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**Paul J. Nahin: Dr. Euler's Fabulous Formula**

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## Introduction

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Like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence.

—Keith Devlin writing of  $e^{i\pi} + 1 = 0$ <sup>1</sup>

The nineteenth-century Harvard mathematician Benjamin Peirce (1809–1880) made a tremendous impression on his students. As one of them wrote many years after Peirce's death, "The appearance of Professor Benjamin Peirce, whose long gray hair, straggling grizzled beard and unusually bright eyes sparkling under a soft felt hat, as he walked briskly but rather ungracefully across the college yard, fitted very well with the opinion current among us that we were looking upon a real live genius, who had a touch of the prophet in his make-up."<sup>2</sup> That same former student went on to recall that during one lecture "he established the relation connecting  $\pi$ ,  $e$ , and  $i$ ,  $e^{\pi/2} = \sqrt[i]{i}$ , which evidently had a strong hold on his imagination."<sup>3</sup> He dropped his chalk and rubber (i.e., eraser), put his hands in his pockets, and after contemplating the formula a few minutes turned to his class and said very slowly and impressively, 'Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth.'

Like any good teacher, Peirce was almost certainly striving to be dramatic ("Although we could rarely follow him, we certainly sat up and took notice"), but with those particular words he reached too far. We certainly *can* understand what Peirce always called the "mysterious formula," and we certainly *do* know what it means. But, yes, it *is* still a wonderful, indeed beautiful, expression; no amount of "understanding" can ever diminish

its power to awe us. As one limerick (a literary form particularly beloved by mathematicians) puts it,

*e* raised to the *pi* times *i*,  
 And plus 1 leaves you nought but a sigh.  
 This fact amazed Euler  
 That genius toiler,  
 And still gives us pause, bye the bye.

The limerick puts front-and-center several items we need to discuss pretty soon. What are *e*, *pi*, and *i*, and who was Euler? Now, it is hard for me to believe that there are any literate readers in the world who haven't heard of the transcendental numbers  $e = 2.71828182\dots$  and  $pi = \pi = 3.14159265\dots$ , and of the imaginary number  $i = \sqrt{-1}$ . As for Euler, he was surely one of the greatest of all mathematicians. Making lists of the "greatest" is a popular activity these days, and I would wager that the Swiss-born Leonhard Euler (1707–1783) would appear somewhere among the top five mathematicians of all time on the list made by any mathematician in the world today (Archimedes, Newton, and Gauss would give him stiff competition, but what great company they are!).

Now, before I launch into the particulars of *e*,  $\pi$ , and  $\sqrt{-1}$ , what about the stupefying audacity I displayed in the Preface by declaring  $e^{i\pi} + 1 = 0$  to be "an expression of exquisite beauty"? I didn't do that lightly and, indeed, I have "official authority." In the fall 1988 issue of the *Mathematical Intelligencer*, a scholarly quarterly journal of mathematics sponsored by the prestigious publisher of mathematics books and journals, Springer-Verlag, there was the call for a vote on the most beautiful theorem in mathematics. Readers of the *Intelligencer*, consisting almost entirely of academic and industrial mathematicians, were asked to rank twenty-four given theorems on a scale of 0 to 10, with 10 being the most beautiful and 0 the least. The list contained, in addition to  $e^{i\pi} + 1 = 0$ , such seminal theorems as

- (a) The number of primes is infinite;
- (b) There is no rational number whose square is 2;

- (c)  $\pi$  is transcendental;
- (d) A continuous mapping of the closed unit disk into itself has a fixed point.

A distinguished list, indeed.

The results, from a total of 68 responses, were announced in the summer 1990 issue. Receiving the top average score of 7.7 was  $e^{i\pi} + 1 = 0$ . The scores for the other theorems above, by comparison, were 7.5 for (a), 6.7 for (b), 6.5 for (c), and 6.8 for (d). The lowest ranked theorem (a result in number theory by the Indian genius Ramanujan) received an average score of 3.9. So, it is *official*:  $e^{i\pi} + 1 = 0$  is the most beautiful equation in mathematics! (I hope most readers can see my tongue stuck firmly in my cheek as they read these words, and will not send me outraged e-mails to tell me why their favorite expression is so much *more* beautiful.)

Of course, the language used above is pretty sloppy, because  $e^{i\pi} + 1 = 0$  is actually *not* an equation. An equation (in a single variable) is a mathematical expression of the form  $f(x) = 0$ , for example,  $x^2 + x - 2 = 0$ , which is true only for certain values of the variable, that is, for the *solutions* of the equation. For the just cited quadratic equation, for example,  $f(x)$  equals zero for the two values of  $x = -2$  and  $x = 1$ , *only*. There is no  $x$ , however, to solve for in  $e^{i\pi} + 1 = 0$ . So, it isn't an equation. It isn't an identity, either, like Euler's identity  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ , where  $\theta$  is *any* angle, not just  $\pi$  radians. That's what an *identity* (in a single variable) is, of course, a statement that is *identically* true for *any* value of the variable. There isn't any variable at all, anywhere, in  $e^{i\pi} + 1 = 0$ : just five constants. (Euler's identity is at the heart of this book and it will be established in Chapter 1.) So,  $e^{i\pi} + 1 = 0$  isn't an equation and it isn't an identity. Well, then, what *is* it? It is a *formula* or a *theorem*.

More to the point for us, here, isn't semantics but rather the issue I first raised in the Preface, that of *beauty*. What could it possibly mean to say a mathematical statement is "beautiful"? To that I reply, what does it mean to say a kitten asleep, or an eagle in flight, or a horse in full gallop, or a laughing baby, or . . . is beautiful? An easy answer is that it is all simply in the eye of the beholder (the ultimate "explanation," I suppose, for the popularity of Jackson Pollock's drip paintings), but I think (at least

in the mathematical case) that there are deeper possibilities. The author of the *Intelligencer* poll (David Wells, the writer of a number of popular mathematical works), for example, offered several good suggestions as to what makes a mathematical expression *beautiful*.

To be beautiful, Wells writes, a mathematical statement must be simple, brief, important, and, obvious when it is stated but perhaps easy to overlook otherwise, *surprising*. (A similar list was given earlier by H. E. Huntley in his 1970 book *The Divine Proportion*.) I think Euler's identity (and its offspring  $e^{i\pi} + 1 = 0$ ) scores high on all four counts, and I believe you will think so too by the end of this book. Not everyone agrees, however, which should be no surprise—there is *always* someone who doesn't agree with *any* statement! For example, in his interesting essay "Beauty in Mathematics," the French mathematician François Le Lionnais (1901–1984) starts off with high praise, writing of  $e^{i\pi} + 1 = 0$  that it

[E]stablishes what appeared in its time to be a fantastic connection between the most important numbers in mathematics, 1,  $\pi$ , and  $e$  [for some reason 0 and  $i$  are ignored by Le Lionnais]. It was generally considered "the most important formula of mathematics."<sup>4</sup>

But then comes the tomato surprise, with a very big splat in the face: "Today the intrinsic reason for this compatibility has become so obvious[!] that the same formula seems, if not insipid, at least entirely natural."

Well, good for François and his fabulous powers of insight (or is it hindsight?), but such a statement is rightfully greeted with the same skepticism that most mathematicians give to claims from those who say they can "see geometrical shapes in the fourth dimension." Such people only *think* they do. They are certainly "seeing things," all right, but I doubt very much it's the true geometry of hyperspace. When you are finished here,  $e^{i\pi} + 1 = 0$  *will* be "obvious," but borderline *insipid*? Never!

At this point, for completeness, I should mention that the great English mathematician G. H. Hardy (1887–1947) had a very odd view of what constitutes beauty in mathematics: to be beautiful, mathematics

must be *useless*! That wasn't a sufficient condition but, for the ultra-pure Hardy, it was a necessary one. He made this outrageous assertion in his famous 1940 book *A Mathematician's Apology*, and I can't believe there is a mathematician today (no matter how pure) who would subscribe to Hardy's conceit. Indeed, I think Hardy's well-known interest and expertise in Fourier series and integrals, mathematics *impossible by 1940 for practical, grease-under-the-fingernails electrical engineers to do without* (as you'll see in chapters 5 and 6), is proof enough that his assertion was nonsense even as he wrote it. To further illustrate how peculiar was Hardy's thinking on this issue, he called the physicists James Clerk Maxwell (1831–1879), and Dirac, "real" mathematicians. That is comical because Maxwell's equations for the electromagnetic field are what make the oh-so-*useful* gadgets of radios and cell-phones possible, and Dirac always gave much credit to his undergraduate training in electrical engineering as being the inspiration behind his very nonrigorous development of the impulse "function" in quantum mechanics!<sup>5</sup>

As a counterpoint to mathematical beauty, it may be useful to mention, just briefly, an example of mathematical *ugliness*. Consider the 1976 "proof" of the four-color theorem for planar maps. The theorem says that four colors are both sufficient *and necessary* to color all possible planar maps so that countries that share a border can have different colors.<sup>6</sup> This problem, which dates from 1852, defied mathematical attack until two mathematicians at the University of Illinois programmed a computer to automatically "check" many hundreds of specific special cases. The details are unimportant here—my point is simply that this particular "proof" is almost always what mathematicians think of when asked "What is an example of ugly mathematics?" If this seems a harsh word to use, let me assure you that I am not the first to do so. The two Illinois programmers themselves have told the story of the reaction of a mathematician friend when informed they had used a computer:<sup>7</sup> the friend "exclaimed in horror, 'God would never permit the best proof of such a beautiful theorem to be so ugly.'"

Although nearly all mathematicians believe the result, nearly all dislike how it was arrived at because the computer calculations hide from view so much of the so-called "solution." As the English mathematician who first started the four-color problem on its way into history, Augustus

De Morgan (1806–1871), wrote in his book *Budget of Paradoxes*, “Proof requires a **person** who can give and a **person** who can receive” (my emphasis). There is no mention here of an automatic machine performing hundreds of millions of intermediate calculations (requiring *weeks* of central processor time on a supercomputer) that not even a single person has ever completely waded through.<sup>8</sup>

Before leaving computer proofs, I should admit that there is one way such an approach *could* result in beautiful mathematics. Imagine that, unlike in the four color problem, the computer discovered one or more counter-examples to a proposed theorem. Those specific counter-examples could then be verified in the traditional manner by as many independent minds as cared to do it. An example of this, involving Euler, dates from 1769.<sup>9</sup> The disproof of a statement, by presenting a specific counter-example, is perhaps the most convincing of all methods (the counter-example’s origin in a computer analysis is irrelevant once we have the counter-example in-hand), and is generally thought by mathematicians to be a beautiful technique.

There are, of course, lots of beautiful mathematical statements that I think might give  $e^{i\pi} + 1 = 0$  a run for its money but weren’t on the original *Intelligencer* list. Just to give you a couple of examples, consider first the infinite series

$$S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$$

This series is called the *harmonic series*, and the question is whether the sum  $S$  is finite or infinite, that is, does the series *converge* or does it *diverge*? Nearly everyone who sees this for the first time thinks  $S$  should be finite (a mathematician would say  $S$  *exists*) because each new term is smaller than the previous term. Indeed, the terms are tending toward zero, which *is, indeed, a necessary* condition for the series to converge to a finite sum—but it isn’t a *sufficient* condition. For a series to converge, the terms must not only go to zero, they must go to zero *fast enough* and, in the case of the harmonic series, they do not. (If the signs of the

harmonic series alternate *then* the sum *is* finite:  $\ln(2)$ .) Thus, we have the beautiful, *surprising* statement that,

$$\lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{1}{n} = \infty,$$

which has been known since about 1350. This theorem should, I think, have been on the original *Intelligencer* list.<sup>10</sup>

By the way, the proof of this beautiful theorem is an example of a beautiful mathematical *argument*. The following is not the original proof (which is pretty slick, too, but is more widely known and so I won't repeat it here<sup>11</sup>). We'll start with the *assumption* that the harmonic series converges, i.e., that its sum  $S$  is some finite number. Then,

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right) \\ &= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) \\ &= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) + \frac{1}{2}S. \end{aligned}$$

So,

$$\frac{1}{2}S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots.$$

That is, the sum of just the *odd* terms alone is one-half of the total sum. Thus, the sum of just the *even* terms alone must be the other half of  $S$ . Therefore, the *assumption* that  $S$  exists has led us to the conclusion that

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots.$$

But this equality is clearly not true since, term by term, the left-hand side is larger than the right-hand side ( $1 > \frac{1}{2}$ ,  $\frac{1}{3} > \frac{1}{4}$ ,  $\frac{1}{5} > \frac{1}{6}$ ,  $\cdots$ ). So, our initial assumption that  $S$  exists must be wrong, that is,  $S$  does *not* exist,

and the harmonic series must diverge. This beautiful argument is called a *proof by contradiction*.

The most famous proof by contradiction is Euclid's proof of theorem (a) on the original *Intelligencer* list. I remember when I first saw (while still in high school) his demonstration of the infinity of the primes; I was thrilled by its elegance and its *beauty*. For me, a proof by contradiction became one of the signatures of a beautiful mathematical demonstration. When Andrew Wiles (1953–) finally cracked the famous problem of Fermat's last theorem in 1995, it was with a proof "by contradiction." And the proof I'll show you in chapter 3 of the irrationality of  $\pi^2$ , using Euler's formula, is a proof "by contradiction."

Celebrity intellectual Marilyn vos Savant ("world's highest IQ") is not impressed by this line of reasoning, however, as she rejects *any* proof by contradiction. As she wrote in her now infamous (and famously embarrassing) book on Wiles's proof,

But how can one ever really prove anything by contradiction? Imaginary numbers are one example. The square root of  $+1$  is a real number because  $+1 \times +1 = +1$ ; however, the square root of  $-1$  is imaginary because  $-1$  times  $-1$  would also equal  $+1$ , instead of  $-1$ . This appears to be a contradiction. [The "contradiction" escapes me, and I have no absolutely idea why she says this.] Yet it is accepted, and imaginary numbers are used routinely. But how can we justify using them to *prove* a contradiction?

This is, of course, as two reviewers of her book put it, an example of "inane reasoning" (the word *drivel* was also used to describe her book),<sup>12</sup> and so let me assure you that proof by contradiction *is* most certainly a valid technique.

So, imagine my surprise when I read two highly respected mathematicians call such a demonstration "a wiseguy argument!" They obviously meant that in a humorous way, but the phrase still brought me up short. I won't give Euclid's proof of the infinity of the primes here (you can find it in any book on number theory), but rather let me repeat what Philip Davis and Reuben Hersh said in their acclaimed 1981 book, *The Mathematical Experience*, about the traditional proof by contradiction of theorem (b) in the *Intelligencer* list. To prove that  $\sqrt{2}$  is not rational, let's

assume that it *is*. That is, *assume* (as did Pythagoras some time in the sixth century B.C.) that there are two integers  $m$  and  $n$  such that

$$\sqrt{2} = \frac{m}{n}.$$

We can further assume that  $m$  and  $n$  have no common factors, because if they *do* then simply cancel them and then call what remains  $m$  and  $n$ .

So, squaring,  $2n^2 = m^2$ , and thus  $m^2$  is even. But that means that  $m$  itself is even, because you can't get an even  $m^2$  by squaring an odd integer (any odd integer has the form  $2k + 1$  for  $k$  some integer, and  $(2k + 1)^2 = 4k^2 + 4k + 1$ , which is odd). But, since  $m$  is even, then there must be an integer  $r$  such that  $m = 2r$ . So,  $2n^2 = 4r^2$  or,  $n^2 = 2r^2$ , which means  $n^2$  is even. And so  $n$  is even. To summarize, we have deduced that  $m$  and  $n$  are *both* even integers *if* the integers  $m$  and  $n$  exist. But, we started by assuming that  $m$  and  $n$  have no common factors (in particular, two even numbers have the common factor 2), and so our assumption that  $m$  and  $n$  exist has led to a logical contradiction. Thus,  $m$  and  $n$  do *not* exist! What a beautiful proof—it uses only the concept that the integers can be divided into two disjoint sets, the evens and the odds.

Davis and Hersh don't share my opinion, however, and besides the "wiseguy" characterization they suggest that the proof also has a problem "with its emphasis on logical inexorableness that seems heavy and plodding." Well, to that all I can say is all proofs should have such a problem! But what is very surprising to me is what they put forth as a better proof. They start as before, until they reach  $2n^2 = m^2$ . Then, they say, imagine that whatever  $m$  and  $n$  are, we factor them into their prime components. Thus, for  $m^2$  we would have a sequence of *paired* primes (because  $m^2 = m \cdot m$ ), and similarly for  $n^2$ . I now quote their dénouement:

But (aha!) in  $2n^2$  there is a 2 that has no partner.

Contradiction.

Huh? *Why* a contradiction? Well, because they are invoking (although they never say so explicitly) a theorem called "the fundamental theorem of arithmetic," which says the factorization of any integer (in the realm of the ordinary integers) into a product of primes is *unique*. They do sort of admit this point, saying "Actually, we have elided some formal details." Yes, I think so!

Davis and Hersh claim their proof would be preferred to the Pythagorean one by “nine professional mathematicians out of ten [because it] exhibits a higher level of aesthetic delight.” I suppose they might well be right, but *I* think the unstated unique factorization result a pretty big hole to jump over. It isn’t hard to prove for the ordinary integers, but it isn’t either trivial or, I think, even obvious — indeed, it isn’t difficult to create different realms of real integers in which it isn’t even true!<sup>13</sup> I therefore have a very big problem with that “aha!” It certainly is a huge step beyond just the concept of evenness and oddness, which is all that the Pythagorean proof uses.

For my second example of a beautiful mathematical expression, this one due to Euler, consider the infinite product expansion of the sine function:

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

You don’t have to know much mathematics to “know” that this is a pretty amazing statement (which may account for why many think it is “pretty,” too). I think a high school student who has studied just algebra and trigonometry would appreciate this. As an illustration of this statement’s significance, it is only a few easy steps from it to the conclusion that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} = 1.644934 \dots,$$

a beautiful line of mathematics in its own right (I’ll show you a way to do it, different from Euler’s derivation, later in the book).<sup>14</sup> Failed attempts to evaluate  $\sum_{n=1}^{\infty} (1/n^2)$  had been frustrating mathematicians ever since the Italian Pietro Mengoli (1625–1686) first formally posed the problem in 1650, although many mathematicians must have thought of this next obvious extension beyond the harmonic series long before Mengoli; Euler finally cracked it in 1734.<sup>15</sup> All this *is* beautiful stuff but, in the end, I still think  $e^{i\pi} + 1 = 0$  is the best. This is, in part, because you can *derive* the infinite product expression for  $\sin(x)$  via the intimate link between  $\sin(x)$  and  $i = \sqrt{-1}$ , as provided by Euler’s identity (see note 14 again).

Let me end this little essay with the admission that perhaps mathematical beauty *is* all in the eye of the beholder, just like a Jackson Pollock painting. At the end of his 1935 presidential address to the London Mathematical Society, for example, the English mathematician G. N. Watson stated that a particular formula gave him “a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing Day, Night, Evening, and Dawn which Michelangelo has set over the tombs of Guiliano de’Medici and Lorenzo de’Medici.”<sup>16</sup> Wow, that’s quite a thrill!

In a series of lectures he gave to general audiences at the Science Museum in Paris in the early 1980s, the Yale mathematician Serge Lang tried to convey what *he* thought beautiful mathematics is, using somewhat less dramatic imagery than Watson’s.<sup>17</sup> He never gave a formal definition, but several times he said that, whatever it was, he knew it when he saw it because it would give him a “chill in the spine.” Lang’s phrase reminds me of Supreme Court Justice Potter Stewart, who, in a 1964 decision dealing with pornography, wrote his famous comment that while he couldn’t define it he “knew it when he saw it.” Perhaps it is the same at the other end of the intellectual spectrum, as well, with beautiful mathematics.

Being able to appreciate beautiful mathematics is a privilege, and many otherwise educated people who can’t sadly understand that they are “missing out” on something precious. In autobiographical recollections that he wrote in 1876 for his children, Charles Darwin expressed his feelings on this as follows:

During the three years which I spent at Cambridge my time was wasted, as far as academical studies were concerned . . . I attempted mathematics, and even went during the summer of 1828 with a private tutor . . . but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, *for men thus endowed seem to have an extra sense* [my emphasis].<sup>18</sup>

I started this section with a limerick, so let me end with one. I think, if you read this book all the way through, then, contrary to Professor Peirce, you'll agree with the following (although I suspect the first two lines don't really apply to *you!*):

I used to think math was no fun,  
'Cause I couldn't see how it was done.  
Now Euler's my hero  
For I now see why zero,  
Equals  $e^{\pi i} + 1$ .

Okay, enough with the bad poetry. Let's begin the good stuff. Let's do some "complex" mathematics.