Introduction

Theory of differences and sums of quantities

Definitions and preliminary notions

(1.) A function of a given variable is defined as any arithmetic expression involving this variable, irrespective of how it appears in it.

Thus x, $a + bx$, $(c - 3dx^3 + fx^4)^5$, $(a + fx^p + gx^q)^r$ etc. are functions of x.

Consider X an arbitrary function of x, and define X' as what becomes of X when x is replaced by $x+k$; then $X'-X$ represents the variation of X when x increases by k. $X' - X$ is called *the difference of* X. Thus, although strictly speaking, one may not talk about the difference of one quantity, we will adopt this commonly used expression; it means the difference between this quantity, considered in an arbitrary state, and the same quantity, considered in another arbitrary state.

We use the letter d to represent the difference of an arbitrary quantity or function. It will not be used for any other purpose to avoid any confusion. Thus, instead of $X' - X$, we write dX or $d(X)$.

And to express, at the same time, the amount by which the quantity x varies, we thus write $d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}$ to express the *difference of* X *when* x *varies by an amount of* k.

We consider increasing quantities here; we will see later what happens when considering decreasing quantities.

Assume the function whose variation or difference under consideration is a function of several variables, x, y or z, whose respective variations are k, l, m; denoting this function by P, we write its difference as $d(P)\begin{pmatrix} x & : & y & : & z \\ k & : & l & : & m \end{pmatrix}$, which means the *difference of* P *when* x *varies by an amount of* k*,* y *by an amount of* l*, and* z *by an amount of* m.

Applying to $X' - X$ the same ideas as above, assume that x is replaced by $x + k'$ in $X' - X$. Then X' becomes X''', and X becomes X''. Then $(X''' - X'') - (X' - X)$ is known as the *second difference of* X, because it is the difference between two successive differences of X.

The second difference will be denoted $dd(X)\ldots\begin{pmatrix} x \\ k, k' \end{pmatrix}$, which means the *second difference of* X*, when* x *varies first by* k *and then by* k'.

(2.) We will very soon give the rules to determine first differences. But we show right now that the second differences are determined by applying to first differences the same rules as those that generate them.

Indeed, the quantity $(X''' - X'') - (X' - X)$ can also be written as follows, $(X''' - X') - (X'' - X)$. Since by assumption X''' is what becomes of X' when substituting $x + k'$ for x and, likewise, X'' is what becomes of X, we therefore obtain $X''' - X' = d(X')\ldots\begin{pmatrix} x \\ k' \end{pmatrix}$ and $X'' - X = d(X)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}$; so $(X''' - X') - (X'' - X)$ or $(X''' - X'') - (X' - X) = d(X')\ldots\begin{pmatrix} x \\ k' \end{pmatrix} - d(X)\ldots\begin{pmatrix} x \\ k' \end{pmatrix} = d(X' - X)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}$. However, $X' - X = d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}$, therefore $(X''' - X'') - (X' - X)$ or

$dd(X)\ldots\begin{pmatrix} x \\ k, k' \end{pmatrix} = d\left(d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}\right)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}$.

That is, we must first compute $d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}$ to obtain $dd(X)\ldots\begin{pmatrix} x \\ k, k' \end{pmatrix}$: We must first take the difference of x, when x varies by an amount of k; we then take the difference of the resulting expression, when x varies by an amount of k'.

(3.) The order of the variation of x (whether x varies by an amount of k in the first difference and k' in the second, or vice versa) makes no difference. Indeed, $(X''' - X'') - (X' - X)$ contains $X''' - X'' = d(X'')\ldots\begin{pmatrix} x \\ k \end{pmatrix}$; it also contains $X' - X = d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}$. Therefore $(X''' - X'') - (X' - X)$ or $dd(X)\ldots\begin{pmatrix} x \\ k, k' \end{pmatrix} = d(X'')\ldots\begin{pmatrix} x \\ k \end{pmatrix} - d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix} = d(X'' - X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}$. But, by definition, $X'' - X = d(X)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}$. Thus

$$\begin{aligned} d(X'' - X)\ldots\begin{pmatrix} x \\ k \end{pmatrix} &= d(X'')\ldots\begin{pmatrix} x \\ k \end{pmatrix} - d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix} \\ &= d\left(d(X\ldots\begin{pmatrix} x \\ k' \end{pmatrix}\right)\ldots\begin{pmatrix} x \\ k \end{pmatrix}. \end{aligned}$$

Thus $dd(X)\ldots\begin{pmatrix} x \\ k, k' \end{pmatrix} = d(d(X)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}\ldots\begin{pmatrix} x \\ k \end{pmatrix}$, but we also just saw that $dd(X)\ldots\begin{pmatrix} x \\ k, k' \end{pmatrix} = d\left(d(X)\ldots\begin{pmatrix} x \\ k \end{pmatrix}\right)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}$; thus $d\left(d(X)\ldots\begin{pmatrix} k \\ x \end{pmatrix}\right)\ldots\begin{pmatrix} x \\ k' \end{pmatrix} = d\left(d(X)\ldots\begin{pmatrix} x \\ k' \end{pmatrix}\right)\ldots\begin{pmatrix} x \\ k \end{pmatrix}$.

Assume the function under consideration contains several variables x, y, z, etc., whose first variation is k, l, m, etc., respectively; we call the second difference of this function (whose name I assume to be P)

$dd(P)\ldots\left(\begin{matrix} x \\ k,k' \end{matrix} : \begin{matrix} y \\ l,l' \end{matrix} : \begin{matrix} z \\ m,m' \end{matrix} \text{ etc.}\right)$.

(4.) To have an idea of the third difference, imagine that x is replaced by $x + k''$ in $(X''' - X'') - (X' - X)$. Then if $X^{\mathrm{VII}}, X^{\mathrm{VI}}, X^{\mathrm{V}}, X^{\mathrm{IV}}$ are what becomes of X''', X'', X' and X with this substitution, the quantity $((X^{\mathrm{VII}} - X^{\mathrm{VI}}) - (X^{\mathrm{V}} - X^{\mathrm{IV}})) - ((X''' - X'') - (X' - X))$ is what is called the *third difference* of X, because it is the difference of two second differences. If k, k', k'' are the successive variations of x, the third difference is written $d^3(X)\ldots\left(\begin{matrix} x \\ k,k',k'' \end{matrix}\right)$. It is easy to see how this extends to the definition of the fourth, fifth and further differences.

About the way to compute the differences of quantities

(5.) Once the algebraic expression of a quantity is given, it is very easy to compute its difference. For example, assume we want to compute the difference of x^3 when x varies by k; we just have to evaluate $(x + k)^3$ and subtract x^3. This difference is $3kx^2 + 3k^2x + k^3$. Computing the difference of a quantity is known as *differentiating this quantity.*

(6.) The differentiation rules are simply the common rules provided by algebra to compute the power of a binomial expression. But to ease and speed up this computation, we give the following rule, already known for other purposes. It is known that the expansion of the binomial $(x + k)$ to the mth power, is $x^m + mx^{m-1}k + m\frac{m-1}{2}x^{m-2}k^2 + m\frac{m-1}{2}\frac{m-2}{3}k^3 +$ etc.

Paying attention to the rules by which those terms are derived from one another, we see that their construction can be performed by using the following rule:

Write on the first line	x^m
Under this line, write	m
Multiply by this exponent, and, diminishing the exponent of x by one unit, replace the factor x that currently misses by the factor k, and get in the second line	$mx^{m-1}k$
Under this line, write one half of the current exponent of x; that is,	$\frac{m-1}{2}$
Multiply by the latter, and, diminishing the current exponent of x by one unit, replace the new missing x factor by a new k factor, and get in the third line	$m\frac{m-1}{2}x^{m-2}k^2$
Under this line, write the third of the current exponent of x; that is,	$\frac{m-2}{3}$
Multiply by the latter, and, diminishing the x exponent by one unit, replace the x factor that is missing again by a new factor k, and get in the fourth line	$m\frac{m-1}{2}\frac{m-2}{3}x^{m-3}k^3$

Keep multiplying according to the same process, successively by one fourth, one fifth, etc. of the exponent of x, and keep lowering the exponent of x by one unit. Replace the missing x factor by a k factor. Then the value of $(x+k)^m$ is the sum of the first, second, third, fourth etc. lines, until the line where the exponent of x becomes 0 which is obvious by comparison with the first formula.

(7.) Therefore it is sufficient to omit the first line in the result from the preceding rule to obtain the difference of x^m where x varies by an amount of k, that is, to obtain the value of $(x+k)^m - x^m$.

(8.) Since the polynomial $Ax^p + Bx^q + Cx^r$ only consists of terms of the form x^m, computing the difference of such a polynomial can be done by simply applying the rule above given for x^m.

Thus, to obtain the difference of $x^3 - 5x^2 + 3x - 6$, where x varies by an amount of k, I write as follows:

First line	x^3	$-$	$5x^2$	$+$	$3x$	$-$	6
Exponent of x	3		2		1		0
Second line	$3x^2k$	$-$	$10xk$	$+$	$3k$		
Half of exponents of x	$\frac{2}{2}$		$\frac{1}{2}$		$\frac{0}{2}$		
Third line	$3xk^2$	$-$	$5k^2$				
Third of exponents of x	$\frac{1}{3}$		$\frac{0}{3}$				
Fourth line	k^3						

Thus $d(x^3 - 5x^2 + 3x - 6) \ldots \begin{pmatrix} x \\ k \end{pmatrix} = 3x^2k + 3xk^2 - 10xk + k^3 - 5k^2 + 3k$, which is the sum of lines 2, 3 and 4.

(9.) We can use the same rule to differentiate quantities involving several variables. Thus, we can compute $d(x^3y^2) \ldots \begin{pmatrix} x & : & y \\ k & : & l \end{pmatrix}$ using the method below, by writing successively under each variable its exponent, then one half of its exponent, one third of it, etc. of its according to the line number being computed.

First line	x^3	y^2							
	3	2							
Second line	$3x^2$	y^2k	$+$	$2x^3$	yl				
	$\frac{2}{2}$	$\frac{2}{2}$		$\frac{3}{2}$	$\frac{1}{2}$				
Third line	$3x$	y^2k^2	$+$	$3x^2$	ykl	$+$	$3x^2$	ykl	$+ \; x^3l^2$
or	$3x$	y^2k^2	$+$	$6x^2$	ykl	$+$	x^3l^2		
	$\frac{1}{3}$	$\frac{2}{3}$		$\frac{2}{3}$	$\frac{1}{3}$		$\frac{3}{3}$		
Fourth line	y^2k^3	$+$	$2x$	yk^2l	$+$	$4x$	$yk^2l + 2x^2kl^2 + x^2kl^2$		
or	y^2k^3	$+$	$6x$	yk^2l	$+$	$3x^2kl^2$			
	$\frac{2}{4}$		$\frac{1}{4}$	$\frac{1}{4}$		$\frac{2}{4}$			

$$
\begin{array}{lllllllll}
\text{Fifth line} & \frac{1}{2}yk^3l & + & \frac{3}{2}yk^3l & + & \frac{3}{2}xk^2l^2 & + & \frac{3}{2}xk^2l^2 \\
\text{or} & 2yk^3l & + & 3xk^2l^2 & & & & \\
 & \frac{1}{5} & & \frac{1}{5} & & & & \\
\hline
\text{Sixth line} & \frac{2}{5}k^3l^2 & + & \frac{3}{5}k^3l^2 & & & & \\
\text{or} & k^3l^2 & & & & & &
\end{array}
$$

Thus $d(x^3, y^2)\ldots\begin{pmatrix} x & : & y \\ k & & l \end{pmatrix} = 3x^2y^2k + 2x^3yl + 3xy^2k^2 + 6x^2ykl + x^3l^2 + y^2k^3 + 6xyk^2l + 3x^2kl^2 + 2yk^3l + 3xk^2l^2 + k^3l^2.$

(10.) The same rule applies to functions of two variables: Simply compare the result of $(x+k)^m \times (y+l)^n$ found with this rule, with the result of the expansion of this quantity using ordinary rules of algebra. These indeed lead to

$$
\begin{aligned}
&x^my^n + mx^{m-1}y^nk + m.\tfrac{m-1}{2}x^{m-2}y^nk^2 + m.\tfrac{m-1}{2}.\tfrac{m-2}{3}.x^{m-3}y^nk^3,\ \text{etc.} \\
&\quad +nx^my^{n-1}l + mnx^{m-1}y^{n-1}kl + mn.\tfrac{m-1}{2}.x^{m-2}y^{n-1}k^2l,\ \text{etc.} \\
&\quad +n.\tfrac{n-1}{2}.x^my^{n-2}l^2 + mn.\tfrac{n-1}{2}.x^{m-1}y^{n-2}kl^2,\ \text{etc.} \\
&\quad +n.\tfrac{n-1}{2}.\tfrac{n-2}{3}.x^my^{n-3}l^3,\ \text{etc.}
\end{aligned}
$$

By applying our rule, we find as follows:

$$
\begin{array}{lllllll}
\text{First line.} & x^m & y^n, & & & & \\
 & m & n & & & & \\
\hline
\text{Second line.} & mx^{m-1} & y^nk & + & nx^m & y^{n-1}l, & \\
 & \frac{m-1}{2} & \frac{n}{2} & & \frac{m}{2} & \frac{n-1}{2} &
\end{array}
$$

$$
\begin{array}{lllllll}
\text{Third line.} & m.\frac{m-1}{2}.x^{m-2} & y^nk^2+ & \frac{mn}{2}.x^{m-1} & y^{n-1}kl+ & \frac{mn}{2}.x^{m-1} & y^{n-1}kl \\
 & & + & n.\frac{n-1}{2}.x^m & y^{n-2}l^2 & & \\
\text{or} & m.\frac{m-1}{2}.x^{m-2} & y^nk^2+ & mnx^{m-1} & y^{n-1}kl+ & n.\frac{n-1}{2}.x^m & y^{n-2}l^2, \\
 & \frac{m-2}{3} & \frac{n}{3} & \frac{m-1}{3} & \frac{n-1}{3} & \frac{m}{3} & \frac{n-2}{3} \\
\hline
\end{array}
$$

Fourth line

$$
\begin{aligned}
&m.\tfrac{m-1}{2}.\tfrac{m-2}{3}.x^{m-3}y^nk^3 + \tfrac{mn}{3}.\tfrac{m-1}{2}.x^{m-2}y^{n-1}k^2l \\
&+mn.\tfrac{m-1}{3}.x^{m-2}y^{n-1}k^2l + mn.\tfrac{n-1}{3}.x^{m-1}y^{n-2}kl^2 \\
&+\tfrac{mn}{3}.\tfrac{n-1}{2}.x^{m-1}y^{n-2}kl^2 + n.\tfrac{n-1}{2}.\tfrac{n-2}{3}x^my^{n-3}l^3;
\end{aligned}
$$

or

$$
\begin{aligned}
&m.\tfrac{m-1}{2}.\tfrac{m-2}{3}.x^{m-3}y^nk^3 + mn.\tfrac{m-1}{2}.x^{m-2}y^{n-1}k^2l \\
&+mn.\tfrac{n-1}{2}.x^{m-1}y^{n-2}kl^2 + n.\tfrac{n-1}{2}.\tfrac{n-2}{3}.x^my^{n-3}l^3,\ \text{etc.}
\end{aligned}
$$

We therefore see that the sum of the first, second, third and fourth lines gives exactly the same result.

(11.) We can use the same method to show that the same rule can be applied to an arbitrary number of variables.

We have demonstrated in (2) that it is enough to apply the same rules to first differences to obtain second differences, and that this also holds true for third, fourth, etc. differences; thus the method to compute arbitrary differences reduces to the only rule given in (4). Consider for example the computation of second differences: We want to compute the value of $dd(x^3+2x^2y-3xy+2xy^2-2x+3y+6)\ldots\left(\begin{array}{ccc} x & : & y \\ k,k' & & l,l' \end{array}\right)$. I write as follows:

$$\begin{array}{llllllllll}
\text{First line} & x^3+ & 2x^2 & y- & 3x & y+ & 2y^2- & 2x+ & 3y+ & 6\\
 & 3 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 0\\ \hline
\text{Second line} & 3x^2k+ & 4x & yk+ & 2x^2l- & 3yk- & 3xl+ & 4yl- & 2k+ & 3l\\
 & \frac{2}{2} & \frac{1}{2} & \frac{1}{2} & \frac{2}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{0}{2} & \frac{0}{2}\\ \hline
\text{Third line} & 3xk^2+ & 2yk^2+ & 2xkl+ & 2xkl- & \frac{3}{2}kl- & \frac{3}{2}kl+ & 2l^2 & & \\
\text{or} & 3xk^2+ & 2yk^2+ & 4xkl- & 3kl+ & 2l^2 & & & & \\
 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{0}{3} & \frac{0}{3} & & & & \\ \hline
\text{Fourth line} & k^3 & + & \frac{2}{3}k^2l & + & \frac{4}{3}k^2l & & & & \\
\text{or} & k^3 & + & 2k^2l & & & & & &
\end{array}$$

Therefore

$$d(x^3+2x^2y-3xy+2y^2-2x+3y+6)\ldots\left(\begin{array}{ccc} x & : & y \\ k & & l \end{array}\right)$$

$$\left.\begin{array}{ccccccc}
=3x^2k & +4x & yk & +2x^2l & -3yk & -3xl & -2k\\
\vdots & \vdots & \vdots & \vdots & +4yl & +3xk^2 & +3l\\
\vdots & \vdots & \vdots & \vdots & +2yk^2 & +4xkl & -3kl\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & +2l^2\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & +2k^2l\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & +k^3\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
2 & 1 & 1 & 2 & 1 & 1 & 0
\end{array}\right\}\begin{array}{c}\text{First line}\\ \text{for the second}\\ \text{difference}\end{array}$$

$$\begin{array}{llllll}
\hline
\text{Second line} & 6xkk' & +4ykk' & +4xkl' & -3kl' & -3lk'\\
 & \vdots & \vdots & +4xk'l & +4ll' & +3k^2k'\\
 & \vdots & \vdots & \vdots & +2k^2l' & +4kk'l\\
 & \vdots & \vdots & \vdots & \vdots & \vdots\\
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{0}{2} & \frac{0}{2}\\ \hline
\text{Third line} & 3kk'^2 & + & 2kk'l' & + & 2kk'l'\\
 & & & & + & 2k'^2l\\
\text{or} & 3kk'^2 & + & 4kk'l' & + & 2k'^2l.
\end{array}$$

Therefore

$$\begin{array}{l}
dd(x^3+2x^2y-3xy+2y^2-2x+3y+6)\ldots\left(\begin{array}{ccc} x & : & y \\ k,k' & & l,l' \end{array}\right)\\
=6xkk'+4ykk'+4xkl'+4xk'l+3kk'^2+2k'^2l\\
+2k^2l'+4kk'l+4kk'l'+3k^2k'+4ll'-3kl'-3lk'.
\end{array}$$

A general and fundamental remark

(12.) Whatever the number of variables entering in the quantity to be differentiated, and whatever the dimension these variables can reach, either alone or together, we can generally observe that:

1. If T is the highest dimension reached by these variables, either alone or together, then $T-1$ is the highest dimension these variables reach in the first difference, since the rule prescribes to reduce the exponent of the variable of interest by one unit.

   Consequently, $T-2$ is the highest degree of the variables in the second difference; $T-3$ is the highest degree of the variables in the third difference; and in general, $T-n$ is the highest dimension of the variables in the difference of order n. Thus, if the order of the difference has the same exponent as that of the highest dimension of the variables, the degree of the variables in the difference is zero; that is, the difference contains no more variables and is only a function of their respective variations.

   For example, $d(ax+by+c)\ldots\begin{pmatrix} x & : & y \\ k & & l \end{pmatrix} = ak+bl$; we see that x and y do not enter in the difference, but their respective variations k and l do.

   Likewise, the above rule yields

   $$\begin{aligned} & dd(ax^2+bxy+cy^2+ex+fy+g)\ldots\begin{pmatrix} x & : & y \\ k,k' & & l,l' \end{pmatrix} \\ = \; & 2akk'+bkl'+bk'l+2cll', \end{aligned}$$

   where we see that x and y have vanished and only their respective variations k, k' and l, l' remain.

2. If there are constant quantities in the function to be differentiated, that is, if there are terms where no variables are present, these terms will not be found in the first derivative, and therefore not in the subsequent differentials either; indeed, the rule prescribes to multiply them by the exponent of the variable, which is zero in that case.

3. The terms where the variables do not exceed, either together or separately, the first dimension, are not to be found in the second difference, since they all become constant by the process of the first differentiation; consequently they will disappear in the second differentiation. For example, assume we must differentiate the quantity $ax^2+bxy+cy^2+ex+fy+g$ twice; the quantity g is not present in the first difference, which is $2axk+byk+bxl+2cyl+ek+fl+ak^2+bkl+cl^2$. Likewise, the terms ex and fy do not appear in the second difference, which is $2akk'+bkl'+bk'l+2cll'$. Indeed, during the first differentiation, these terms become ck and fl. Since these terms are constant, they cannot be found in the following difference.

Likewise, the terms where the variables do not exceed, either together or separately, the dimension 2 do not appear in the third difference; in general, the terms where the degree of the variables does not exceed, either together or separately, the dimension $n-1$, disappear in the difference of order n.

The differentiations we have to perform later are all, or almost all, of the order of the total dimensions of the quantities involved; thus it is appropriate to present here the simplifications that the observations we just made can bring to the usage of the differentiation method.

Reductions that may apply to the general rule to differentiate quantities when several differentiations must be made

(13.) The terms where the variables do not exceed, either together or alone, the dimension $n-1$ cannot be found in the differential of order n; thus the calculations can be considerably simplified, if we strictly follow the general rule we first gave.

This simplification is about rejecting all terms of all dimensions from 0 to $n-1$ included before any computations are performed; n is the planned number of differentiations.

Thus, if we must differentiate twice the quantity $ax^2+bxy+cy^2+ex+fy+g$, the question reduces to differentiating twice the quantity $ax^2+bxy+cy^2$.

If we must differentiate twice the quantity $ax^3+bx^2y+cx^2z+exy^2+fxyz+gxz^2+ky^3+ly^2z+myz^2+nz^3+px^2+qxy+rxz+a'y^2+b'yz+c'z^2+e'x+f'y+g'z+h'$, the question reduces to differentiating twice the quantity $ax^3+bx^2y+cx^2z+exy^2+fxyz+gxz^2+ky^3+ly^2z+myz^2+nz^3+px^2+qxy+rxz+a'y^2+b'yz+c'z^2$.

And if the differentiation were to be performed three times, the question would reduce to differentiating three times the quantity $ax^3+bx^2y+cx^2z+exy^2+fxyz+gxz^2+ky^3+ly^2z+myz^2+nz^3$.

(14.) This simplification is not the only one resulting from the previous observations. After rejecting the various terms that cannot enter the differential, we proceed with the differentiation of the remaining terms; we observe that when computing the various parts that we have called *lines*, it is superfluous to perform computations beyond the line number $T-n+2$, where T is the total dimension of the quantity that we want to differentiate and n is the number of differentiations to be performed.

Indeed, the total dimension diminishes by one unit in every line starting from the second line; when reaching the line $T-n+2$, the dimension is $n-1$; so it is clear that the lines computed beyond this point disappear through successive differentiations, because their dimension is less than $n-1$. It is therefore useless to consider them.

So, if the degree of the differential equals that of the total dimension of the quantity to be differentiated, (i) we must keep only the terms with the highest dimension, and (ii) we need not go beyond the second line for each differentiation.

For example, assume we need to differentiate three times the quantity $x^3 - 3xyz + 2y^3 - x^2 + 2xz - y + 2z - 2$:

1. We reject the dimensions 2, 1 and 0, which reduce this quantity to $x^3 - 3xyz + 2y^3$.

2. We take, in the first difference, the second line only, yielding $3x^2k - 3yzk - 3xzl - 3xym + 6y^2l$.

3. We take, in the second difference, the second line only, yielding $6xkk' - 3zkl' - 3ykm' - 3zlk' - 3xlm' - 3ymk' - 3xml' + 12yll'$.

4. We take, in the third difference, the second line only, yielding $6kk'k'' - 3kl'm'' - 3km'l'' - 3lk'm'' - 3lm'k'' - 3mk'l'' - 3ml'k'' + 12ll'l''$ as the third difference.

Remarks about the differences of decreasing quantities

(15.) Until now we have assumed that each variable was increasing. If, conversely, they were all decreasing, it would not be necessary to establish different rules, but simply to make a slight change in the computed lines.

Indeed, if x becomes $x - k$ instead of $x + k$, there is no other difference between these expressions than k becoming $-k$.

Concerning the differential, there is another change, because we must differentiate x^n, for example; in the first case, we must expand $(x+k)^n - x^n$ and in the second case we must expand $x^n - (x-k)^n$.

If we had to expand $(x-k)^n - x^n$, we would clearly have nothing else to do than to differentiate x^n according to the preceding rules, and making x vary by the quantity $-k$ instead of k.

Thus, in the case of $x^n - (x-k)^n$, we should differentiate x^n, making x vary by the amount $-k$; we should then change the sign of all lines of the result, or, alternatively, we should write along each part of the result, with the sign of one line opposite to that yielded by the differentiation obtained by making x vary by an amount $-k$.

(16.) Thus we see that, in general, the differential of a function is different when its variables are increasing quantities from the same differential when all variables are decreasing. There are, however, two cases when these differentials are the same. The first case is when the variations of the variables are infinitesimally small. The second is when the quantity must be differentiated as many times as the size of its exponent of the highest dimension.

This last case is the only one of interest to this work: Thus, in the differentiations we will perform later, we will not need to examine whether the variables are increasing or decreasing. We will differentiate following the rules we have first given.

About certain quantities that must be differentiated through a simpler process than that resulting from the general rule

(17.) The principles that we just elicited are general and could even, with slight changes, be applied to fractional and irrational quantities. They can be applied to convert functions of several variables into series, and to many other objects. But our goal is not to discuss these applications. We will only consider rational quantities that can be differentiated faster than through the general rule: We consider only those that will be useful to us later on.

Assume we must differentiate a quantity such as $(x+a).(x+a+b).(x+a+2b).(x+a+3b)\dots(x+a+(n-1)b)$, where n is the number of factors and x varies by a quantity b; the differential is $nb.(x+a+b).(x+a+2b).(x+a+3b)\dots(x+a+(n-1)b)$, where $n-1$ is the number of factors in arithmetic progression.

But if the variation is $-b$, the differential is $nb(x+a).(x+a+b).(x+a+2b)\dots(x+a+(n-2)b)$, where $n-1$ is the number of factors in arithmetic progression.

Indeed,

$$\begin{aligned}
& d[(x+a).(x+a+b).(x+a+2b)\dots(x+a+(n-1)b)]\dots\begin{pmatrix} x \\ b \end{pmatrix} \\
= & (x+a+b).(x+a+2b)(x+a+3b)\dots(x+a+nb) \\
& -(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b) \\
= & [(x+a+b)(x+a+2b)(x+a+3b)\dots(x+a+(n-1)b)] \\
& \times(x+a+nb-x-a) \\
= & nb(x+a+b)(x+a+2b)(x+a+3b)\dots(x+a+(n-1)b).
\end{aligned}$$

Likewise,

$$\begin{aligned}
& d[(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b)]\dots\begin{pmatrix} x \\ -b \end{pmatrix} \\
= & (x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b) \\
& -(x+a-b)(x+a)(x+a+b)\dots(x+a+(n-2)b) \\
= & [(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-2)b)] \\
& \times(x+a+(n-1)b-x-a+b) \\
= & nb(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-2)b).
\end{aligned}$$

About sums of quantities

(18.) Imagine that P is an arbitrary function of one or many variables x, y, z, etc., and that, giving successively to each of these variables the values k, l, m, etc., k', l', m', etc., k'', l'', m'', etc. respectively, the quantity P becomes successively P', P'', P''', etc. the sum $P+P'+P''+P'''+$ etc. is what we will call *sum of* P, and we will write it as $\int P$.

We will not attempt, by far, to deal with this matter to the whole extent that it deserves. Our purposes only require a very specific branch of this theory and we will restrict ourselves to it.

We therefore consider rational functions of a single variable, with no variable divider.

We will also assume that the variable increases or decreases by equal amounts.

About sums of quantities whose factors grow arithmetically

(19.) Those products are usually represented by

$$(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b),$$

where n is the number of factors.

Substituting for x the quantities $(x-b)$, $(x-2b)$, $(x-3b)$, etc., the quantities of interest become

$$\begin{array}{l}(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b),\\(x+a-b)(x+a)(x+a+b)\dots(x+a+(n-2)b),\\(x+a-2b)(x+a-b)(x+a)\dots(x+a+(n-3)b),\\(x+a-3b)(x+a-2b)(x+a-b)\dots(x+a+(n-4)b),\end{array}$$

etc.

Let P be the sum of all these products, and P' the sum of all these products, except the first. We have $P-P'=(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b)$. But $P-P'=d(P)\dots\left(\begin{array}{c}x\\-b\end{array}\right)$. Thus $d(P)\dots\left(\begin{array}{c}x\\-b\end{array}\right)=(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b)$.

Finding P therefore reduces to *finding the function whose difference is* $(x+a)(x+a+b)(x+a+2b)\dots(x+a+(n-1)b)$, *when* x *varies by* $-b$.

Considering what was said in (17), it is easy to see that this function is

$$\frac{1}{(n+1)b}(x+a)(x+a+b)(x+a+2b)\dots(x+a+nb),$$

where $n+1$ is the number of factors.[1]

Thus $P=\frac{1}{(n+1)b}(x+a)(x+a+b)(x+a+2b)\dots(x+a+nb)$.

Remarks

(20.) First, we have supposed that the variation of x was precisely equal to the difference b present in the progression of the factors. We will soon see how to determine this sum when this variation is any other quantity.

(21.) Second, from (12) the constant terms present in a quantity to be differentiated vanish in the difference; thus it follows that a constant must always be added to the quantity to be summed. From a computational standpoint, this constant is arbitrary, since the differential is always the same. But in each question, this constant has a specific value, which is easily found by the conditions of the problem.

[1] One must be careful, when comparing with what was said in (17), that what was denoted n in (17) is now $n+1$.

From now on, we write this constant as C. Thus the value of P we have found is, more generally,

$$P = \frac{1}{(n+1)b}(x+a)(x+a+b)(x+a+2b)\dots(x+a+nb)+C.$$

To give an example about the way to determine this constant C, assume we need to compute the sum of the products $2\times4\times6$, $4\times6\times8$, $6\times8\times10$, $8\times10\times12$ until $14\times16\times18$; we therefore have $(x+a).(x+a+b).(x+a+2b) = 14\times16\times18$ and $n = 3$.

Assume $a = b = 2$; we will have $x = 12$. Thus $P = \frac{1}{4\times2}14\times16\times18\times20+C$.

But we want the sum only from $2\times4\times6$; comparing this product with $(x+a).(x+a+b).(x+a+2b)$, we have $x = 0$; thus when $x = 0$, the sum P becomes $2\times4\times6$; we therefore have $2\times4\times6 = \frac{1}{4.2}\times2\times4\times6\times8+C$, or $C = 48-48 = 0$. The sum is therefore simply $\frac{1}{4.2}\times14\times16\times18\times20$, that is, 10080. It is easy to check this result by adding the products together.

If instead of assuming that $a = 2$ we had assumed that $a = 0$, then we would have had $x = 14$ as the final value of x, and $x = 2$ as its initial value; the sum would then be $P = \frac{1}{4.2}\times14\times16\times18\times20+C$. To determine the constant C, we could rely on the condition that the sum P must become $2\times4\times6 = \frac{1}{4.2}\times2\times4\times6\times8+C$ when $x = 2$. Thus $C = 0$, and P is again 10,080 as expected.

About sums of rational quantities with no variable divider

(22.) For the sake of clarity, assume first that we must sum a simple quantity, such as x^3 or mx^3. The question asked is ill-posed, because we must know by which amounts x increases or decreases. Assume therefore that x decreases by equal amounts of amplitude b.

Then the true meaning of the question is the following: Assuming that x becomes $x-b$, $x-2b$, $x-3b$, successively, compute the sum of the quantities mx^3, $m(x-b)^3$, $m(x-2b)^3$, $m(x-3b)^3$, etc.

To answer this question, I reduce it to the one solved in (19), by bringing mx^3 back to the form $(x+b).(x+2b).(x+3b)$, etc.

I therefore write $mx^3 = A(x+b).(x+2b).(x+3b)+B(x+b).(x+2b)+C(x+b)+D$. I then obtain:

$$\begin{aligned} mx^3 \quad = \quad & Ax^3+6Abx^2+11Ab^2x+6Ab^3 \\ & \qquad\qquad +Bx^2+3Bbx+2Bb^2 \\ & \qquad\qquad\qquad\qquad +Cx+Cb \\ & \qquad\qquad\qquad\qquad\qquad +D. \end{aligned}$$

Since this equality must hold true for any value of x, I conclude that $A = m$, $6Ab+B = 0$, $11Ab^2+3Bb+C = 0$, $6Ab^3+2Bb^2+Cb+D = 0$; that is, $A = m$, $B = -6mb$, $C = +7mb^2$, $D = -mb^3$; thus

$$mx^3 = m(x+b).(x+2b).(x+3b)-6mb.(x+b).(x+2b)+7mb^2(x+b)-mb^3.$$

The value of mx^3 is therefore composed of four parts, each of which is of the form we know to sum, from (19). We therefore easily find, through what was already established in (19), that

$$\int mx^3 = \frac{m}{4b}.(x+b).(x+2b).(x+3b).(x+4b)$$
$$-2m(x+b).(x+2b).(x+3b)$$
$$+\frac{7mb}{2}.(x+b).(x+2b) - mb^2(x+b) + C,$$

where C is a constant (21).

(23.) Consider the quantity $mx^3 + nx^2 + px + q$: We see that each term reduces to the form $(x+b).(x+2b).(x+3b)$, etc., as we have seen with mx^3. Thus the total can also be reduced to that form. Therefore, I can sum a quantity such as $mx^3 + nx^2 + px + q$ by writing

$$\begin{aligned} mx^3 + nx^2 + px + q &= A(x+b).(x+2b).(x+3b) \\ &+ B(x+b).(x+2b) \\ &+ C(x+b) + D, \end{aligned}$$

by determining the coefficients A, B, C, D, and by equating the coefficients of the same powers of x in the left- and right-hand side of the equations. Thus I only have to compute the sum of the quantity $A.(x+b).(x+2b).(x+3b) + B.(x+b).(x+2b) + C.(x+b) + D$, which is easy to do from what was said in (19), and which is

$$\begin{aligned} & \tfrac{A}{4b}.(x+b).(x+2b).(x+3b).(x+4b) \\ + & \tfrac{B}{3b}.(x+b).(x+2b).(x+3b) \\ + & \tfrac{C}{2b}.(x+b).(x+2b) \\ + & \tfrac{D}{b}.(x+b) + C, \end{aligned}$$

where A, B, C and D must be replaced by their values.

(24.) Looking at the form of the sum in this example and the previous one, we see that computing these sums can be made simpler. We do not need to bring the proposed quantity back to the form $(x+b).(x+2b).(x+3b).$, etc., since the sum is of the same form; we can immediately determine the coefficients of the sum as follows. Let us go back to the example of mx^3.

(25.) I assume

$$\begin{aligned} \int mx^3 &= A.(x+b).(x+2b).(x+3b).(x+4b) \\ &+ B.(x+b).(x+2b).(x+3b) \\ &+ C.(x+b).(x+2b) \\ &+ D.(x+b) + C \end{aligned}$$

right away; to obtain the coefficients, I differentiate each one of the terms (17) and I get

$$\begin{aligned} mx^3 &= 4Ab.(x+b).(x+2b).(x+3b) \\ &= 3Bb.(x+b).(x+2b) \\ &= 2Cb.(x+b) + Db, \end{aligned}$$

that is,

$$\begin{aligned} mx^3 = {} & 4Abx^3 + 24Ab^2x^2 + 44Ab^3x + 24Ab^4 \\ & +3Bbx^2 + 9Bb^2x + 6Bb^3 \\ & +2Cbx + 2Cb^2 \\ & +Db. \end{aligned}$$

I therefore obtain

$$4Ab = m, \quad 24Ab^2 + 3Bb = 0,$$
$$44Ab^3 + 9Bb^2 + 2Cb = 0,$$
$$24Ab^4 + 6Bb^3 + 2Cb^2 + Db = 0;$$

thus $A = m/4b$, $B = -2m$, $C = 7mb/2$, $D = -mb^2$; this leads to the same expression as computed earlier for $\int mx^3$.

(26.) In general we see that in order to integrate a rational polynomial without variable divider, such as $ax^p + bx^q + cx^r +$ etc., we first write

$$\begin{aligned} & \int (ax^p + bx^q + cx^r + \text{, etc. }) \\ = \; & A.(x+b).(x+2b).(x+3b)\ldots(x+(p+1).b) \\ + \; & B.(x+b).(x+2b).(x+3b)\ldots(x+pb) \\ + \; & C.(x+b).(x+2b).(x+3b)\ldots(x+(p-1).b) \\ + \; & D.(x+b).(x+2b).(x+3b)\ldots(x+(p-2).b) + \cdots \\ + \; & P.(x+b).(x+2b) + Q.(x+b) + C, \end{aligned}$$

where we assume that p is the largest of the exponents p, q, r, etc.; we then compute the coefficients as we have discussed earlier.

If we had $(ax^p + bx^q + cx^r + \text{etc.})^k$, we would recast the problem to the preceding problem by expanding this power of a polynomial.

(27.) We now see, as promised in (20), how we can compute the sum of

$$(x+a).(x+a+b).(x+a+2b)\ldots(x+a+(n-1)b)$$

when x increases or decreases by steps other than b. For example, if k is the stepsize by which x grows, we write

$$\begin{aligned} & \int (x+a).(x+a+b).(x+a+2b)\ldots(x+a+(n-1)b) \\ & = A.(x+k).(x+2k).(x+3k)(x+(n+1)k) \\ & +B.(x+k).(x+2k).(x+3k)\ldots(x+n.k) \\ & +C.(x+k).(x+2k)\ldots(x+(n-1)k) + \cdots \\ & +Q.(x+k) + C. \end{aligned}$$

(28.) If we wanted to compute the value of $\int Ax^m$ when $m = 0$, it follows from what we have just said that this value would be $A(x+b)$. Indeed, since m is zero, the question reduces to computing the sum of A from a given value of x to another value of x. Thus if $x + b$ represents the range over which we compute the sum of A, the sum is $A.(x+b)$.