{R}\big\} = \{f \in \mathbb{T} : f = f_{\prec}\}$$

and thus $\mathcal{O}_{\mathbb{T}} = \mathbb{R} + \mathcal{o}_{\mathbb{T}}$. Its very definition shows that $\mathcal{O}_{\mathbb{T}}$ is *existentially* definable in the differential field $\mathbb{T}$. However, $\mathcal{O}_{\mathbb{T}}$ is *not* universally definable in the differential field $\mathbb{T}$: Corollary 16.2.6. In light of the model completeness conjecture discussed below, it is therefore advisable to add the valuation as an extra primitive, and so in the rest of this introduction *we construe* $\mathbb{T}$ *as an ordered and valued differential field, with valuation given by* $\mathcal{O}_{\mathbb{T}}$. By a *valued differential field* we mean throughout a differential field K equipped with a valuation ring of K that contains the prime subfield $\mathbb{Q}$ of K.

Grid-based transseries

When referring to transseries we have in mind the *well-based transseries of finite logarithmic and exponential depth* of [190], also called *logarithmic-exponential series* in [112]. The construction of the field $\mathbb{T}$ in Appendix A allows variants, and we briefly comment on one of them.

Each transseries f is an infinite sum $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where each $\mathfrak{m}$ is a transmonomial and $f_{\mathfrak{m}} \in \mathbb{R}$. The *support* of such a transseries f is the set $\operatorname{supp}(f)$ of transmonomials $\mathfrak{m}$ for which the coefficient $f_{\mathfrak{m}}$ is nonzero. For instance, the transmonomials in the support of the transseries φ of example (1) are

$$\mathrm{e}^{\mathrm{e}^x},\ \mathrm{e}^{\frac{\mathrm{e}^x}{\log x}+\frac{\mathrm{e}^x}{(\log x)^2}+\frac{\mathrm{e}^x}{(\log x)^3}+\cdots},\ x^{11},$$
$$1,\ \frac{1}{x},\ \frac{1}{x\log x},\ \dots,\ \frac{1}{x^2},\ \frac{1}{x^3},\dots,\ \mathrm{e}^{-x},\ \mathrm{e}^{-x^2},\ \dots.$$

By imposing various restrictions on the kinds of permissible supports, the construction from Appendix A yields various interesting differential subfields of $\mathbb{T}$.

To define multiplication on $\mathbb{T}$, supports should be *well-based*: every nonempty subset of the support of a transseries f should contain an asymptotically dominant element. So well-basedness is a minimal requirement on supports. A much stronger condition on $\operatorname{supp}(f)$ is as follows: there are transmonomials $\mathfrak{m}$ and $\mathfrak{n}_1,\dots,\mathfrak{n}_k \in \prec_{\mathbb{T}}$ $(k \in \mathbb{N})$ such that

$$\operatorname{supp} f \subseteq \{\mathfrak{m}\,\mathfrak{n}_1^{i_1}\cdots\mathfrak{n}_k^{i_k} : i_1,\dots,i_k \in \mathbb{N}\}.$$

Supports of this kind are called *grid-based*. Imposing this constraint all along, the construction from Appendix A builds the differential subfield $\mathbb{T}_{\mathrm{g}}$ of *grid-based* transseries of $\mathbb{T}$. Other suitable restrictions on the support yield other interesting differential subfields of $\mathbb{T}$.

The differential field $\mathbb{T}_{\mathrm{g}}$ of grid-based transseries has been studied in detail in [194]. In particular, that book contains a kind of algorithm for solving *algebraic differential equations* over $\mathbb{T}_{\mathrm{g}}$. These equations are of the form

$$P\big(y,\dots,y^{(r)}\big) = 0, \tag{3}$$

where $P \in \mathbb{T}_{\mathrm{g}}[Y,\dots,Y^{(r)}]$ is a nonzero polynomial in Y and a finite number of its formal derivatives $Y',\dots,Y^{(r)}$. We note here that by combining results from [194] and the present volume, any solution $y \in \mathbb{T}$ to (3) is actually grid-based. Thus transseries outside $\mathbb{T}_{\mathrm{g}}$ such as $\varphi(x)$ from (1) or $\zeta(x) = 1 + 2^{-x} + 3^{-x} + \cdots$ are differentially transcendental over $\mathbb{T}_{\mathrm{g}}$; see the *Notes and comments* to Section 16.2 for more details, and Grigor′ev-Singer [155] for an earlier result of this kind.

Model completeness

One reason that "geometric" fields like $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$ are more manageable than "arithmetic" fields like $\mathbb{Q}$ is that the former are *model complete*; see Appendix B for this and other basic model-theoretic notions used in this volume. A consequence of the model completeness of $\mathbb{R}$ is that any finite system of polynomial equations over $\mathbb{R}$ (in any number of unknowns) with a solution in an ordered field extension of $\mathbb{R}$, has a solution in $\mathbb{R}$ itself. By the $\mathbb{R}$-version of (2) we can also allow polynomial inequalities in such a system. (A related fact: if such a system has real algebraic coefficients, then it has a real algebraic solution.)

For a more geometric view of model completeness we first specify an algebraic subset of $\mathbb{R}^n$ to be the set of common zeros,

$$\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : P_1(y) = \cdots = P_k(y) = 0\},$$

of finitely many polynomials $P_1, \dots, P_k \in \mathbb{R}[Y_1, \dots, Y_n]$. Define a subset of $\mathbb{R}^m$ to be *subalgebraic* if it is the image of an algebraic set in $\mathbb{R}^n$ for some $n \geqslant m$ under the projection map

$$(y_1, \dots, y_n) \mapsto (y_1, \dots, y_m) : \mathbb{R}^n \to \mathbb{R}^m.$$

Then a consequence of the model completeness of $\mathbb{R}$ is that the complement in $\mathbb{R}^m$ of any subalgebraic set is again subalgebraic. Model completeness of $\mathbb{R}$ is a little stronger in that only polynomials with integer coefficients should be involved.

A nice analogy between $\mathbb{R}$ and $\mathbb{T}$ is the following intermediate value property, announced in [193] and established for $\mathbb{T}_{\mathrm{g}}$ in [194]: Let $P(Y) = p(Y, \dots, Y^{(r)})$ be a differential polynomial over $\mathbb{T}$, that is, with coefficients in $\mathbb{T}$, and let f, h be transseries with $f < h$; then $P(g)$ takes on all values strictly between $P(f)$ and $P(h)$ for transseries g with $f < g < h$. Underlying this opulence of $\mathbb{T}$ is a more robust property that we call *newtonianity,* which is analogous to henselianity for valued fields. The fact that $\mathbb{T}$ is newtonian implies, for instance, that any differential equation $y' = Q(y, y', \dots, y^{(r)})$ with $Q \in x^{-2}\mathcal{O}_{\mathbb{T}}[Y, Y', \dots, Y^{(r)}]$ has an infinitesimal solution $y \in \mathcal{o}_{\mathbb{T}}$. The definition of "newtonian" is rather subtle, and is discussed later in this introduction.

Another way that $\mathbb{R}$ and $\mathbb{T}$ are similar concerns the factorization of linear differential operators: any linear differential operator $A = \partial^r + a_1\partial^{r-1} \cdots + a_r$ of order $r \geqslant 1$ with coefficients $a_1, \dots, a_r \in \mathbb{T}$, is a product of such operators of order one and order two, with coefficients in $\mathbb{T}$. Moreover, any linear differential equation $y^{(r)} + a_1 y^{(r-1)} + \cdots + a_r y = b$ $(a_1, \dots, a_r, b \in \mathbb{T})$ has a solution $y \in \mathbb{T}$ (possibly $y = 0$). In particular, every transseries f has a transseries integral g, that is, $f = g'$. (It is noteworthy that a *convergent* transseries can very well have a *divergent* transseries as an integral; for example, the transmonomial $\frac{e^x}{x}$ has as an integral the divergent transseries $\sum_{n=0}^{\infty} n! \frac{e^x}{x^{n+1}}$. The analytic aspects of transseries are addressed by Écalle's theory of *analyzable functions* [120], where genuine functions are associated to transseries such as $\sum_{n=0}^{\infty} n! \frac{e^x}{x^{n+1}}$, using the process of accelero-summation, a far reaching generalization of Borel summation; these analytic issues are not addressed in the present volume.)

These strong closure properties make it plausible to conjecture that $\mathbb{T}$ is model complete, as a valued differential field. This and some other conjectures to be mentioned in this introduction go back some 20 years, and are proved in the present volume. To state model completeness of $\mathbb{T}$ geometrically we use the terms d-*algebraic* and d-*polynomial* to abbreviate *differential-algebraic* and *differential polynomial* and we define a d-*algebraic set* in $\mathbb{T}^n$ to be the set of common zeros,

$$\{f = (f_1, \dots, f_n) \in \mathbb{T}^n : P_1(f) = \cdots = P_k(f) = 0\}$$

of some d-polynomials $P_1, \dots, P_k$ in differential indeterminates $Y_1, \dots, Y_n$,

$$P_i(Y_1, \dots, Y_n) = p_i\big(Y_1, \dots, Y_n, Y_1', \dots, Y_n', Y_1'', \dots, Y_n'', Y_1''', \dots, Y_n''', \dots\big)$$

over $\mathbb{T}$. We also define an *H-algebraic set* to be the intersection of a d-algebraic set with a set of the form

$$\{y = (y_1, \dots, y_n) \in \mathbb{T}^n : y_i \in \mathcal{O}_{\mathbb{T}} \text{ for all } i \in I\} \quad \text{where } I \subseteq \{1, \dots, n\},$$

and we finally define a subset of $\mathbb{T}^m$ to be *sub-H-algebraic* if it is the image of an H-algebraic set in $\mathbb{T}^n$ for some $n \geqslant m$ under the projection map

$$(f_1, \dots, f_n) \mapsto (f_1, \dots, f_m) \colon \mathbb{T}^n \to \mathbb{T}^m.$$

It follows from the model completeness of $\mathbb{T}$ that the complement in $\mathbb{T}^m$ of any sub-H-algebraic set is again sub-H-algebraic, in analogy with Gabrielov's "theorem of the complement" for real subanalytic sets [145]. (The model completeness of $\mathbb{T}$ is a little stronger: it is equivalent to this "complement" formulation where the defining d-polynomials of the d-algebraic sets involved have integer coefficients.) A consequence is that for subsets of $\mathbb{T}^m$,

$$\text{sub-}H\text{-algebraic} \;=\; \text{definable in } \mathbb{T}.$$

The usual model-theoretic approach to establishing that a given structure is model complete consists of two steps. (There is also a preliminary choice to be made of *primitives*; our choice for $\mathbb{T}$: its ring operations, its derivation, its ordering, and its valuation.) The first step is to record the basic compatibilities between primitives; "basic" here means in practice that they are also satisfied by the *substructures* of the structure of interest. For the more familiar structure of the ordered field $\mathbb{R}$ of real numbers, these basic compatibilities are the ordered field axioms. The second and harder step is to find some closure properties satisfied by our structure that together with these basic compatibilities can be shown to imply *all* its elementary properties. In the model-theoretic treatment of $\mathbb{R}$, it turns out that this job is done by the closure properties defining *real closed fields*: every positive element has a square root, and every odd degree polynomial has a zero.

H-fields

For $\mathbb{T}$ we try to capture the first step of the axiomatization by the notion of an *H-field.* We chose the prefix H in honor of E. Borel, H. Hahn, G. H. Hardy, and F. Hausdorff, who pioneered our subject about a century ago [55, 162, 164, 171], and who share the initial H, except for Borel. To define H-fields, let K be an ordered differential field (with constant field C) and set

$$\begin{aligned} \mathcal{O} &:= \{a \in K : |a| \leqslant c \text{ for some } c > 0 \text{ in } C\} \qquad \text{(a convex subring of } K\text{)}, \\ \mathcal{o} &:= \{a \in K : |a| < c \text{ for all } c > 0 \text{ in } C\}. \end{aligned}$$

These notations should remind the reader of Landau's big O and small o. The elements of $\mathcal{o}$ are thought of as *infinitesimal,* the elements of $\mathcal{O}$ as *bounded,* and those of $K \setminus \mathcal{O}$ as *infinite.* Note that $\mathcal{O}$ is definable in the ordered differential field K, and is a valuation ring of K with (unique) maximal ideal $\mathcal{o}$. We define K to be an *H-field* if it satisfies the two conditions below:

(H1) for all $a \in K$, if $a > C$, then $a' > 0$,

(H2) $\mathcal{O} = C + \smallo$.

By (H2) the constant field C can be identified canonically with the residue field $\mathcal{O}/\smallo$ of $\mathcal{O}$. As we did with $\mathbb{T}$ we construe an H-field K as an *ordered valued differential field.* An H-field K is said to have *small derivation* if $\partial\smallo \subseteq \smallo$ (and thus $\partial\mathcal{O} \subseteq \smallo$). If K is an H-field and $a \in K$, $a > 0$, then K with its derivation ∂ replaced by $a\partial$ is also an H-field. Such changes of derivation play a major role in our work.

Among H-fields with small derivation are $\mathbb{T}$ and its ordered differential subfields containing $\mathbb{R}$, and any Hardy field containing $\mathbb{R}$. Thus $\mathbb{R}(x)$, $\mathbb{R}(x, \mathrm{e}^x, \log x)$ as well as Hardy's larger field of logarithmico-exponential functions are H-fields.

Closure properties

Let $\mathrm{Th}(\boldsymbol{M})$ be the first-order theory of an $\mathcal{L}$-structure $\boldsymbol{M}$, that is, $\mathrm{Th}(\boldsymbol{M})$ is the set of $\mathcal{L}$-sentences that are true in $\boldsymbol{M}$; see Appendix B for details. In terms of H-fields, we can now make the model completeness conjecture more precise, as was done in [19]:

$$\mathrm{Th}(\mathbb{T}) \;=\; \text{model companion of the theory of } H\text{-fields with small derivation},$$

where $\mathbb{T}$ is construed as an ordered and valued differential field. This amounts to adding to the earlier model completeness of $\mathbb{T}$ the claim that any H-field with small derivation can be embedded as an ordered valued differential field into some ultrapower of $\mathbb{T}$. Among the consequences of this conjecture is that any finite system of algebraic differential equations over $\mathbb{T}$ (in several unknowns) has a solution in $\mathbb{T}$ whenever it has one in some H-field extension of $\mathbb{T}$. It means that the concept of "H-field" is intrinsic to the differential field $\mathbb{T}$. It also suggests studying systematically the extension theory of H-fields: A. Robinson taught us that for a theory to have a model companion at all—a rare phenomenon—is equivalent to certain embedding and extension properties of its class of models. Here it helps to know that H-fields fall under the so-called *differential-valued fields* (abbreviated as d-*valued fields* below) of Rosenlicht, who began a study of these valued differential fields and their extensions in the early 1980s; see [364]. (A d-*valued* field is defined to be a valued differential field such that $\mathcal{O} = C + \smallo$, and $a'b \in b'\smallo$ for all $a, b \in \smallo$; here $\mathcal{O}$ is the valuation ring with maximal ideal $\smallo$, and C is the constant field.) Most of our work is actually in the setting of valued differential fields where no field ordering is given, since even for H-fields the valuation is a more robust and useful feature than its field ordering.

Besides developing the extension theory of H-fields we need to isolate the relevant *closure properties* of $\mathbb{T}$. First, $\mathbb{T}$ is real closed, but that property does not involve the derivation. Next, $\mathbb{T}$ is closed under integration and, by its very construction, also under exponentiation. In terms of the derivation this gives two natural closure properties of $\mathbb{T}$:

$$\forall a \exists b\,(a = b'), \qquad \forall a \exists b\,(b \neq 0\;\&\; ab = b').$$

An H-field K is said to be *Liouville closed* if it is real closed and satisfies these two sentences; cf. Liouville [260, 261]. So $\mathbb{T}$ is Liouville closed. It was shown in [19] that any H-field has a *Liouville closure,* that is, a minimal Liouville closed H-field extension. If K is a Hardy field containing $\mathbb{R}$ as a subfield, then it has a unique Hardy field extension that is also a Liouville closure of K, but it can happen that an H-field K has two Liouville closures that are not isomorphic over K; it cannot have more than two. Understanding this "fork in the road" and dealing with it is fundamental in our work. Useful notions in this connection are *comparability classes*, *groundedness*, and *asymptotic integration.* We discuss this briefly below for H-fields. (Parts of Chapters 9 and 11 treat these notions for a much larger class of valued differential fields.) Later in this introduction we encounter an important but rather hidden closure property, called ω*-freeness*, which rules over the fork in the road. Finally, there is the very powerful closure property of newtonianity that we already mentioned earlier.

Valuations and asymptotic relations

Let K be an H-field, let a, b range over K, and let $v\colon K \to \Gamma_\infty$ be the (Krull) valuation on K associated to $\mathcal{O}$, with value group $\Gamma = v(K^\times)$ and $\Gamma_\infty := \Gamma \cup \{\infty\}$ with $\Gamma < \infty$. Recall that Γ is an ordered abelian group, additively written as is customary in valuation theory. Then

$$va < vb \quad \Longleftrightarrow \quad |a| > c|b| \text{ for all } c > 0 \text{ in } C.$$

Thinking of elements of K as germs of functions at $+\infty$, we also adopt Hardy's notations from asymptotic analysis:

$$a \succ b, \qquad a \succcurlyeq b, \qquad a \prec b, \qquad a \preccurlyeq b, \qquad a \asymp b, \qquad a \sim b$$

are defined to mean, respectively,

$$va < vb, \quad va \leqslant vb, \quad va > vb, \quad va \geqslant vb, \quad va = vb, \quad v(a-b) > va.$$

(Some of these notations from [165] actually go back to du Bois-Reymond [48].) Note that $a \succ 1$ means that a is infinite, that is, $|a| > C$, and $a \prec 1$ means that a is infinitesimal, that is, $a \in \mathcal{o}$. It is crucial that the asymptotic relations above can be differentiated, provided we restrict to nonzero a, b with $a \not\asymp 1$, $b \not\asymp 1$:

$$a \succ b \Longleftrightarrow a' \succ b', \qquad a \asymp b \Longleftrightarrow a' \asymp b', \qquad a \sim b \Longleftrightarrow a' \sim b'.$$

For $a \neq 0$ we let $a^\dagger := a'/a$ be its logarithmic derivative, so $(ab)^\dagger = a^\dagger + b^\dagger$ for $a, b \neq 0$. Elements $a, b \succ 1$ are said to be *comparable* if $a^\dagger \asymp b^\dagger$; if K is a Hardy field containing $\mathbb{R}$ as subfield, or $K = \mathbb{T}$, this is equivalent to the existence of an $n \geqslant 1$ such that $|a| \leqslant |b|^n$ and $|b| \leqslant |a|^n$. Comparability is an equivalence relation on the set of infinite elements of K, and the comparability classes $\mathrm{Cl}(a)$ of K are totally ordered by $\mathrm{Cl}(a) \leqslant \mathrm{Cl}(b) :\Longleftrightarrow a^\dagger \preccurlyeq b^\dagger$.

EXAMPLE. For $K = \mathbb{T}$, set $e_0 = x$ and $e_{n+1} = \exp(e_n)$. Then the sequence $(\mathrm{Cl}(e_n))$ is strictly increasing and cofinal in the set of comparability classes. More important are the ℓ_n defined recursively by $\ell_0 = x$, and $\ell_{n+1} = \log \ell_n$. Then the sequence $\mathrm{Cl}(\ell_0) >$

$\mathrm{Cl}(\ell_1) > \mathrm{Cl}(\ell_2) > \cdots > \mathrm{Cl}(\ell_n) > \cdots$ is coinitial in the set of comparability classes of $\mathbb{T}$. For later use it is worth noting at this point that

$$\ell_n^\dagger = \frac{1}{\ell_0 \cdots \ell_n}, \qquad -\ell_n^{\dagger\dagger} = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}.$$

We call K *grounded* if K has a smallest comparability class. Thus $\mathbb{T}$ is not grounded. If $\Gamma^>$ contains an element α such that for every $\gamma \in \Gamma^>$ we have $n\gamma \geqslant \alpha$ for some $n \geqslant 1$, then K is grounded; this condition on Γ is in particular satisfied if $\Gamma \neq \{0\}$ and Γ has finite archimedean rank. If K is grounded, then K has only one Liouville closure (up to isomorphism over K).

The H-field K is said to have *asymptotic integration* if K satisfies $\forall a \exists b (a \asymp b')$, equivalently, $\{vb' : b \in K\} = \Gamma_\infty$. It is obvious that every Liouville closed H-field has asymptotic integration; in particular, $\mathbb{T}$ has asymptotic integration. In general, at most one $\gamma \in \Gamma$ lies outside $\{vb' : b \in K\}$; if K is grounded, then such a γ exists, by results in Section 9.2, and so K cannot have asymptotic integration.

STRATEGY AND MAIN RESULTS

Model completeness of $\mathbb{T}$ concerns finite systems of algebraic differential equations over $\mathbb{T}$ with asymptotic side conditions in several differential indeterminates.

Robinson's strategy for establishing model completeness applied to $\mathbb{T}$ requires us to move beyond $\mathbb{T}$ to consider H-fields and their extensions. If we are lucky—as we are in this case—it will suffice to consider extensions of H-fields by one element y at a time. This leads to equations $P(y) = 0$ with an asymptotic side condition $y \prec g$. Here $P \in K\{Y\}$ is a univariate differential polynomial with coefficients in an H-field K with $g \in K^\times$, and $K\{Y\} = K[Y, Y', Y'', \dots]$ is the differential domain of d-polynomials in the differential indeterminate Y over K. The key issue: when is there a solution in some H-field extension of K? A detailed study of such equations in the special case $K = \mathbb{T}_{\mathrm{g}}$ and where we only look for solutions in $\mathbb{T}_{\mathrm{g}}$ itself was undertaken in [194], using an assortment of techniques (for instance, various fixpoint theorems) heavily based on the particular structure of $\mathbb{T}_{\mathrm{g}}$. Generalizing these results to suitable H-fields is an important guideline in our work.

Differential Newton diagrams

Let K be an H-field, and consider a d-algebraic equation with asymptotic side condition,

$$P(y) = 0, \qquad y \prec g, \tag{4}$$

where $P \in K\{Y\}$, $P \neq 0$, and $g \in K^\times$; we look for nonzero solutions in H-field extensions of K. For the sake of concreteness we take $K = \mathbb{T}_{\mathrm{g}}$ and look for nonzero solutions in $\mathbb{T}_{\mathrm{g}}$, focusing on the example below:

$$\mathrm{e}^{-\mathrm{e}^x} y^2 y'' + y^2 - 2xyy' - 7\,\mathrm{e}^{-x} y' - 4 + \tfrac{1}{\log x} = 0, \qquad y \prec x. \tag{5}$$

We sketch briefly how [194] goes about solving (5). First of all, we need to find the possible *dominant terms* of solutions y. This is done by considering possible cancellations. For example, y^2 and -4 might be the terms of least valuation in the left side of (5), with all other terms having greater valuation, so negligible compared to y^2 and -4. This yields a cancellation $y^2 \sim 4$, so $y \sim 2$ or $y \sim -2$, giving 2 and -2 as potential dominant terms of a solution y.

Another case: $\mathrm{e}^{-\mathrm{e}^x} y^2 y''$ and y^2 are the terms of least valuation. Then we get a cancellation $\mathrm{e}^{-\mathrm{e}^x} y^2 y'' \sim -y^2$, that is, $y'' \sim -\mathrm{e}^{\mathrm{e}^x}$, which leads to $y \sim -\mathrm{e}^{\mathrm{e}^x}/\mathrm{e}^{2x}$. But this possibility is discarded, since (5) also requires $y \prec x$. (On the other hand, if the asymptotic condition in (5) had been $y \prec \mathrm{e}^{\mathrm{e}^x}$, we would have kept $-\mathrm{e}^{\mathrm{e}^x}/\mathrm{e}^{2x}$ as a potential dominant term of a solution y.)

What makes things work in these two cases is that the cancellations arise from terms of *different* degrees in $y, y', y'', \dots$. Such cancellations are reminiscent of the more familiar setting of algebraic equations where the dominant monomials of solutions can be read off from a *Newton diagram* and the corresponding dominant coefficients are zeros of the corresponding *Newton polynomials*; see Section 3.7. This method still works in our d-algebraic setting, for cancellations among terms of different degrees, but requires the construction of so-called *equalizers*.

A different situation arises for cancellations between terms of the *same* degree. Consider for example the case that y^2 and $-2xyy'$ have least valuation among the terms in the left side of (5), with all other terms of higher valuation. Then $y^2 \sim 2xyy'$, so $y^\dagger \sim \frac{1}{2x}$. Now $y^\dagger = \frac{1}{2x}$ gives $y = cx^{1/2}$ with $c \in \mathbb{R}^\times$, but the weaker condition $y^\dagger \sim \frac{1}{2x}$ only gives $y = ux^{1/2}$ with $u \neq 0$, $u^\dagger \prec x^{-1}$, that is, $|v(u)| < |v(x)|/n$ for all $n \geqslant 1$. Substituting $ux^{1/2}$ for y in (5) and considering u as the new unknown, the condition on $v(u)$ forces $u \asymp 1$, so after all we do get $y \sim cx^{1/2}$ with $c \in \mathbb{R}^\times$, giving $cx^{1/2}$ as a potential dominant term of a solution y. It is important to note that here an *integration constant* c gets introduced.

Manipulations as we just did are similar to rewriting an equation $H(y) = 0$ with *homogeneous* nonzero $H \in K\{Y\}$ of positive degree as a (Riccati) equation $R(y^\dagger) = 0$ with R of *lower* order than H.

This technique can be shown to work in general for cancellations among terms of the same degree, provided we are also allowed to transform the equation to an equivalent one by applying a suitable iteration of the *upward shift* $f(x) \mapsto f(\mathrm{e}^x)$. (For reasonable H-fields K one can apply instead compositional conjugation by positive active elements; see below for *compositional conjugation* and *active*.)

Having determined a possible dominant term $f = c\mathfrak{m}$ of a solution of (4), where $c \in \mathbb{R}^\times$ and $\mathfrak{m}$ is a transmonomial, we next perform a so-called *refinement*

$$(6) \qquad P(f+y) = 0, \qquad y \prec f$$

of (4). For instance, taking $f = 2$, the equation (5) transforms into

$$\begin{aligned} &\mathrm{e}^{-\mathrm{e}^x} y^2 y'' + y^2 - 2xyy' + 4\,\mathrm{e}^{-\mathrm{e}^x} yy'' \\ &+ 4y - (4x + 7\,\mathrm{e}^{-x})y' + 4\,\mathrm{e}^{-\mathrm{e}^x} y'' + \tfrac{1}{\log x} = 0, \qquad y \prec 2. \end{aligned}$$

Now apply the same procedure to this refinement, to find the "next" term.

Roughly speaking, this yields an infinite process to obtain all possible asymptotic expansions of solutions to any asymptotic equation. How do we make this into a finite process? For this, it is useful to introduce the *Newton degree* of (4). This notion is similar to the Weierstrass degree of a multivariate power series and corresponds to the degree of the asymptotically significant part of the equation. If the Newton degree is 0, then (4) has no solution. The Newton degree of (5) turns out to be 2: this has to do with the fact that $\mathrm{e}^{-\mathrm{e}^x} y^2 y'' \prec y^2$ whenever $y \prec x$. We shall return soon to the precise definition of Newton degree for differential polynomials over rather general H-fields. As to the resolution of asymptotic equations over $K = \mathbb{T}_{\mathrm{g}}$, the following key facts were established in [194]:

- The Newton degree stays the same or decreases under refinement.
- If the Newton degree of the refinement (6) equals that of (4), we employ so-called *unravelings*; these resemble the *Tschirnhaus transformations* that overcome similar obstacles in the algebraic setting. Combining unravelings with refinements as described above, we arrive after finitely many steps at an asymptotic equation of Newton degree 0 or 1.
- The H-field $\mathbb{T}_{\mathrm{g}}$ is newtonian, that is, any asymptotic equation over $\mathbb{T}_{\mathrm{g}}$ of Newton degree 1 has a solution in $\mathbb{T}_{\mathrm{g}}$.

All in all, we have for any given asymptotic equation over $\mathbb{T}_{\mathrm{g}}$ a more or less finite procedure for gaining an overview of the entire space of solutions in $\mathbb{T}_{\mathrm{g}}$.

To define the Newton degree of an asymptotic equation (4) over rather general H-fields, we first need to introduce the *dominant part* of P and then, based on a process called *compositional conjugation,* the *Newton polynomial* of P.

The dominant part

Let K be an H-field. We extend the valuation v of K to the integral domain $K\{Y\}$ by setting

$$vP \;=\; \min\{va:\ a \text{ is a coefficient of } P\},$$

and we extend the binary relations $\asymp$ and $\sim$ on K to $K\{Y\}$ accordingly. It is also convenient to fix a monomial set $\mathfrak{M}$ in K, that is, a subset $\mathfrak{M}$ of $K^{>}$ that is mapped bijectively by v onto the value group Γ of K. This allows us to define the *dominant part* $D_P(Y)$ of a nonzero d-polynomial $P(Y)$ over K to be the unique element of $C\{Y\} \subseteq K\{Y\}$ with $P \sim \mathfrak{d}_P D_P$, where $\mathfrak{d}_P \in \mathfrak{M}$ is the *dominant monomial* of P determined by $P \asymp \mathfrak{d}_P$. (Another choice of monomial set would just multiply D_P by some positive constant.) For $K = \mathbb{T}$ we always take the set of transmonomials as our monomial set.

EXAMPLE 1. Let $K = \mathbb{T}$. For $P = x^5 + (2 + \mathrm{e}^x)Y + (3\,\mathrm{e}^x + \log x)(Y')^2$, we have $\mathfrak{d}_P = \mathrm{e}^x$ and $D_P = Y + 3(Y')^2$. For $Q = Y^2 - 2xYY'$ we have $D_Q = -2YY'$.

For K with small derivation we can use D_P to get near the zeros $a \asymp 1$ of P: if $P(a) = 0$, $a \asymp 1$, then $D_P(c) = 0$ where c is the unique constant with $a \sim c$. We need to understand, however, the behavior of $P(a)$ not only for $a \asymp 1$, that is, $va = 0$, but also for "sufficiently flat" elements $a \in K$, that is, for va approaching $0 \in \Gamma$. For instance, in $\mathbb{T}$, the iterated logarithms

$$\ell_0 = x, \quad \ell_1 = \log x, \quad \ell_2 = \log\log x, \ \dots$$

satisfy $v(\ell_n) \to 0$ in $\Gamma_{\mathbb{T}}$ and likewise $v(1/\ell_n) \to 0$. The *dominant term* $\mathfrak{d}_P D_P$ of P often provides a good approximation for P when evaluating at sufficiently flat elements, but not always: for $K = \mathbb{T}$ and Q as in Example 1 we note that for $y = \ell_2$ we have: $y^2 = \ell_2^2 \succ 2xyy' = 2\ell_2/\ell_1$, so $Q(y) \sim y^2 \not\asymp (\mathfrak{d}_Q D_Q)(y)$.

In order to approximate $P(y)$ by $(\mathfrak{d}_P D_P)(y)$ for sufficiently flat y, we need one more ingredient: *compositional conjugation.* For $K = \mathbb{T}$ and Q as in Example 1, this amounts to a change of variables $x = \mathrm{e}^{\mathrm{e}^{\widetilde{x}}}$, so that $Q(y) = y^2 - 2y(dy/d\widetilde{x})\,\mathrm{e}^{-\widetilde{x}}$ for $y \in \mathbb{T}$. With respect to this new variable $\widetilde{x}$, the dominant term Y^2 of the adjusted d-polynomial $Y^2 - 2YY'\,\mathrm{e}^{-\widetilde{x}}$ is then an adequate approximation of Q when evaluating at sufficiently flat elements of $\mathbb{T}$. Such changes of variable do not make sense for general H-fields, but as it turns out, compositional conjugation is a good substitute.

Compositional conjugation

We define this for an arbitrary differential field K. For $\phi \in K^\times$ we let K^ϕ be the differential field obtained from K by replacing its derivation ∂ by the multiple $\phi^{-1}\partial$. Then a differential polynomial $P(Y) \in K\{Y\}$ defines the same function on the common underlying set of K and K^ϕ as a certain differential polynomial $P^\phi(Y) \in K^\phi\{Y\}$: for $P = Y'$, we have $P^\phi(Y) = \phi Y'$ (since over K^ϕ we evaluate Y' according to the derivation $\phi^{-1}\partial$), for $P = Y''$ we have $P^\phi(Y) = \phi' Y' + \phi^2 Y''$ (with $\phi' = \partial\phi$), and so on. This yields a ring isomorphism

$$P \mapsto P^\phi \colon K\{Y\} \to K^\phi\{Y\}$$

that is the identity on the common subring $K[Y]$. It is also an automorphism of the common underlying K-algebra of $K\{Y\}$ and $K^\phi\{Y\}$, and studied as such in Chapter 12. We call K^ϕ the *compositional conjugate of K by ϕ,* and P^ϕ the *compositional conjugate of P by ϕ.* Note that K and K^ϕ have the same constant field C. If K is an H-field and $\phi \in K^>$, then so is K^ϕ. It pays to note how things change under compositional conjugation, and what remains invariant.

The Newton polynomial

Suppose now that K is an H-field with asymptotic integration. For $\phi \in K^>$ we say that ϕ is *active* (in K) if $\phi \succcurlyeq a^\dagger$ for some nonzero $a \not\asymp 1$ in K; equivalently, the derivation $\phi^{-1}\partial$ of K^ϕ is small. Let $\phi \in K^>$ range over the active elements of K in what follows, fix a monomial set $\mathfrak{M} \subseteq K^>$ of K, and let $P \in K\{Y\}$, $P \neq 0$. The dominant part D_{P^ϕ} of P^ϕ lies in $C\{Y\}$, and we show in Section 13.1 that it eventually stabilizes as ϕ varies: there is a differential polynomial $N_P \in C\{Y\}$ and an active $\phi_0 \in K^>$ such that for all $\phi \preccurlyeq \phi_0$,

$$D_{P^\phi} = c_\phi N_P, \qquad c_\phi \in C^>.$$

We call N_P the *Newton polynomial* of P. It is of course only determined up to a factor from $C^>$, but this ambiguity is harmless. The (total) degree of N_P is called the *Newton degree* of P.

EXAMPLE 2. Let $K = \mathbb{T}$. Then $f \in K^>$ is active iff $f \succcurlyeq \ell_n^\dagger = \frac{1}{\ell_0\ell_1\cdots\ell_n}$ for some n. If P is as in Example 1, then for each ϕ,

$$P^\phi = x^5 + (2+\mathrm{e}^x)Y + \phi^2(3\,\mathrm{e}^x + \log x)(Y')^2,$$

so $D_{P^\phi} = Y$ if $\phi \prec 1$. This yields $N_P = Y$, so P has Newton degree 1. It is an easy exercise to show that for $Q = Y^2 - 2xYY'$ we have $N_Q = Y^2$.

A crucial result in [194] (Theorem 8.6) says that if $K = \mathbb{T}_{\mathrm{g}}$, then $N_P \in \mathbb{R}[Y](Y')^{\mathbb{N}}$. A major step in our work was to isolate a robust class of H-fields K with asymptotic integration for which likewise $N_P \in C[Y](Y')^{\mathbb{N}}$ for all nonzero $P \in K\{Y\}$. This required several completely new tools to be discussed below.

The special cuts γ, λ and ω

Recall that ℓ_n denotes the nth iterated logarithm of x in $\mathbb{T}$, so $\ell_0 = x$ and $\ell_{n+1} = \log \ell_n$. We introduce the elements

$$\begin{aligned}
\gamma_n &= \ell_n^\dagger &&= \frac{1}{\ell_0\cdots\ell_n}\\
\lambda_n &= -\gamma_n^\dagger &&= \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}\\
\omega_n &= -2\lambda_n' - \lambda_n^2 &&= \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2}
\end{aligned}$$

of $\mathbb{T}$. As $n \to \infty$ these elements approach their formal limits

$$\begin{aligned}
\gamma_{\mathbb{T}} &= \frac{1}{\ell_0\ell_1\ell_2\cdots}\\
\lambda_{\mathbb{T}} &= \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \frac{1}{\ell_0\ell_1\ell_2} + \cdots\\
\omega_{\mathbb{T}} &= \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \frac{1}{\ell_0^2\ell_1^2\ell_2^2} + \cdots,
\end{aligned}$$

which for now are just suggestive expressions. Indeed, our field $\mathbb{T}$ of transseries *of finite logarithmic and exponential depth* does not contain any pseudolimit of the pseudocauchy sequence (λ_n), nor of the pseudocauchy sequence (ω_n). There are, however, immediate H-field extensions of $\mathbb{T}$ where such pseudolimits exist, and if we let $\lambda_{\mathbb{T}}$ be such a pseudolimit of (λ_n), then in some further H-field extension we have an element suggestively denoted by $\exp(\int -\lambda_{\mathbb{T}})$ that can play the role of $\gamma_{\mathbb{T}}$.

Even though $\gamma_{\mathbb{T}}$, $\lambda_{\mathbb{T}}$ and $\omega_{\mathbb{T}}$ are not in $\mathbb{T}$, we can take them as elements of some H-field extension of $\mathbb{T}$, as indicated above, and so we obtain sets

$$\begin{aligned}\Gamma(\mathbb{T}) &= \{a \in \mathbb{T} : a > \gamma_{\mathbb{T}}\} \\ \Lambda(\mathbb{T}) &= \{a \in \mathbb{T} : a < \lambda_{\mathbb{T}}\} \\ \Omega(\mathbb{T}) &= \{a \in \mathbb{T} : a < \omega_{\mathbb{T}}\}\end{aligned}$$

that can be shown to be definable in $\mathbb{T}$. For instance,

$$\begin{aligned}\Gamma(\mathbb{T}) &= \{a \in \mathbb{T} : \forall b \in \mathbb{T}\, (b \succ 1 \Rightarrow a \neq b^\dagger)\} \\ &= \{-a' : a \in \mathbb{T},\, a \geqslant 0\}.\end{aligned}$$

In other words, $\gamma_{\mathbb{T}}$, $\lambda_{\mathbb{T}}$ and $\omega_{\mathbb{T}}$ realize *definable cuts* in $\mathbb{T}$.

For any ungrounded H-field $K \neq C$ we can build a sequence (ℓ_ρ) of elements $\ell_\rho \succ 1$, indexed by the ordinals ρ less than some infinite limit ordinal, such that

$$\sigma > \rho \Rightarrow \ell_\sigma^\dagger \prec \ell_\rho^\dagger, \qquad v(\ell_\rho) \to 0 \text{ in } \Gamma.$$

These ℓ_ρ play in K the role that the iterated logarithms ℓ_n play in $\mathbb{T}$. In analogy with $\mathbb{T}$ they yield the elements

$$\gamma_\rho := \ell_\rho^\dagger, \qquad \lambda_\rho := -\gamma_\rho^\dagger, \qquad \omega_\rho := -2\lambda_n' - \lambda_n^2,$$

of K, and (λ_ρ) and (ω_ρ) are pseudocauchy sequences. As with $\mathbb{T}$ this gives rise to definable sets $\Gamma(K)$, $\Lambda(K)$ and $\Omega(K)$ in K. The fact mentioned earlier that $\mathbb{T}$ does not contain $\gamma_{\mathbb{T}}$, $\lambda_{\mathbb{T}}$ or $\omega_{\mathbb{T}}$ turns out to be very significant: in general, we have

$$\gamma_K \in K \quad \Rightarrow \quad \lambda_K \in K \quad \Rightarrow \quad \omega_K \in K$$

and each of the four mutually exclusive cases

$$\gamma_K \in K, \qquad \gamma_K \notin K \,\&\, \lambda_K \in K, \qquad \lambda_K \notin K \,\&\, \omega_K \in K, \qquad \omega_K \notin K$$

can occur; see Section 13.9. Here we temporarily abuse notations, since we should explain what we mean by $\gamma_K \in K$ and the like; see the next subsections.

On gaps and forks in the road

Let K be an H-field. We say that an element $\gamma \in K$ is a *gap* in K if for all $a \in K$ with $a \succ 1$ we have

$$a^\dagger \succ \gamma \succ (1/a)'.$$

The existence of such a gap is the formal counterpart to the informal statement that $\gamma_K \in K$. If K has a gap γ, then γ has no primitive in K, so K is not closed under integration. If K has trivial derivation (that is, $K = C$), then K has a gap $\gamma = 1$.

There are also K with $K \neq C$ (even Hardy fields) that have a gap. Not having a gap is equivalent to being grounded or having asymptotic integration.

We already mentioned the result from [19] that K may have two Liouville closures that are not isomorphic over K (but fortunately not more than two). Indeed, if K has a gap γ, then in one Liouville closure all primitives of γ are infinitely large, whereas in the other γ has an infinitesimal primitive. Even if K has no gap, the above fork in the road can arise more indirectly: Assume that K has asymptotic integration and $\lambda \in K$ is such that for all $a \in K^\times$ with $a \prec 1$,

$$a'^\dagger < -\lambda < a^{\dagger\dagger}.$$

Then K has no element $\gamma \neq 0$ with $\lambda = -\gamma^\dagger$, but K has an H-field extension $K\langle\gamma\rangle$ generated by an element γ with $\lambda = -\gamma^\dagger$, and any such γ is a gap in $K\langle\gamma\rangle$. It follows again that K has two Liouville closures that are not K-isomorphic.

For real closed K with asymptotic integration, the existence of such an element λ corresponds to the informal statement that $\gamma_K \notin K$ & $\lambda_K \in K$. We define K to be *λ-free* if K has asymptotic integration and satisfies the sentence

$$\forall a \exists b \big[b \succ 1 \ \&\ a - b^{\dagger\dagger} \succcurlyeq b^\dagger \big].$$

It can be shown that for real closed K with asymptotic integration, λ-freeness is equivalent to the nonexistence of an element λ as above. More generally, K is λ-free iff K has asymptotic integration and (λ_ρ) has no pseudolimit in K.

The property of ω-freeness

Even λ-freeness might not prevent a fork in the road for some d-algebraic extension. Let K be an H-field, and define

$$\omega = \omega_K : K \to K, \qquad \omega(z) := -2z' - z^2.$$

Assume that K is λ-free and $\omega \in K$ is such that for all $b \succ 1$ in K,

$$\omega - \omega(b^{\dagger\dagger}) \prec (b^\dagger)^2.$$

Then the first-order differential equation $\omega(z) = \omega$ admits no solution in K, but K has an H-field extension $K\langle\lambda\rangle$ generated by a solution $z = \lambda$ to $\omega(z) = \omega$ such that $K\langle\lambda\rangle$ is no longer λ-free (and with a fork in its road towards Liouville closure).

For λ-free K the existence of an element ω as above corresponds to the informal statement that $\lambda_K \notin K$ & $\omega_K \in K$. We say that K is *ω-free* if no such ω exists, more precisely, K has asymptotic integration and satisfies the sentence

$$\forall a \exists b \big[b \succ 1 \ \&\ a - \omega(b^{\dagger\dagger}) \succcurlyeq (b^\dagger)^2 \big].$$

(It is easy to show that if K is ω-free, then it is λ-free.) For K with asymptotic integration, ω-freeness is equivalent to the pseudocauchy sequence (ω_ρ) not having a pseudolimit in K. Thus $\mathbb{T}$ is ω-free. More generally, if K has asymptotic integration and is a union of grounded H-subfields, then K is ω-free by Corollary 11.7.15.

Much deeper and very useful is that if K is an ω-free H-field and L is a d-algebraic H-field extension of K, then L is also ω-free and has no comparability class smaller than all those of K; this is part of Theorem 13.6.1. Thus the property of ω-freeness is very robust: if K is ω-free, then forks in the road towards Liouville closure no longer occur, even for d-algebraic H-field extensions of K (Corollary 13.6.2). There are, however, Liouville closed H-fields that are not ω-free; see [22].

Another important consequence of ω-freeness is that Newton polynomials of differential polynomials then take the same simple shape as those over $\mathbb{T}_{\mathrm{g}}$:

THEOREM 1. *If K is ω-free and $P \in K\{Y\}$, $P \neq 0$, then $N_P \in C[Y](Y')^{\mathbb{N}}$.*

The proof in Chapter 13 depends heavily on Chapter 12, where we determine the invariants of certain automorphism groups of polynomial algebras in infinitely many variables $Y_0, Y_1, Y_2, \dots$ over a field of characteristic zero.

The function ω and the notion of ω-freeness are closely related to second order linear differential equations over K. More precisely (Riccati), for $y \in K^\times$, $4y'' + fy = 0$ is equivalent to $\omega(z) = f$ with $z := 2y^\dagger$; so the second-order linear differential equation $4y'' + fy = 0$ reduces in a way to a first-order (but non-linear) differential equation $\omega(z) = f$. (The factor 4 is just for convenience, to get simpler expressions below.)

EXAMPLE. The differential equation $y'' = -y$ has no solution $y \in \mathbb{T}^\times$, whereas the Airy equation $y'' = xy$ has two $\mathbb{R}$-linearly independent solutions in $\mathbb{T}$ [308, Chapter 11, (1.07)]. Indeed, in Sections 11.7 and 11.8 we show that for $f \in \mathbb{T}$, the differential equation $4y'' + fy = 0$ has a solution $y \in \mathbb{T}^\times$ if and only if $f < \omega_{\mathbb{T}}$, that is, $f < \omega_n = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2}$ for some n. This fact reflects classical results [167, 184] on the question: for which logarithmico-exponential functions f (in Hardy's sense) does the equation $4y'' + fy = 0$ have a non-oscillating real-valued solution (more precisely, a nonzero solution in a Hardy field)?

Newtonianity

This is the most consequential elementary property of $\mathbb{T}$. An ω-free H-field K is said to be *newtonian* if every d-polynomial $P(Y)$ over K of Newton degree 1 has a zero in $\mathcal{O}$. This turns out to be the correct analogue for valued differential fields like $\mathbb{T}$ of the property of being henselian for a valued field. We chose the adjective *newtonian* since it is this property that allows us to develop in Chapter 13 a Newton diagram method for differential polynomials. It is good to keep in mind that the role of newtonianity in the results of Chapters 14, 15, and 16 is more or less analogous to that of henselianity in the theory of valued fields and as the key condition in the Ax-Kochen-Eršov results.

We already mentioned the result from [194] that $\mathbb{T}_{\mathrm{g}}$ is newtonian. That $\mathbb{T}$ is newtonian is a consequence of the following analogue in Chapter 15 of the familiar valuation-theoretic fact that spherically complete valued fields are henselian:

THEOREM 2. *If K is an H-field, $\partial K = K$, and K is a directed union of spherically complete grounded H-subfields, then K is* (ω*-free and*) *newtonian.*

EXAMPLE. Let $K = \mathbb{T}$ and consider for $\alpha \in \mathbb{R}$ the differential polynomial

$$P(Y) = Y'' - 2Y^3 - xY - \alpha \in \mathbb{T}\{Y\}.$$

For $\phi \in \mathbb{T}^\times$ we have $(Y'')^\phi = \phi^2 Y'' + \phi' Y'$ for $\phi \in \mathbb{T}^\times$, so

$$P^\phi \;=\; \phi^2 Y'' + \phi' Y' - 2Y^3 - xY - \alpha.$$

Now $\phi^2, \phi' \prec 1 \prec x$ for active $\phi \prec 1$ in $\mathbb{T}^>$. Hence $N_P \in \mathbb{R}^\times Y$, so P has Newton degree 1. Thus the Painlevé II equation $y'' = 2y^3 + xy + \alpha$ has a solution $y \in \mathcal{O}_{\mathbb{T}}$. (It is known that P has a zero $y \preccurlyeq 1$ in the differential subfield $\mathbb{R}(x)$ of $\mathbb{T}$ iff $\alpha \in \mathbb{Z}$; see for example [156, Theorem 20.2].)

The main results of Chapter 14 amount for H-fields to the following:

THEOREM 3. *If K is a newtonian ω-free H-field with divisible value group, then K has no proper immediate* d*-algebraic H-field extension.*

COROLLARY 1. *Let K be a real closed newtonian ω-free H-field, and let $K^{\mathrm{a}} = K[\mathrm{i}]$ (where $\mathrm{i}^2 = -1$) be its algebraic closure. Then:*

(i) *each* d*-polynomial in $K^{\mathrm{a}}\{Y\}$ of positive degree has a zero in K^{a};*

(ii) *each linear differential operator in $K^{\mathrm{a}}[\partial]$ of positive order is a composition of such operators of order* 1*;*

(iii) *each* d*-polynomial in $K\{Y\}$ of odd degree has a zero in K; and*

(iv) *each linear differential operator in $K[\partial]$ of positive order is a composition of such operators of order* 1 *and order* 2*.*

THEOREM 4. *If K is an ω-free H-field with divisible value group, then K has an immediate* d*-algebraic newtonian H-field extension, and any such extension embeds over K into every ω-free newtonian H-field extension of K.*

An extension of K as in Theorem 4 is minimal over K and thus unique up to isomorphism over K. We call such an extension a *newtonization* of K.

THEOREM 5. *If K is an ω-free H-field, then K has a* d*-algebraic newtonian Liouville closed H-field extension that embeds over K into every ω-free newtonian Liouville closed H-field extension of K.*

An extension of K as in Theorem 5 is minimal over K and thus unique up to isomorphism over K. We call such an extension a *Newton-Liouville closure* of K.

The main theorems

We now come to the results in Chapter 16, which in our view justify this volume. First, the various elementary conditions we have discussed axiomatize a model complete theory. To be precise, construe H-fields in the natural way as $\mathcal{L}$-structures where $\mathcal{L} := \{0, 1, +, -, \cdot, \partial, \leqslant, \preccurlyeq\}$, and let T^{nl} be the $\mathcal{L}$-theory whose models are the newtonian ω-free Liouville closed H-fields.

THEOREM 6. *T^{nl} is model complete.*

The theory T^{nl} is not complete and has exactly two completions, namely $T^{\text{nl}}_{\text{small}}$ (small derivation) and $T^{\text{nl}}_{\text{large}}$ (large derivation). Thus newtonian ω-free Liouville closed H-fields with small derivation have the same elementary properties as $\mathbb{T}$.

Every H-field with small derivation can be embedded into a model of $T^{\text{nl}}_{\text{small}}$; thus Theorem 6 yields the strong version of the model completeness conjecture from [19] stated earlier in this introduction. As $T^{\text{nl}}_{\text{small}}$ is complete and effectively axiomatized, it is decidable. In particular, there is an algorithm which, for any given d-polynomials $P_1, \dots, P_m$ in indeterminates $Y_1, \dots, Y_n$ with coefficients from $\mathbb{Z}[x]$, decides whether there is a tuple $y \in \mathbb{T}^n$ such that $P_1(y) = \cdots = P_m(y) = 0$. Such an algorithm with $\mathbb{T}$ replaced by its differential subring $\mathbb{R}[[x^{-1}]]$ is due to Denef and Lipshitz [101], but no such algorithm can exist with $\mathbb{T}$ replaced by $\mathbb{R}((x^{-1}))$ or by any of various other natural H-subfields of $\mathbb{T}$ [20, 155].

Theorem 6 is the main step towards an elimination of quantifiers, in a slightly extended language: Let $\mathcal{L}^{\iota}_{\Lambda,\Omega}$ be $\mathcal{L}$ augmented by the unary function symbol ι and the unary predicates Λ, Ω, and extend T^{nl} to the $\mathcal{L}^{\iota}_{\Lambda,\Omega}$-theory $T^{\text{nl},\iota}_{\Lambda,\Omega}$ by adding as defining axioms for these new symbols the universal closures of

$$\begin{aligned} &\big[a \neq 0 \longrightarrow a \cdot \iota(a) = 1\big] \ \&\ \big[a = 0 \longrightarrow \iota(a) = 0\big], \\ &\Lambda(a) \ \longleftrightarrow\ \exists y\big[y \succ 1 \ \&\ a = -y^{\dagger\dagger}\big], \\ &\Omega(a) \ \longleftrightarrow\ \exists y\big[y \neq 0 \ \&\ 4y'' + ay = 0\big]. \end{aligned}$$

For a model K of T^{nl} this makes the sets $\Lambda(K)$ and $\Omega(K)$ downward closed with respect to the ordering of K. For example, for $f \in \mathbb{T}$,

$$\begin{aligned} f \in \Lambda(\mathbb{T}) \iff f < \lambda_n &= \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} \quad \text{for some } n, \\ f \in \Omega(\mathbb{T}) \iff f < \omega_n &= \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2} \quad \text{for some } n, \end{aligned}$$

that is, $\Lambda(\mathbb{T})$ and $\Omega(\mathbb{T})$ are the cuts in $\mathbb{T}$ determined by $\lambda_{\mathbb{T}}$, $\omega_{\mathbb{T}}$ introduced earlier. We can now state what we view as the main result of this volume:

THEOREM 7. *The theory $T^{\text{nl},\iota}_{\Lambda,\Omega}$ admits elimination of quantifiers.*

We cannot omit here either Λ or Ω. In Chapter 16 we do include for convenience one more unary predicate I in $\mathcal{L}^{\iota}_{\Lambda,\Omega}$: for a model K of T^{nl} and $a \in K$,

$$\mathrm{I}(a) \ \longleftrightarrow\ \exists y\big[a \preccurlyeq y' \ \&\ y \preccurlyeq 1\big] \ \longleftrightarrow\ a = 0 \vee \big[a \neq 0 \ \&\ \neg\Lambda(-a^{\dagger})\big],$$

where the first equivalence is the defining axiom for I, and the second shows that I is superfluous in Theorem 7. We note here that this predicate I governs the solvability of first-order linear differential equations *with asymptotic side condition.* More precisely, for K as above and $f \in K$, $g, h \in K^{\times}$, the following are equivalent:

(a) there exists $y \in K$ with $y' = fy + g$ and $y \prec h$;

(b) $\big[(f - h^{\dagger}) \in \mathrm{I}(K)$ and $(g/h) \in \mathrm{I}(K)\big]$ or $\big[(f - h^{\dagger}) \notin \mathrm{I}(K)$ and $(g/h) \prec f - h^{\dagger}\big]$.

This equivalence is part of Corollary 11.8.12 and exemplifies Theorem 7 (but is not derived from that theorem, nor used in its proof).

In the proof of Theorem 7, and throughout the construction of suitable H-field extensions, the predicates I, Λ and Ω act as switchmen. Whenever a fork in the road occurs due to the presence of a gap γ, then $\mathrm{I}(\gamma)$ tells us to take the branch where $\int \gamma \preccurlyeq 1$, while $\neg\mathrm{I}(\gamma)$ forces $\int \gamma \succ 1$. Likewise, the predicates Λ and Ω control what happens when adjoining elements γ and λ with $\gamma^\dagger = -\lambda$ and $\omega(\lambda) = \omega$.

From the above defining axioms for Λ and Ω it is clear that these predicates are (uniformly) existentially definable in models of T^{nl}. By model completeness of T^{nl} they are also uniformly *universally* definable in these models; Section 16.5 deals with such algebraic-linguistic issues.

Next we list some more intrinsic consequences of our elimination theory.

COROLLARY 2. *Let K be a newtonian ω-free Liouville closed H-field, and suppose the set $X \subseteq K^n$ is definable. Then X has empty interior in K^n (with respect to the order topology on K and the product topology on K^n) if and only if for some nonzero $P \in K\{Y_1, \dots, Y_n\}$ we have $X \subseteq \{y \in K^n : P(y) = 0\}$.*

In (i) below the intervals are in the sense of the ordered field K.

COROLLARY 3. *Let K be a newtonian ω-free Liouville closed H-field. Then:*

(i) *K is o-minimal at infinity: if $X \subseteq K$ is definable in K, then for some $a \in K$, either $(a, +\infty) \subseteq X$, or $(a, +\infty) \cap X = \emptyset$;*

(ii) *if $X \subseteq K^n$ is definable in K, then $X \cap C^n$ is semialgebraic in the sense of the real closed constant field C of K;*

(iii) *K has* NIP. (*See Appendix B for this very robust property.*)

It is hard to imagine obtaining these results for $K = \mathbb{T}$ without Theorem 7. Item (i) relates to classical bounds on solutions of algebraic differential equations over Hardy fields; see [20, Section 3]. To illustrate item (ii) of Corollary 3, we note that the set of real parameters $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ for which the system

$$\lambda_0 y + \lambda_1 y' + \cdots + \lambda_n y^{(n)} = 0, \qquad 0 \neq y \prec 1$$

has a solution in $\mathbb{T}$ is a semialgebraic subset of $\mathbb{R}^{n+1}$; in fact, it agrees with the set of all $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ such that the polynomial $\lambda_0 + \lambda_1 Y + \cdots + \lambda_n Y^n \in \mathbb{R}[Y]$ has a negative zero in $\mathbb{R}$; see Corollary 11.8.26. To illustrate item (iii), let $Y = (Y_1, \dots, Y_n)$ be a tuple of distinct differential indeterminates; for an m-tuple $\sigma = (\sigma_1, \dots, \sigma_m)$ of elements of $\{\prec, \asymp, \succ\}$ we say that $P_1, \dots, P_m \in \mathbb{T}\{Y\}$ *realize* σ if there exists $a \in \mathbb{T}^n$ such that $P_i(a)\, \sigma_i\, 1$ holds for $i = 1, \dots, m$. Then a special case of (iii) says that for fixed $d, n, r \in \mathbb{N}$, the number of tuples $\sigma \in \{\prec, \asymp, \succ\}^m$ realized by some $P_1, \dots, P_m \in \mathbb{T}\{Y\}$ of degree at most d and order at most r grows only polynomially with m, even though the total number of tuples is 3^m. These manifestations of (ii) and (iii), though instructive, are perhaps a bit misleading, since they can be obtained without appealing to (ii) and (iii).

In the course of proving Theorem 6 we also get:

THEOREM 8. *If K is a newtonian ω-free Liouville closed H-field, then K has no proper* d*-algebraic H-field extension with the same constant field.*

For $K = \mathbb{T}_{\mathrm{g}}$ this yields: every $f \in \mathbb{T} \setminus \mathbb{T}_{\mathrm{g}}$ is d-transcendental over $\mathbb{T}_{\mathrm{g}}$.

We can also enlarge $\mathbb{T}$. For example, the series $\sum_{n=0}^{\infty} e_n^{-1}$, with e_n the nth iterated exponential of x, does not lie in $\mathbb{T}$ but does lie in a certain completion $\mathbb{T}^{\mathrm{c}}$ of $\mathbb{T}$. This completion $\mathbb{T}^{\mathrm{c}}$ is naturally an ordered valued differential field extension of $\mathbb{T}$, and by Corollary 14.1.6 we have $\mathbb{T} \preccurlyeq \mathbb{T}^{\mathrm{c}}$.

ORGANIZATION

Here we discuss the somewhat elaborate organization of this volume into chapters, some technical ingredients not mentioned so far, and some material that goes beyond the setting of H-fields. Indeed, the supporting algebraic theory deserves to be developed in a broad way, and there are more notions to keep track of than one might expect.

Background chapters

To make our work more accessible and self-contained, we provide in the first five chapters background on commutative algebra, valued abelian groups, valued fields, differential fields, and linear differential operators. This material has many sources, and we thought it would be convenient to have it available all in one place. In addition we have an appendix with the construction of $\mathbb{T}$, and an appendix exposing the (small) part of model theory that we need.

The basic facts on Hahn products, pseudocauchy sequences and spherical completeness in these early chapters are used throughout the volume. Some readers might prefer to skip in a first reading cauchy sequences, completeness (for valued abelian groups and valued fields) and step-completeness, which are not needed for the main results in this volume (but see Corollary 14.1.6). Some parts, like Sections 2.3 and 5.4, fit naturally where we put them, but are mainly intended for use in the next volume. On the other hand, Section 5.7 on compositional conjugation is elementary and frequently referred to in subsequent chapters, but this material seems virtually absent from the literature.

Valued differential fields

We also profited from examining arbitrary valued differential fields K with small derivation, that is, $\partial \smallo \subseteq \smallo$ for the maximal ideal $\smallo$ of the valuation ring $\mathcal{O}$ of K. This yields the continuity of the derivation ∂ with respect to the valuation topology and gives $\partial\mathcal{O} \subseteq \mathcal{O}$, and so induces a derivation on the residue field. To our surprise, we could establish in Chapters 6 and 7 some useful facts in this very general setting when the induced derivation on the residue field is nontrivial, for example the Equalizer Theorem 6.0.1. We need this result in deriving an "eventual" version of it for ω-free H-fields in Chapter 13, which in turn is crucial in obtaining our main results, via its role in constructing an appropriate Newton diagram for d-polynomials.

Asymptotic couples

A useful gadget is the *asymptotic couple* of an H-field K. This is the value group Γ of K equipped with the map $\gamma \mapsto \gamma^\dagger \colon \Gamma^{\neq} \to \Gamma$ defined by: if $\gamma = vf$, $f \in K^\times$, then $\gamma^\dagger = v(f^\dagger)$. This map is a valuation on Γ, and we extend it to a map $\Gamma \to \Gamma_\infty$ by setting $0^\dagger := \infty$. Two key facts are that $\alpha^\dagger < \beta + \beta^\dagger$ for all $\alpha, \beta > 0$ in Γ, and $\alpha^\dagger \geqslant \beta^\dagger$ whenever $0 < \alpha \leqslant \beta$ in Γ. The condition on an H-field of having small derivation can be expressed in terms of its asymptotic couple; the same holds for having a gap, for being grounded, and for having asymptotic integration, but not for being ω-free.

Asymptotic couples were introduced by Rosenlicht [364] for d-valued fields. In Chapter 6 we assign to *any* valued differential field with small derivation an asymptotic couple, with good effect. Asymptotic couples play also an important role in Chapters 9, 10, 11, 13, and 16.

Differential-henselian fields

Valued differential fields with small derivation include the so-called *monotone* differential fields defined by the condition $a' \preccurlyeq a$. In analogy with the notion of a *henselian* valued field, Scanlon [382] introduced *differential-henselian* monotone differential fields. Using the Equalizer Theorem we extend this notion and basic facts about it to arbitrary valued differential fields with small derivation in Chapter 7. (We abbreviate *differential-henselian* to d-*henselian.*) This material plays a role in Chapter 14, using the following relation between d-*henselian* and *newtonian*: an ω-free H-field K is newtonian iff for every active $\phi \in K^>$ the compositional conjugate K^ϕ is d-henselian, with the valuation v on K^ϕ replaced by the coarser valuation $\pi \circ v$ where $\pi \colon v(K^\times) = \Gamma \to \Gamma/\Delta$ is the canonical map to the quotient of Γ by its convex subgroup

$$\Delta := \{\gamma \in \Gamma : \gamma^\dagger > v\phi\}.$$

We pay particular attention to two special cases: $v(C^\times) = \{0\}$ (few constants), and $v(C^\times) = \Gamma$ (many constants). The first case is relevant for newtonianity, the second case is considered in a short Chapter 8, where we present Scanlon's extension of the Ax-Kochen-Eršov theorems to d-henselian valued fields with many constants, and add some things on definability.

While d-henselianity is defined in terms of solving differential equations in one unknown, it implies the solvability of suitably non-singular systems of n differential equations in n unknowns: this is shown at the end of Chapter 7, and has a nice consequence for newtonianity: Proposition 14.5.7.

Asymptotic differential fields

To keep things simple we confined most of the exposition above to H-fields, but this setting is a bit too narrow for various technical reasons. For example, a differential subfield of an H-field with the induced ordering is not always an H-field, and passing to an algebraic closure like $\mathbb{T}[i]$ destroys the ordering, though $\mathbb{T}[i]$ is still a d-valued field. On occasion we also wish to change the valuation of an H-field or d-valued field by coarsening. For all these reasons we introduce in Chapter 9 the class of *asymptotic differential fields*, which is larger and more flexible than Rosenlicht's class of d-valued fields. Many basic facts about H-fields and d-valued fields do have good analogues for asymptotic differential fields. This is shown in Chapter 9, which also contains a lot of basic material on asymptotic couples. Chapter 10 deals more specifically with H-fields.

Immediate extensions

Indispensable for attaining our main results is the fact that every H-field with divisible value group and with asymptotic integration has a spherically complete immediate H-field extension. This is part of Theorem 11.4.1, and proving it about five years ago removed a bottleneck. It provides the only way known to us of extending every H-field to an ω-free H-field. Possibly more important than Theorem 11.4.1 itself are the tools involved in its proof. In view of the theorem's content, it is ironic that models of T^{nl} are never spherically complete, in contrast to all prior positive results on elementary theories of valued fields with or without extra structure, cf. [28, 29, 41, 131, 382].

The differential Newton diagram method

Chapters 13 and 14 present the differential Newton diagram method in the general context of asymptotic fields that satisfy suitable technical conditions, such as ω-freeness. Before tackling these chapters, the reader may profit from first studying our exposition of the Newton diagram method for ordinary one-variable polynomials over henselian valued fields of equicharacteristic zero in Section 3.7. Some of the issues encountered there (for example, the *unraveling* technique) appear again, albeit in more intricate form, in the differential context of these chapters. In the proofs of a few crucial facts about the special cuts λ and ω in Chapter 13 we use some results from the preceding Chapter 12 on triangular automorphisms. Chapter 12 is a bit special in being essentially independent of the earlier chapters.

Proving newtonianity

Chapter 15 contains the proof of Theorem 2, and thus establishes that $\mathbb{T}$ is a model of our theory $T^{\mathrm{nl}}_{\mathrm{small}}$. This theorem is also useful in other contexts: In [43], Berarducci and Mantova construct a derivation on Conway's field **No** of surreal numbers [92, 150] turning it into a Liouville closed H-field with constant field $\mathbb{R}$. From Theorem 2 and the completeness of T^{nl} it follows that **No** with this derivation and $\mathbb{T}$ are elementarily equivalent, as we show in [24].

Quantifier elimination

In Chapter 16 we first prove Theorem 6 on model completeness, next we consider H-fields equipped with a $\Lambda\Omega$-structure, and then deduce Theorem 7 about quantifier elimination with various interesting consequences, such as Corollaries 2 and 3. The introduction to this chapter gives an overview of the proof and the role of various embedding and extension results in it.

THE NEXT VOLUME

The present volume focuses on achieving quantifier elimination (Theorem 7), and so we left out various things we did since 1995 that were not needed for that. In a second volume we intend to cover these things as required for developing our work further. Let us briefly survey some highlights of what is to come.

Linear differential equations

We plan to consider linear differential equations in much greater detail, comprising the corresponding differential Galois theory, in connection with constructing the linear surjective closure of a differential field, factoring linear differential operators over suitable algebraically closed d-valued fields, and explicitly constructing the Picard-Vessiot extension of such an operator. Concerning the latter, the complexification $\mathbb{T}[i]$ of $\mathbb{T}$ is no longer closed under exponential integration, since oscillatory "transmonomials" such as e^{ix} are not in $\mathbb{T}[i]$. Adjoining these oscillatory transmonomials to $\mathbb{T}[i]$ yields a d-valued field that contains a Picard-Vessiot extension of $\mathbb{T}$ for each operator in $\mathbb{T}[\partial]$.

Hardy fields

We also wish to pay more attention to Hardy fields, and this will bring up analytic issues. For example, every Hardy field containing $\mathbb{R}$ can be shown to extend to an ω-free Hardy field. Using methods from [195], we also hope to prove that it always extends to a newtonian ω-free Hardy field. Indeed, that paper proves among other things the following pertinent result (formulated here with our present terminology): Let $\mathbb{T}_{\mathrm{g}}^{\mathrm{da}}$ consist of the grid-based transseries that are d-algebraic over $\mathbb{R}$. Then $\mathbb{T}_{\mathrm{g}}^{\mathrm{da}}$ is a newtonian ω-free Liouville closed H-subfield of $\mathbb{T}_{\mathrm{g}}$ and is isomorphic over $\mathbb{R}$ to a Hardy field containing $\mathbb{R}$.

Embedding into fields of transseries

Another natural question we expect to deal with is whether every H-field can be given some kind of transserial structure. This can be made more precise in terms of the axiomatic definition of a *field of transseries* in terms of a transmonomial group $\mathfrak{M}$ in Schmeling's thesis [388]. For instance, one axiom there is that for all $\mathfrak{m} \in \mathfrak{M}$ we have $\operatorname{supp} \log \mathfrak{m} \subseteq \mathfrak{M}^{\succ}$. We hope that any H-field can be embedded into such a field of transseries. This would be a natural counterpart of Kaplansky's theorem [209] embedding certain valued fields into Hahn fields, and would make it possible to think of H-field elements as generalized transseries.

More on the model theory of $\mathbb{T}$

In the second volume we hope to deal with further issues around $\mathbb{T}$ of a model-theoretic nature: for example, identifying the induced structure on its value group (conjectured to be given by its H-couple, as specified in [18]); and determining the definable closure of a subset of a model of T^{nl}, in order to get a handle on what functions are definable in $\mathbb{T}$.

A by-product of the present volume is a full description of several important 1-types over a given model of T^{nl}, but the entire space of such 1-types remains to be surveyed. Theorem 8 suggests that the model-theoretic notions of *non-orthogonality to* C or C*-internality* may be significant for models of T^{nl}; see also [25].

FUTURE CHALLENGES

We now discuss a few more open-ended avenues of inquiry.

Differentiation and exponentiation

The restriction to $\mathcal{O}_{\mathbb{T}}$ of the exponential function on $\mathbb{T}$ is easily seen to be definable in $\mathbb{T}$, but by part (ii) of Corollary 3, the restriction to $\mathbb{R}$ of this exponential function is not definable in $\mathbb{T}$. This raises the question whether our results can be extended to the differential field $\mathbb{T}$ *with exponentiation*, or with some other extra o-minimal structure on it.

Logarithmic transseries

A transseries is *logarithmic* if all transmonomials in it are of the form $\ell_0^{r_0}\cdots\ell_n^{r_n}$ with $r_0,\dots,r_n\in\mathbb{R}$. (See Appendix A.) The logarithmic transseries make up an ω-free newtonian H-subfield $\mathbb{T}_{\log}$ of $\mathbb{T}$ that is not Liouville closed. We conjecture that $\mathbb{T}_{\log}$ as a valued differential field is model complete. The asymptotic couple of $\mathbb{T}_{\log}$ has been successfully analyzed by Gehret [146], and turns out to be model-theoretically tame, in particular, has NIP [147]. (There is also the notion of a transseries being *exponential.* The exponential transseries form a real closed H-subfield $\mathbb{T}_{\exp}$ of $\mathbb{T}$ in which the set $\mathbb{Z}$ is existentially definable, see [20]. It follows that the differential field $\mathbb{T}_{\exp}$ does not have a reasonable model theory: it is as complicated as so-called *second-order arithmetic.*)

Accelero-summable transseries

The paper [195] on transserial Hardy fields yields on the one hand a method to associate a genuine function to a suitable formal transseries, and in the other direction also provides means to associate concrete asymptotic expansions to elements of Hardy fields. We expect that more can be done in this direction.

Écalle's theory of analyzable functions has a more canonical procedure that associates a function to an *accelero-summable* transseries. These transseries make up an H-subfield $\mathbb{T}_{\mathrm{as}}$ of $\mathbb{T}$. This procedure has the advantage that it does not only preserve the ordered field structure, but also composition, functional inversion, and several other operations. In its full generality, however, Écalle's theory requires sophisticated analytic tools, and is beyond the scope of this volume. It is clear that $\mathbb{T}_{\text{as}}$ is analytically more important than $\mathbb{T}$, but the latter might help in understanding the former. The H-subfield $\mathbb{T}_{\text{as}}$ of $\mathbb{T}$ contains $\mathbb{R}$, is ω-free and Liouville closed. Is it newtonian? In view of Theorem 8, a positive answer would confirm Écalle's belief [120, p. 148] that any solution in $\mathbb{T}$ of an algebraic differential equation $P(Y) = 0$ over $\mathbb{T}_{\text{as}}$ with $P \neq 0$ lies in $\mathbb{T}_{\text{as}}$.

Beyond H-fields

The derivation of a differentially closed field K cannot be continuous with respect to a nontrivial valuation on K; see Section 10.7. This sets a limit for the study of valued differential fields with a reasonable interaction between valuation and derivation. However, one may close off the d-valued field $\mathbb{T}[i]$ under exponential integration, by adding oscillatory transmonomials recursively. This results in valued differential fields of *complex transseries* over which a version of the Newton diagram method for $\mathbb{T}_{\text{g}}$ goes through; see [192]. It would be interesting to find out more about the model theory of these rich valued differential fields.

A HISTORICAL NOTE ON TRANSSERIES

The differential field of transseries was first defined and extensively used in Écalle's solution of Dulac's problem, which is about plane analytic vector fields. Its solution shows in particular that a plane polynomial vector field admits only a finite number of limit cycles. (A *limit cycle* of a planar vector field is a periodic trajectory with an annular neighborhood not containing any other periodic trajectory.) It was long believed that in 1923 Dulac [115] had given a proof of this finiteness statement, until Il′yashenko [197] found a gap in 1981: Dulac was operating with asymptotic expansions of germs of functions as if they faithfully represented these germs. To justify this in Dulac's case is not easy: it was done independently by Écalle [120] and Il′yashenko [198], and required fundamental new ideas (and hundreds of pages). We briefly sketch here the role of transseries in Écalle's approach.

Suppose towards a contradiction that some polynomial vector field on $\mathbb{R}^2$ has infinitely many limit cycles. Classical facts about planar vector fields such as the Poincaré-Bendixson Theorem allow us to reduce to the case where infinitely many of these limit cycles accumulate at a so-called *polycycle* of the vector field; see [199, Theorem 24.22] for details. Such a polycycle σ consists of finitely many trajectories

$$S_1 \to S_2,\ \dots\ , S_{r-1} \to S_r,\ S_r \to S_1 \qquad \text{(the edges)}$$

between singularities $S_1, \dots, S_r$ (the vertices) of the vector field; see Figure 0.1 where $r = 3$. Draw lines $\ell_1, \dots, \ell_r$ that cross these edges $S_r \to S_1$, $S_1 \to S_2$, ..., $S_{r-1} \to S_r$ transversally at points $O_1, \dots, O_r$. For any trajectory φ of the vector field that is sufficiently close to σ we consider the successive points where φ meets

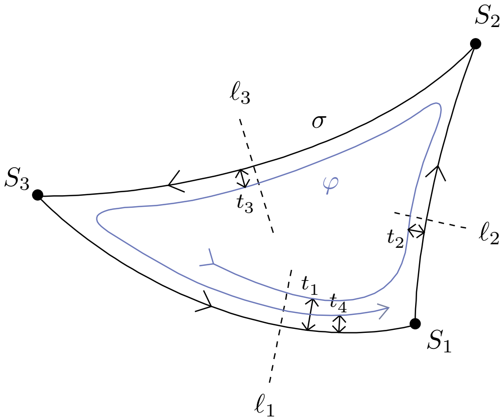

Figure 0.1: A polycycle σ and a close trajectory φ.

$\ell_1, \ell_2, \ldots, \ell_r, \ell_1$ and denote their distances to $O_1, \ldots, O_r, O_1$ by $t_1, t_2, \ldots, t_r, t_{r+1}$. The behavior of the vector field near S_i yields for some $\varepsilon_i > 0$ a real analytic function $g_i \colon (0, \varepsilon_i) \to (0, \infty)$ such that $t_{i+1} = g_i(t_i)$. We have $g_i(t) \to 0$ as $t \to 0^+$ but g_i does not necessarily extend analytically to 0.

The composition $f := g_r \circ \cdots \circ g_1$ is defined on some interval $(0, \varepsilon)$ and is called the *Poincaré return map* of the polycycle σ (relative to our choice of $\ell_1, \ldots, \ell_r$). We have $t_{r+1} = f(t_1)$, for trajectories close enough to σ. Thus $f(t) = t$ corresponds to a periodic trajectory, and so it suffices to show that either f is the identity, or $f(t) \neq t$ for all sufficiently small $t > 0$. One can even ask whether this non-oscillation property holds for Poincaré return maps of polycycles of plane *analytic* (not necessarily polynomial) vector fields; this is *Dulac's problem.*

It is convenient to work at infinity by setting $x = t^{-1}$ and replace these functions $f, g_1, \ldots, g_r$ of t by functions $F, G_1, \ldots, G_r$ of x with $F = G_r \circ \cdots \circ G_1$. Dulac [115] provides formal series expansions $\widetilde{G_i}$ of the G_i, which are rather simple transseries, usually divergent, and which by formal composition yields an often complicated transseries expansion $\widetilde{F} = \widetilde{G_r} \circ \cdots \circ \widetilde{G_1}$ of F.

Écalle is able to reconstitute the germs G_i and F from their formal counterparts $\widetilde{G_i}$ and $\widetilde{F}$ by developing a delicate analytic machinery of *accelero-summation*. More precisely, he constructs an (accelero-summation) operator $\widetilde{G}(x) \mapsto G$ whose domain of definition is a certain differential subfield $\mathbb{T}_{\mathrm{as}}$ of $\mathbb{T}$ and whose values are germs of real analytic functions at $+\infty$; it assigns to each $\widetilde{G_i}$ the germ G_i. Moreover, $\mathbb{T}_{\mathrm{as}}$ is closed under composition, and accelero-summation preserves real constants, addition, multiplication, differentiation, composition, and the (total) field ordering: if $\widetilde{G}(x) \in \mathbb{T}_{\mathrm{as}}$, $\widetilde{G}(x) > 0$, then $G(x) > 0$ for all sufficiently large real x; here the x in $\widetilde{G}(x)$ is an indeterminate, while in $G(x)$ it ranges over real numbers. Applying this operator to

$\widetilde{F}(x) - x$ yields the desired result: either $F(x) = x$ for all large enough x, or $F(x) < x$ for all large enough x, or $F(x) > x$ for all large enough x.

Accelero-summation is very powerful—the solution of the Dulac problem is just one application—and much of Écalle's book [120] consists of developing it in various directions. Unorthodox summations occur already in Euler's study of divergent series ([134, p. 220], see also [454, §5.3]), but even the remarkable generalization of this work by Emil Borel more than a century later [55] is not adequate to reconstitute the above F from $\widetilde{F}$.

Écalle [120, §1.9] also indicates another approach to Dulac's problem in which accelero-summation would play a smaller role. It involves the group of formal Laurent series $x(1 + a_1x^{-1} + a_2x^{-2} + \cdots) \in \mathbb{T}$ (with respect to composition) and the group $\{\dots, \log\log x, \log x, x, \mathrm{e}^x, \mathrm{e}^{\mathrm{e}^x}, \dots\}$ generated under composition by $\mathrm{e}^x \in \mathbb{T}$. An intriguing open question is whether there exist nontrivial relations between these two groups. In other words, is the group they generate under composition their free product?

Dulac's problem is often mentioned in connection with Hilbert's 16th Problem, whose second part asks for a uniform bound (only depending on the degrees of the polynomials involved) on the number of limit cycles of a polynomial vector field in $\mathbb{R}^2$. This remains open, and is part of Smale's list [425] of mathematical problems "for the next century."

The exponential field of transseries (without the derivation) was also introduced independently by Dahn and Göring [96]. Motivated by Tarski's problem on the real exponential field, they saw $\mathbb{T}$ as a candidate for a non-standard model of the theory of this structure. This idea was vindicated in [111], in the wake of Wilkie's solution [465] of the "geometric" part of Tarski's problem.