Introduction

> ... In effect, if one extends these functions by allowing complex values for the arguments, then there arises a harmony and regularity which without it would remain hidden.
>
> *B. Riemann,* 1851

When we begin the study of complex analysis we enter a marvelous world, full of wonderful insights. We are tempted to use the adjectives magical, or even miraculous when describing the first theorems we learn; and in pursuing the subject, we continue to be astonished by the elegance and sweep of the results.

The starting point of our study is the idea of extending a function initially given for real values of the argument to one that is defined when the argument is complex. Thus, here the central objects are functions from the complex plane to itself

$$f : \mathbb{C} \to \mathbb{C},$$

or more generally, complex-valued functions defined on open subsets of $\mathbb{C}$. At first, one might object that nothing new is gained from this extension, since any complex number z can be written as $z = x + iy$ where $x, y \in \mathbb{R}$ and z is identified with the point (x, y) in $\mathbb{R}^2$.

However, everything changes drastically if we make a natural, but misleadingly simple-looking assumption on f: that it is differentiable in the complex sense. This condition is called **holomorphicity**, and it shapes most of the theory discussed in this book.

A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \qquad (h \in \mathbb{C})$$

exists. This is similar to the definition of differentiability in the case of a real argument, except that we allow h to take *complex* values. The reason this assumption is so far-reaching is that, in fact, it encompasses a multiplicity of conditions: so to speak, one for each angle that h can approach zero.

Although one might now be tempted to prove theorems about holomorphic functions in terms of real variables, the reader will soon discover that complex analysis is a new subject, one which supplies proofs to the theorems that are proper to its own nature. In fact, the proofs of the main properties of holomorphic functions which we discuss in the next chapters are generally very short and quite illuminating.

The study of complex analysis proceeds along two paths that often intersect. In following the first way, we seek to understand the universal characteristics of holomorphic functions, without special regard for specific examples. The second approach is the analysis of some particular functions that have proved to be of great interest in other areas of mathematics. Of course, we cannot go too far along either path without having traveled some way along the other. We shall start our study with some general characteristic properties of holomorphic functions, which are subsumed by three rather miraculous facts:

1. Contour integration: If f is holomorphic in Ω, then for appropriate closed paths in Ω

$$\int_\gamma f(z)dz = 0.$$

2. Regularity: If f is holomorphic, then f is indefinitely differentiable.

3. Analytic continuation: If f and g are holomorphic functions in Ω which are equal in an arbitrarily small disc in Ω, then $f = g$ everywhere in Ω.

These three phenomena and other general properties of holomorphic functions are treated in the beginning chapters of this book. Instead of trying to summarize the contents of the rest of this volume, we mention briefly several other highlights of the subject.

- The zeta function, which is expressed as an infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and is initially defined and holomorphic in the half-plane $\mathrm{Re}(s) > 1$, where the convergence of the sum is guaranteed. This function and its variants (the L-series) are central in the theory of prime numbers, and have already appeared in Chapter 8 of Book I, where we proved Dirichlet's theorem. Here we will prove that ζ extends to a meromorphic function with a pole at $s = 1$. We shall see that the behavior of $\zeta(s)$ for $\text{Re}(s) = 1$ (and in particular that ζ does not vanish on that line) leads to a proof of the prime number theorem.

- The theta function

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z},$$

which in fact is a function of the two complex variables z and τ, holomorphic for all z, but only for τ in the half-plane $\text{Im}(\tau) > 0$. On the one hand, when we fix τ, and think of Θ as a function of z, it is closely related to the theory of elliptic (doubly-periodic) functions. On the other hand, when z is fixed, Θ displays features of a modular function in the upper half-plane. The function $\Theta(z|\tau)$ arose in Book I as a fundamental solution of the heat equation on the circle. It will be used again in the study of the zeta function, as well as in the proof of certain results in combinatorics and number theory given in Chapters 6 and 10.

Two additional noteworthy topics that we treat are: the Fourier transform with its elegant connection to complex analysis via contour integration, and the resulting applications of the Poisson summation formula; also conformal mappings, with the mappings of polygons whose inverses are realized by the Schwarz-Christoffel formula, and the particular case of the rectangle, which leads to elliptic integrals and elliptic functions.