I understand it, this is a beautiful and intricate story relating the signal (sic) contributions of this volume with significant paths through twentieth-century engineering, mathematics, and science. We shall tell parts of the story, in a mix of vignettes and perspectives, as appetizers for the seven course feast that follows. To fix ideas we describe Figure 1 in Section 2. The figure itself is meant as a mise-en-scène to get started. The technical definitions in Section 2 associated with the figure can be omitted by the reader without substantial damage to our storyline. They can also be examined more carefully while reading the descriptive parts of the Introduction related to them.

2. Definitions

The problem of *signal representation* at the top of Figure 1 is to provide effective decompositions of given signals f in terms of harmonics. The terms "effective" and "harmonics" are problem-specific notions.

To define the next level of Figure 1, recall that Fourier series $S(f)$ of 1-periodic functions f have the form

$$S(f)(x) = \sum_{n\in\mathbb{Z}} c_n(f)\, e_n(-x), \tag{2.1}$$

where $e_n(x) = e^{2\pi i x n}$. Equation (2.1) is an example of a *discrete representation*, with harmonics $\{e_n\}_{n\in\mathbb{Z}}$, since we are representing the signal f as a sum (over the integers $\mathbb{Z}$). In a *continuous integral representation* the right side of (2.1) is replaced by an integral such as the Calderón reproducing formula (4.2) below.

These continuous integral representations frequently depend on an underlying locally compact group. The *affine group* in Figure 1 is the underlying group associated with the Calderón reproducing formula or continuous wavelet transform. Section II of the volume is devoted to this topic, and the articles by Grossmann, Morlet, and Paul and Feichtinger and Gröchenig (both reprinted in this volume) are particularly important, cf., our preliminary remarks in Section 4.2 of this introduction.

Formally, the Fourier transform $\hat{f}\colon \mathbb{R}^d \to \mathbb{C}$ of $f\colon \mathbb{R}^d \to \mathbb{C}$ is defined as $\hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\gamma}\, dx$. τ_y denotes translation defined by $(\tau_y f)(x) = f(x-y)$. e_y denotes modulation defined by $(e_y f)(x) = e^{2\pi i x\cdot y} f(x)$. The next part of Figure 1 is described in the following definitions.

Definition 2.1 (Gabor and wavelet systems)

1. Let $\psi \in L^2(\mathbb{R})$. The associated *wavelet* or *affine system* is the sequence $\{\psi_{m,n} : (m,n) \in \mathbb{Z}\times\mathbb{Z}\}$, where $\psi_{m,n}$ is defined by

$$\psi_{m,n}(t) = 2^{m/2}\psi\left(2^m t - n\right). \tag{2.2}$$

Clearly,

$$\widehat{\psi_{m,n}}(\gamma) = 2^{-m/2}\left(e_{-n}\widehat{\psi}\right)(\gamma/2^m).$$

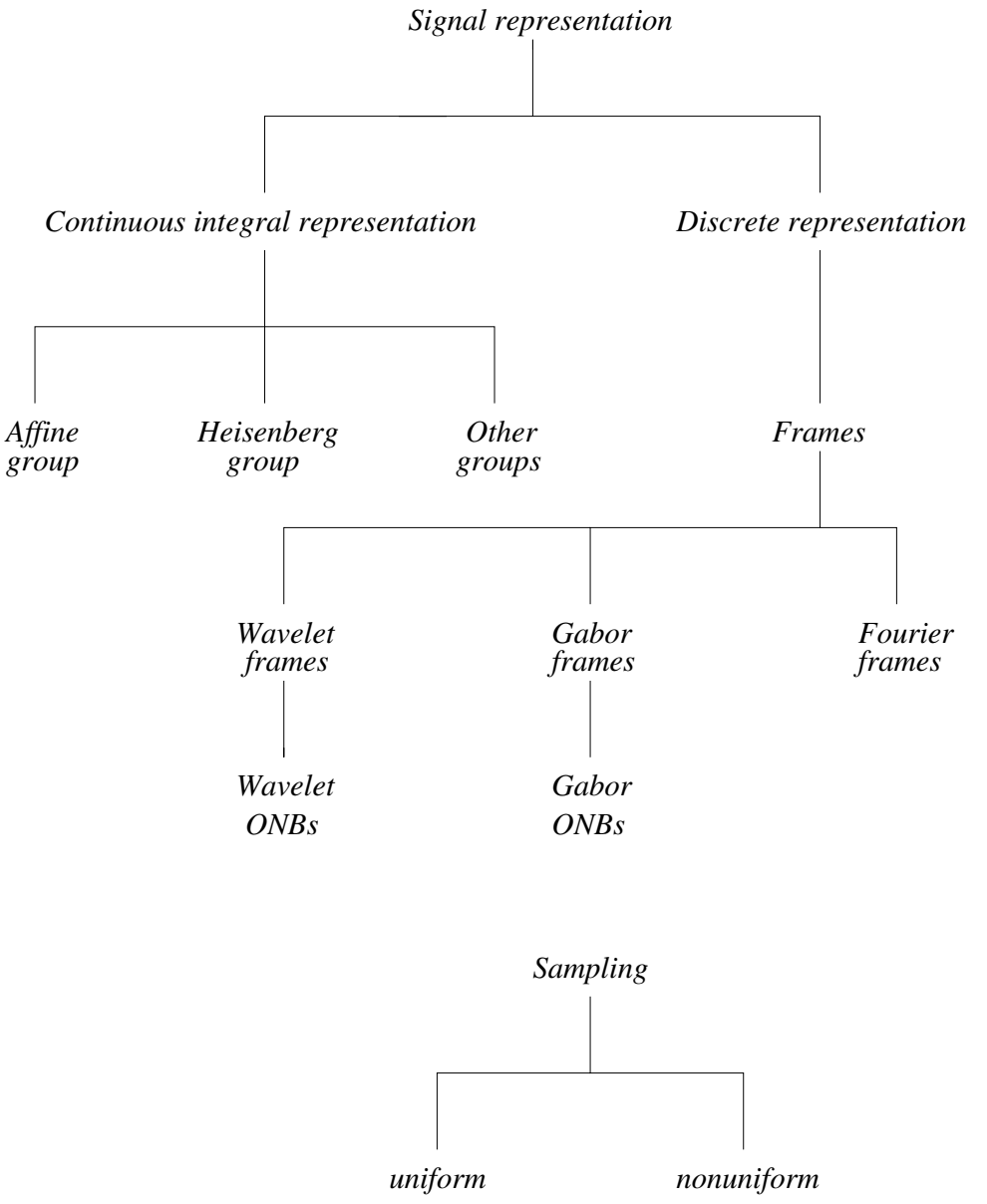

Figure 1. Signal representation

2. Let $g \in L^2(\mathbb{R})$ and let $a, b > 0$. The associated *Gabor* or *Weyl-Heisenberg system* is the sequence $\{g_{m,n} : (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$, where $g_{m,n}$ is defined by

$$g_{m,n}(t) = e^{2\pi i t m b} g(t - na).$$

Clearly,

$$\widehat{g_{m,n}}(\gamma) = \tau_{mb}(e_{-na}\widehat{g})(\gamma).$$

3. Let $\Lambda \subseteq \mathbb{R}$ be countable and let $R > 0$. The associated *Fourier system* is $\{e_\lambda : \lambda \in \Lambda\}$ considered as a subset of $L^2[-R, R]$.

Definition 2.2 (Bases and frames)

Let H be a separable Hilbert space and let $\{x_n : n \in \mathbb{Z}\} \subseteq H$ be a sequence in H.

1. The sequence $\{x_n\}$ is a *basis* or *Schauder basis* for H if each $x \in H$ has a unique decomposition
$$x = \sum_{n\in\mathbb{Z}} c_n(x)\, x_n \quad \text{in } H.$$
A basis $\{x_n\}$ for H is an *orthonormal basis* (ONB) for H if it is *orthonormal*.

2. A basis $\{x_n\}$ for H is an *unconditional basis* for H if

   $\exists\, C > 0$ such that $\forall\, F \subseteq \mathbb{Z}$, where $\operatorname{card} F < \infty$, and $\forall\, b_n, c_n \in \mathbb{C}$, where $n \in F$ and $|b_n| < |c_n|$,
$$\Big\| \sum_{n\in F} b_n x_n \Big\| \le C \Big\| \sum_{n\in F} c_n x_n \Big\|.$$
An unconditional basis is a *bounded unconditional basis* for H if
$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, n \in \mathbb{Z}, \quad A \le \|x_n\| \le B.$$

3. A basis for H is a *Riesz basis* if there is a bounded invertible operator on H mapping $\{x_n\}$ onto an ONB for H.

4. The sequence $\{x_n\}$ is a *frame* for H if there are $A, B > 0$ such that
$$\forall\, x \in H, \quad A\,\|x\|^2 \le \sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \le B\,\|x\|^2.$$
The constants A and B are *frame bounds*, and a frame is *tight* if $A = B$. A frame is an *exact frame* if it is no longer a frame whenever any of its elements is removed.

Remark 2.3. Frames give rise to discrete representations; see, for example, the articles by Duffin and Schaeffer, Daubechies, Grossmann, and Meyer, and Daubechies (all in this volume). It is natural to analyze wavelet, Gabor, and Fourier frames. The theme of this volume is the wavelet case in both the discrete and continuous setting. However, the Gabor and Fourier cases play a role even when the theme is wavelets.

Finally, the bottom of Figure 1 is meant to indicate that *sampling formulas* are discrete representations associated with various frame decompositions. For example, the Classical Sampling Formula (going back to Cauchy, see [BF01, Chapter 1]),
$$f = T \sum_{n\in\mathbb{Z}} f(nT)\, \tau_{nT} s, \tag{2.3}$$

is a discrete representation of functions f in the Paley-Wiener space of Ω-bandlimited functions, where $2T\Omega \leq 1$, and where the sampling function s is a $1/(2T)$-bandlimited function satisfying some natural properties. In the special case that $2T\Omega = 1$ and $\hat{s} = 1$ on $[-\Omega, \Omega]$, equation (2.3) gives rise to the so-called *Shannon wavelet* ONB for $L^2(\mathbb{R})$; and, in this case, the wavelet decomposition of functions $f \in L^2(\mathbb{R})$ that are not Ω-bandlimited provides an interpretation of aliasing error. The sequence $\{nT\}$ in (2.3) indicates *uniform sampling*; see Figure 1.

3. Frames

3.1 General frames

From the point of view of harmonic analysis, many of us learned about frames from the 1952 article of Duffin and Schaeffer reprinted in this volume, and then from the influential book by Young [You80], now deservedly enjoying a revised first edition. Duffin and Schaeffer defined frames in the Hilbert space setting, but their basic examples were Fourier frames; see Section 3.2.

From a functional analytic point of view, in 1921 Vitali (1875–1932) [Vit21] proved that, if $\{x_n\}$ is a tight frame with $A = B = 1$ and with $\|x_n\| = 1$ for all n, then $\{x_n\}$ is an ONB. Actually, Vitali's result is stronger for the setting $H = L^2[a, b]$ in which he dealt.

In 1936 Köthe [Köt36] proved that bounded unconditional bases are exact frames, and the converse is straightforward. Also, the category of Riesz bases is precisely that of exact frames. Thus, the following three notions are equivalent: Riesz bases, exact frames, and bounded unconditional bases. Besides the article by Duffin and Schaeffer, Bari's characterization of Riesz bases [Bar51] is fundamental in this realm of ideas. From my point of view, her work has all the more impact because it was motivated in part by her early research, with others in the Russian school, in analyzing Riemann's sets of uniqueness for trigonometric series.

Frames have also been studied in terms of the celebrated Naimark dilation theorem (1943), a special case of which asserts that any frame can be obtained by "compression" from a basis. The rank 1 case of Naimark's theorem is the previous assertion for tight frames. The finite decomposition rank 1 case of Naimark's theorem antedates Naimark's paper, and it is due to Hadwiger [Had40] and Gaston Julia [Jul42]. This is particularly interesting in light of modern applications of finite normalized tight frames in communications theory. Because they will arise later, we mention Chandler Davis' use of Walsh functions to give explicit constructions of dilations [Dav77]. Davis [Dav79] also provides an in-depth perspective on the results referred to in this paragraph.

Other applications of Naimark's theorem in the context of frames include feasibility issues for von Neumann measurements in quantum signal processing.

3.2 Fourier frames

Fourier frames go back to Dini (1880) and his book on Fourier series [Din80, pages 190 ff]. There he gives Fourier expansions in terms of the set $\{e_\lambda\}$ of harmonics, where each λ is a

solution of the equation

$$x \cos \pi x + a \sin \pi x = 0. \tag{3.1}$$

Equation (3.1) was chosen because of a problem in mathematical physics from Riemann's (1826–1866) and later Riemann-Weber's classical treatise [Rie76, 158–167]. Dini (1845–1918) returned to this topic in 1917, just before his death, with a significant generalization including Fourier frames that are not ONBs [Din54].

The inequalities defining a Fourier frame were explicitly written by Paley and Wiener [PW34, 115, inequalities (30.56)]. The book by Paley and Wiener [PW34] (and to a lesser extent a stability theorem by G. D. Birkhoff [Bir17]) had tremendous influence on mid-twentieth century harmonic analysis. Although nonharmonic Fourier series expansions were developed, the major effort in the study of Fourier systems emanating from [PW34] addressed completeness problems of sequences $\{e_\lambda\} \subseteq L^2[-R, R]$, that is, determining when the closed linear span of $\{e_\lambda\}$ is all of $L^2[-R, R]$. This culminated in the profound work of Beurling and Malliavin in 1962 and 1966 ([BM62], [BM67], [Koo96]; see [BF01, Chapter 1] for a technical overview).

A landmark in this intellectual journey to the heights of Beurling-Malliavin is the article by Duffin and Schaeffer. In retrospect, their paper was underappreciated when it appeared in 1952. The authors defined Fourier frames as well as the general notion of a frame for a Hilbert space H. They emphasized that frames $\{x_n\} \subseteq H$ provide discrete representations $x = \sum a_n x_n$ in norm, as opposed to the previous emphasis on completeness. They understood that the Paley-Wiener theory for Fourier systems is equivalent to the theory of exact Fourier frames. (We noted above that Paley and Wiener used precisely the inequalities defining Fourier frames.) Duffin and Schaeffer also knew that generally they were dealing with overcomplete systems, a useful feature in noise reduction problems.

The next step on this path created by Duffin and Schaeffer is the article by Daubechies, Grossmann, and Meyer reprinted in this volume. From the point of view of the affine and Heisenberg groups (see Figure 1), and inspired by Duffin and Schaeffer, the article by Daubechies, Grossmann, and Meyer establishes the basic theory of wavelet and Gabor frames. Given the nature of this volume, I shall say nothing about Gabor systems except to the extent that they have an impact on wavelet theory — which they do. On the other hand, the wavelet frame results of Daubechies, Grossmann, and Meyer allow us to segue into a broader discussion of wavelets in Section 4.

Before closing Section 3, I'd like to make a brief personal reminiscence about Richard Duffin. In 1990 I spoke about frames and some of their applications at the University of Pittsburgh. Pesi Masani (1919–1999) and Duffin (1909–1996) sat in the front row, both magisterial in their own ways. Masani, an expert on Hilbert spaces, understood the relevance of Naimark's theorem.

Duffin was amused and surprised that frames had reemerged as a tool and theory in time-scale and time-frequency analysis. We talked mathematics through dinner and much later, spirited on by a salubrious beverage or two. I was proud that we "hit it off." There were napkins on which to write (I wish I had kept his calculations), and he told me about networks and his student Raoul Bott, from whom I had taken topology in 1961. Mostly, he was very interested in discussing mathematics and at a genuinely technical level. He was 81

years old! Amazing. Duffin published his last paper with Hans Weinberger in the *Journal of Fourier Analysis and Applications* [DW97] in a lengthy issue on frames that was dedicated to his memory.

4. Wavelet Theory

4.1 A broad and selective outline

What is a *wavelet*? We have already answered this question by defining wavelet systems, frames, and ONBs, and we have commented on the continuous wavelet transform. For example, in the case of wavelet ONBs, $\psi \in L^2(\mathbb{R})$ is a *wavelet* if the sequence $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$, where $\psi_{m,n}$ was defined in (2.2), is orthonormal and if

$$\forall\, f \in L^2(\mathbb{R}), \quad f = \sum_{m,n\in\mathbb{Z}} \langle f, \psi_{m,n}\rangle \psi_{m,n} \quad \text{in } L^2\text{-norm},$$

where the inner products $\langle f, \psi_{m,n}\rangle$ are the *wavelet coefficients*.

What is *wavelet theory*? This is a much bigger question, one that was first addressed in [Mey90] and [Dau92]. It continues to be answered in diverse ways, as existing methods interact with other branches of mathematics, as the first rush of wavelet results has had a chance to regroup and evolve and mature, and as new applications have tested and created wavelet-based algorithms. Wavelet theory has developed into an imposing mathematical edifice with vitality and depth, as well as with emerging limitations and baroque tendencies. The scope of its applicability exhibits a similar effectiveness and limitation. Furthermore, it is a relief to assert that all that glitters is not a wavelet! Not that anyone ever said that wavelets were a panacea, but, as indicated earlier, there was definitely a period of overprescription of them.

The first wavelet ONB was constructed by Haar in his 1909 dissertation (translated in this volume). The *Haar wavelet* h for the setting of $L^2(\mathbb{R})$ is defined as

$$h(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2), \\ -1 & \text{if } t \in [1/2, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Haar's work was followed by Walsh's 1923 construction [Wal23] of ONBs in terms of so-called *Walsh functions*. Actually, about 1900 and without any interest in ONBs, engineers (especially, J. A. Barrett) designed transposition schemes in open-wired lines based on Walsh functions, and they used these schemes to minimize channel "crosstalk." The sequence of Walsh functions on $\mathbb{R}$ is the prototype of *wavelet packets*, just as the Haar wavelet system $\{h_{m,n}\}$ is the prototype of wavelet ONBs on $\mathbb{R}$. The theory of wavelet packets is due to Coifman, Meyer, and Wickerhauser, for example, [Wic94].

It would be nice to give a sequential litany of waveleteers, beginning with Haar and Walsh and taking us to Meyer and Daubechies. Unfortunately, it didn't happen linearly. It is true that the construction of wavelet ONBs is part of a program to construct unconditional bases for many of the important function spaces in analysis. Hence there is a certain lineage after Walsh from the article of Franklin (reprinted in this volume) in 1928, to Ciesielski [Cie81], to Carleson [Car80], and to the article by Strömberg (reprinted in this volume).

However, there were many other paths to the establishment of wavelet theory, not all of which were fully appreciated by the late 1980s. Briefly, and before the appearance of [Mey90] and [Dau92], there were wavelet-oriented traditions and/or developments in spline and approximation theory, in speech and image processing, and in atomic decompositions and the Calderón reproducing formula.

In any case, Carleson's construction of an unconditional basis for the Hardy space H^1 in 1980 led to the spline wavelet ONBs in 1981 in the article by Strömberg. Besides Carleson's construction, there was also the construction of Billard [Bil72]. Using cardinal B-splines, Battle (reprinted in this volume) and Lemarié independently constructed wavelet ONBs in the late 1980s *in the context of* wavelet theory. Research on spline wavelets continues to the present day. Further, there are natural relationships between other aspects of approximation theory and wavelet theory; see, for example, the article by DeVore, Jawerth, and Popov reprinted in this volume.

In harmonic analysis, Coifman's striking decomposition theorem [Coi74] provided a basic theme in the definitive essay by Coifman and Weiss (reprinted in this volume) for the Hardy spaces H^p, $0 < p \leq 1$. This theory had an influence on the development of wavelet theory at the level of expansions in terms of "atoms" (harmonics) having vanishing moments. It is natural, but not necessary, that the harmonics $\psi_{m,n}$ of a wavelet expansion have vanishing moments.

We shall now give a little more detail about the topics in wavelet theory that we have just sketched.

4.2 The Calderón reproducing formula

As mentioned earlier, the Calderón reproducing formula [Cal64, Section 34] is now synonymous with the so-called *continuous wavelet transform.* Calderón's formula is

$$f(t) = \int_0^\infty (\psi_{1/u} * \psi_{1/u} * f)(t)\, \frac{du}{u}, \tag{4.2}$$

where the dilation ψ_x is defined as $\psi_x(t) = x\psi(tx)$ for $x > 0$. Equation (4.2) has been a major influence from harmonic analysis on wavelet theory, and it can be considered as a continuous ("overcomplete") wavelet decomposition of $f \in L^2(\mathbb{R})$. We have used the term "wavelet" in the sense of describing f as an integral ("sum") whose harmonics are dilates and translates of a fixed function ψ. In fact, equation (4.2) is

$$f(t) = \int_0^\infty \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi_{1/u}(v-w) f(w) dw \right) \psi_{1/u}(v-t)\, \frac{dv\, du}{u},$$

and so the "sum" we have alluded to is the double integral

$$\int_0^\infty \int_{\mathbb{R}} \cdots \frac{dv\, du}{u}$$

taken over the temporal or spatial variable v and the dilation u. In this case the "wavelet coefficients" of f are

$$\int_{\mathbb{R}} \psi_{1/u}(v-w)\, f(w)\, dw, \quad v \in \mathbb{R}, \quad u > 0.$$

The formal verification of Calderón's formula is elementary, as long as ψ satisfies certain properties. In fact, the Fourier transform of the right side of (4.2) is

$$\widehat{f}(\gamma) \int_0^\infty \widehat{\psi}(u\gamma)^2 \, \frac{du}{u};$$

and there are even, real-valued, compactly supported, infinitely differentiable functions ψ for which

$$\forall \, \gamma \in \mathbb{R} \backslash \{0\}, \quad \int_0^\infty \widehat{\psi}(u\gamma)^2 \, \frac{du}{u} = 1.$$

See the masterpiece on this topic by Frazier, Jawerth, and Weiss [FJW91].

4.3 Haar ONB

In Haar's 1909 dissertation (translated in this volume), where he constructs the Haar ONB, his historical perspective includes contributors such as Poisson, Riemann, Cantor, du Bois-Reymond, Fejér (another Weisz in the field!), and Hilbert. We shall trace his result through the twentieth century, where a host of new names and ideas emerges.

Hoping against hope, many nineteenth-century harmonic analysts desired that Fourier series of continuous functions f converge everywhere or even uniformly to f. Of course, du Bois-Reymond's 1872 example [dBR76] dashed these dreams. Haar's dissertation provides a positive solution to Hilbert's question of finding an ONB $\{h_{m,n}\}$, viz., the Haar ONB, for $L^2[0,1]$ for which $f = \sum \langle f, h_{m,n} \rangle \, h_{m,n}$ uniformly for every continuous 1-periodic function on $\mathbb{R}$. It is not unexpected that the Haar system is unbounded in supremum norm.

From the point of view of Fourier series, Lusin's conjecture [Lus13] directed the reaction of the du Bois-Reymond example to the problem of determining if $S(f) = f$ a.e. for $f \in L^2[0,1]$. This problem was solved affirmatively by Carleson [Car66] and then by Charles Fefferman [Fef73]. We mention this because of the influence of these results on a recent and seemingly preordained phase of wavelet theory, viz., the study of wave packets, not to be confused with wavelet packets. On the road to the solution of Lusin's conjecture, there were two significant results by Kolmogorov in the 1920s. Kolmogorov (1922) proved that there exist $f \in L^1[0,1]$ for which $S(f)$ diverges everywhere [Kol26]. Also, with the Lusin conjecture in mind, Kolmogorov proved that if $f \in L^2[0,1]$, then $S(f) = f$ a.e. when the sums are taken over dyadic blocks [Kol24]. The proof is elementary but ingenious. The extension to $L^p[0,1]$, $p > 1$ is deep and is an integral part of the original Littlewood-Paley theory [LP31], [LP37]. Furthermore, such convergence over dyadic blocks has a natural dyadic wavelet interpretation; see [Mey90].

Besides the aforementioned (Sections 3.1 and 4.1) Walsh functions, to which we shall return in Section 4.4, there were two other Haar-related sequels published in 1928, both of which are inspired exclusively by Haar's article. The first is the article by Philip Franklin reprinted in this volume and the second is the article by Juljusz Schauder [Sch28].

Philip Franklin (1898–1965) addressed and solved the problem of constructing an orthonormal basis $\{f_n\}$ of continuous 1-periodic functions on $\mathbb{R}$ such that $f = \sum c_n(f) \, f_n$ uniformly for every continuous 1-periodic function on $\mathbb{R}$. He did this by orthogonalizing the integrals of the Haar functions. At a personal level, Franklin met and became friendly with Norbert Wiener in 1918 at the U.S. Army Proving Ground in Aberdeen, Maryland.

Both were computers, working on noisy hand-computing machines known as "crashers" — a time-invariant scientific bottom line! Wiener became Franklin's colleague at MIT in 1919, and the two later became brothers-in-law.

Franklin's story for this volume serves as background for Strömberg's article, where "Franklin wavelets" ψ are constructed with the property that $\{\psi_{m,n}\}$ is an ONB for $L^2(\mathbb{R}^d)$. Besides the article by Franklin, Strömberg's work had other influences (some mentioned in Section 4.1), including a body of work by Bočkarev, Carleson, Ciesielski, Domsta, Maurey, Pelcynski, Simon, Sjölin, and Wojtaszczyk from the 1960s to 1980s. One of their themes was to prove the equivalence or not of various bases. For example, Ciesielski, Simon, and Sjölin [CSS77] proved that the Haar and Franklin systems are equivalent in $L^p[0,1]$, $1 < p < \infty$. This brings us back to the 1928 article by Schauder, where he proved that the Haar ONB for $L^2[0,1]$ is a basis for $L^p[0,1]$, $p \geq 1$. Schauder's view helped to set the stage long ago for the wonderful wavelet characterization of Besov spaces and, along with Calderón's profound influence, for the wavelet relationship with Littlewood-Paley-Stein theory; for example, see Section 4.6, Peetre's classic [Pee76], the 1987 article by Meyer translated in this volume, and Meyer's treatise [Mey90]. For perspective, it is well to recall that the Besov spaces $B_p^{s,q}$ are generalizations of the Sobolev spaces (e.g., $W^{s,2} = B_1^{s,2}$) and the Hölder spaces $C^s = B_\infty^{s,\infty}$.

It is interesting to note the following inherent property, one might say limitation, of the Haar or any multiresolution analysis ONB in the setting of $L^2(\mathbb{R})$. If we have a discrete wavelet representation $f = \sum \langle f, h_{m,n} \rangle \, h_{m,n}$ in $L^2(\mathbb{R})$, where h is the Haar wavelet defined by (4.1), then there is "leakage" to infinity of the supports of the $h_{m,n}$ with nonvanishing coefficients $\langle f, h_{m,n} \rangle$, even in the case that f is compactly supported (e.g., see [Dau92]).

4.4 Walsh ONB

Concerning the influence and importance of historical perspective, as we mentioned with Haar, Joseph L. Walsh (1895–1973) could wax rhapsodic. For example, in discussing the Riesz-Fischer theorem in the context of the Walsh ONB, he wrote that its "beauty and simplicity ... was, and still is, almost overwhelming" [Bas70]. Walsh was of course aware of Haar's work, and later he advertised Franklin's theorem. On the other hand, he was probably sensitive to priority vis-à-vis the space of orthonormal Rademacher functions, even though the latter was not an ONB. Rademacher [Rad22] published his results in 1922 and Walsh in 1923, but the discoveries were independent. Rademacher's manuscript was received by the editors of *Mathematische Annalen* on October 8, 1921, and Walsh announced his results to the American Mathematical Society at a meeting on February 25, 1922 (his paper [Wal23] is dated May 1922).

Walsh understood some of the essential differences between the Haar and Walsh ONBs, especially concerning oscillation properties analogous to the trigonometric functions. In his recursive definition, Walsh ordered the Walsh functions according to the average number of zero-crossings of these functions on $[0,1]$; this ordering was also used by Kaczmarz [Kac29], and it is referred to as *sequency ordering*. Another natural ordering of the Walsh functions is due to Paley (1932); it is based on the *binary ordering* of indices. To be specific, the sequence $\{r_n : n = 1, \ldots\}$ of *Rademacher functions* on $[0,1)$ is defined by the 1-periodic

functions

$$\forall\, n = 1, \ldots, \qquad r_n(t) = \operatorname{sgn}(\sin 2^n \pi t) \quad \text{on } \mathbb{R}.$$

Paley's binary ordering is based on his theorem asserting that Walsh's original functions w_n can be written as $w_0 = 1$ and

$$\forall\, n = 1, \ldots, \qquad w_n(t) = \prod_{\epsilon_j = 1} r_j(t) \quad \text{on } \mathbb{R},$$

where $n = \sum_{j=1}^{\infty} \epsilon_j 2^{j-1}$ is the binary expansion of n. A third ordering is based on the orthogonal ± 1 matrices of Sylvester (1867) and Hadamard (1893). It is sometimes called the *Kronecker ordering* and is essentially a binary bit inversion of the binary ordering.

Walsh knew that the Haar and Walsh systems were Hadamard transforms of each other, and that $\{w_n\}$ was uniformly bounded, as opposed to the Haar system. He was not aware that $\{w_n\}$ is the discrete dual group of the compact dyadic group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$. ($\mathbb{Z}_2$ is the discrete cyclic group of order 2, not the 2-adic integers.) This duality theorem was proved by Vilenkin [Vil47] and Fine [Fin49]. Paley and Wiener, in their foray into the duality theory of locally compact abelian groups, had announced a similar result at the ICM in Zurich (1932).

The peroration for this review of properties of Walsh functions, especially as compared with the Haar system, is the ultimate distinction (alluded to in Subsection 4.1) between these two systems. This distinction turns out to be a consequence of wavelet theory: the Haar system on $\mathbb{R}$ is a multiresolution analysis wavelet ONB, and the Walsh system on $\mathbb{R}$ is its corresponding family of wavelet packets!

As a personal postscript to this subsection, I was a student of Walsh in potential theory in 1960, and we met again as colleagues at the University of Maryland when we both arrived there in 1965. In the early 1970s he suggested that I study Walsh functions since there were many problems he thought I'd be interested in. Those were the heady days of spectral synthesis, and I remained blissfully ignorant of Walsh functions, especially their applications, until the 1990s. To those less recalcitrant than I, the *Proceedings* from 1970–1974 of *Applications of Walsh Functions*, for example, [SS74], may provide an archetype of things to come and are certainly a quantitative and fascinating portrait of things past.

4.5 Filters and an early patent

On July 29, 1983, Goupillaud, Grossmann, and Morlet filed for a patent on signal representation generators. The patent, based on work of Morlet [Mor81], [MAFG82], was awarded in 1986.

Morlet's idea [Mor81] was to analyze seismic traces by means of sequences of harmonics each having a fixed shape. The trace s can be considered as the real part of a signal f whose Fourier transform is causal, that is, $f \in H^2$. He designed these harmonics to be translates and dilates of a single function ψ. In particular, there are the same number of cycles for high, medium, and low frequencies. The reconstruction of the trace s is then effected by its sequence of "sampled values" $\langle s, \psi_{m,n} \rangle$ (wavelet coefficients) in terms of a wavelet representation. Morlet originally used modulated Gaussians ψ.

As Goupillaud reminded us in 1997, Morlet's set $\{\psi_{m,n}\}$ of harmonics necessarily cannot be orthonormal or linearly independent, because it was vital to achieve noise reduction as well as stable representation in the physical problems being addressed.

Furthermore, Morlet's beautiful idea was first quantified with the proper mathematical tools by Alex Grossmann. This was a significant scientific contribution, fortuitously juxtaposed with the fact that Daubechies was working with Grossmann. Apparently, Roger Balian, who formulated the ONB version of the Balian-Low uncertainty principle for exact Gabor frames, had advised for the Morlet-Grossmann connection, which led to a large body of results including [GGM84] as well as their article reprinted in this volume. Despite my proclivity to extol the many virtues of Gabor systems, these systems arise in the Goupillaud, Grossmann, and Morlet patent as a technological device with shortcomings (inherent undersampling problems in analyzing high frequencies), which are overcome by the Morlet approach.

The first smooth wavelet ONBs for $L^2(\mathbb{R}^d)$ were constructed by Strömberg (reprinted in this volume), for ψ m-times continuously differentiable with exponential decay, and by Meyer (1985), for bandlimited ψ in the Schwartz class on $\mathbb{R}$ and by Lemarié on $\mathbb{R}^d$; see the article by Lemarié and Meyer (translated in this volume). We have already mentioned Daubechies' construction of a compactly supported m times continuously differentiable orthonormal wavelet in 1987. In his Zygmund lectures, Meyer used Mallat's newly packaged (1986) concept of multiresolution analysis (MRA) to prove the Daubechies theorem, see the articles by Mallat reprinted in this volume.

We have introduced MRAs in the same breath while discussing patents because of the signal processing origins of MRAs. These origins are represented by the articles in Section I. At the risk of oversimplification, the article by Burt and Adelson reprinted in this volume and their article [BA83] are a substantial precursor for the structural nature of MRAs, and the remaining articles in Section I provide the decidedly nontrivial nuts and bolts for constructing a wavelet ψ by means of conjugate mirror filters (CMFs) arising in the Fourier analysis of an MRA. Nowadays, the details to clarify and verify the claims in the previous teutonic sentence can be found in many places, but one can do no better than reading the triune treatises by Meyer [Mey90], Daubechies [Dau92], and Mallat [Mal98]. A beautiful component in establishing the relation between MRAs and CMFs is Albert Cohen's equivalence theorem (translated in this volume) for these notions in the case that the Fourier transform of the scaling function is in each of the Sobolev spaces $W^{m,2}(\mathbb{R})$, $m \in \mathbb{N}$. Another gem in this area is the article by Lawton (reprinted in this volume), which constructs compactly supported tight frames for a given trigonometric polynomial CMF.

I had the good fortune to consult for The MITRE Corporation in Washington, D.C., for many years. Consultation for me included an enlightening and respectful exposure to engineering excellence. MITRE's Signal Processing Group was actively involved in designing algorithms for digitizing voice. By 1981, it was not only implementing CMFs from the work of Esteban and Galand and Crochiere, Webber, and Flanagan (both in this volume), but also from a host of related work, for example, Croisier, Esteban, and Galand [CEG76] and Barnwell [BI81]. By 1987, we had discovered (along with many others in industrial or government laboratories) the groundbreaking article by Smith and Barnwell reprinted

in this volume, along with [SBI86] and [SBI87], as well as the article by Vetterli in this volume and his earlier work [Vet84]. Furthermore, besides the article by Vaidyanathan in this volume, his book [Vai93] became a staple in our group.

4.6 Harmonic analysis and wavelets in the 1980s

At this point, and dealing with wavelet theoretic harmonic analysis in the mid-1980s, we have commented on the work of Cohen, Daubechies, Lemarié, Mallat, and Meyer. There was contemporary, as well as comparably creative and fundamental, wavelet theoretic harmonic analysis produced by Frazier and Jawerth and Feichtinger and Gröchenig (both in this volume).

An impetus for the work of Frazier and Jawerth was Uchiyama's smooth atomic decomposition of $L^2(\mathbb{R}^d)$ [Uch82], which is a consequence of the Calderón reproducing formula. Bidisc and parabolic versions appeared before the Euclidean version! Another impetus was Michael Wilson's use of the Calderón formula to obtain a smooth atomic decomposition of $B_1^{0,1}$. This led to the Frazier-Jawerth smooth atomic decomposition of $B_p^{s,q}$, and then to Triebel-Lizorkin spaces. They also replaced these smooth atomic decomposition methods with the ϕ-transform, which allows representation independent of the function being decomposed. They had therefore obtained a discrete reformulation of the Littlewood-Paley-Stein theory! Their work was influenced by variations on Hardy space decompositions due to Coifman-Rochberg [CR80] and Ricci-Taibleson [RT83]. The Frazier-Jawerth article in this volume on wavelet frame representations of Besov spaces was submitted on September 17, 1984. The subsequent results on Triebel-Lizorkin spaces were obtained shortly thereafter, but their papers containing them "grew to maturity" before appearing in 1988 and 1990.

The harmonic analysis background for the article by Frazier and Jawerth was firmly in the realm of the Zygmund-Calderón "school." The harmonic analysis background for the article by Feichtinger and Gröchenig is more abstract and equally important.

When I first tried to understand Feichtinger and Gröchenig's atomic decomposition theory (actually, it was one of their infamous preliminary versions), I was unprepared to plumb its depths. It has aged lucidly — a quality personally sought by said plumber. It remains a creative tour d'horizon, extending its tentacles to Banach frames and flexing its formidable technology into a well-developed methodology to address new problems in wavelet theory and its applications. Feichtinger and Gröchenig use representation theory in a fundamental way. They deal with algebraic and structural aspects of classical topics such as Wiener's Tauberian theory. For example, let $S_0(\mathbb{R}^d)$ be the smallest Segal algebra isometrically invariant under translation and modulation. Then, $S_0(\mathbb{R}^d)$ can be identified with the Wiener amalgam space whose global norm is determined by $\ell^1(\mathbb{Z}^d)$ and whose local norm is determined by the space of absolutely convergent Fourier transforms. Their approach to wavelet theory and the construction of unconditional bases uses their methodology for analyzing coorbit spaces. In particular, if one considers the Schrödinger representation of the Heisenberg group, then $S_0(\mathbb{R}^d)$ is obtained as the coorbit of $L^1(\mathbb{R}^{2d})$. With this approach, and independent of the Calderón-Zygmund theory, they proved that sufficiently structured orthonormal wavelets for $L^2(\mathbb{R}^d)$ give rise to unconditional bases for all of the corresponding coorbit spaces including Besov and Triebel-Lizorkin spaces [FG88].

5. Conclusion

At the beginning of section 4, we asked: What is wavelet theory?

For me the answer is both theoretical and concrete. On the one hand, in harmonic analysis, wavelet theory is a natural continuation in the history of *some* of the ideas that define our subject. On the other hand, wavelet theory has become an effective tool to address *some* problems in engineering, mathematics, and the sciences. As such, it also provides a unifying methodology allowing for the possibility of genuine communication between diverse groups. The articles herein indicate a protean body of knowledge so that my answer to the above question is assuredly only one of many. Similarly, this introduction may seem idiosyncratic to others who have also thought about wavelet theory. I hope I have not crafted a procrustean bed with questionable resemblance to the spirit of the articles that follow.

Finally, just as I began by saying that the articles in this volume needed no introduction, so too the introducers. I shall not resist listing two among my favorite publications by each of them: Jelena Kovačević [VK95] and [GKK01]; Jean-Pierre Antoine [AAG00] and [AKLT00]; Hans Feichtinger [FS03] and [Fei02]; Yves Meyer [Mey72] and [JM96]; Guido Weiss [HW96] and [CCMW02]; and Victor Wickerhauser [Wic94] and [Wic03].

The denouement of wavelet theory may lie in the future, or may have already been integrated in the present, but this volume represents its virtuosic roots.

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