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## Introduction

One of the basic facts of complex analysis is the exactness of the de Rham complex of *sheaves* of analytic differential forms on a smooth complex analytic space. In its turn, its proof is based on the fact that every point of such a space has an open neighborhood isomorphic to an open polydisc, which reduces the verification of the exactness to the classical Poincaré lemma. The latter states that the de Rham complex of *spaces* of analytic differential forms on an open polydisc is exact. Its proof actually works over any non-Archimedean field  $k$  of characteristic zero as well, and so it implies also that the de Rham complex of sheaves of analytic differential forms on a smooth  $k$ -analytic space (as introduced in [Ber1] and [Ber2]) is exact at every point that admits a fundamental system of étale neighborhoods isomorphic to an open polydisc. One can show (Corollary 2.3.3) that a point  $x$  of a smooth  $k$ -analytic space possesses the above property if and only if the non-Archimedean field  $\mathcal{H}(x)$ , associated with the point  $x$ , possesses the property that its residue field  $\widetilde{\mathcal{H}(x)}$  is algebraic over  $\widetilde{k}$  and the group  $|\mathcal{H}(x)^*|/|k^*|$  is torsion.

It is a distinctive feature of non-Archimedean analytic spaces that the subset  $X_{\text{st}}$  of points with the latter property does not coincide with the whole space  $X$ . Notice that  $X_{\text{st}}$  contains the set  $X_0 = \{x \in X \mid |\mathcal{H}(x) : k| < \infty\}$  (the underlying space of  $X$  in rigid analytic geometry) and, in particular, the set of  $k$ -rational points  $X(k) = \{x \in X \mid \mathcal{H}(x) = k\}$ . Although  $X$  is locally arcwise connected, the topology induced on  $X_{\text{st}}$  is totally disconnected and, if the valuation on  $k$  is nontrivial,  $X_{\text{st}}$  is dense in  $X$ . Moreover, if  $X$  is smooth,  $X_{\text{st}}$  is precisely the set of points at which the de Rham complex is exact and, in fact, for every point  $x \notin X_{\text{st}}$  there is a closed one-form, defined in an open neighborhood of  $x$ , that has no primitive at any étale neighborhood of  $x$ .

We now recall that a locally analytic function is a map  $f : X(k) \rightarrow k$  such that, for every point  $x \in X(k)$ , there is an analytic function  $g$  defined on an open neighborhood  $U$  of  $x$  with  $f(y) = g(y)$  for all  $y \in U(k)$ . It is clear that the local behavior of such a function does not determine its global behavior. For example, if its differential is zero, the function is not necessarily constant. On the other hand, for a long time number theorists have been using very natural locally analytic functions possessing certain properties that make them look like analytic ones. An example of such a function (for  $X = \mathbf{G}_m = \mathbf{A}^1 \setminus \{0\}$ ) is a homomorphism  $k^* \rightarrow k$  from the multiplicative to the additive group of  $k$  which extends the homomorphism  $a \mapsto \log(a)$  on the subgroup  $k^1 = \{a \in k^* \mid |a - 1| < 1\}$ , where  $\log(T)$  is the usual logarithm defined by the power series  $-\sum_{i=1}^{\infty} \frac{(1-T)^i}{i}$  (convergent on  $k^1$ ).

Let us assume (till the end of the introduction) that  $k$  is a closed subfield of  $\mathbf{C}_p$ , the completion of the algebraic closure  $\overline{\mathbf{Q}}_p$  of the field of  $p$ -adic numbers  $\mathbf{Q}_p$ .

Then such a homomorphism is uniquely determined by its value at  $p$ , and the homomorphism, whose value at  $p$  is an element  $\lambda \in k$ , is denoted by  $\log^\lambda(T)$  and is called a branch of the logarithm. One of the properties we had in mind states that, if  $X$  is an open annulus in  $\mathbf{A}^1$  with center at zero and the differential of a locally analytic function on  $X(k)$  of the form  $\sum_{i=0}^n f_i \log^\lambda(T)^i$  with  $f_i \in \mathcal{O}(X)$  is equal to zero in an open subset of  $X(k)$ , then the function is a constant and, in fact,  $f_0 \in k$  and  $f_i = 0$  for all  $1 \leq i \leq n$ . Notice also that every one-form on  $X$  with coefficients of the above form has a primitive which is a locally analytic function of the same form.

It was an amazing discovery of R. Coleman ([Col1], [CoSh]) in the early 1980s that there is a way to construct primitives of analytic one-forms and their iterates in the class of locally analytic functions on certain smooth  $k$ -analytic curves, called by him basic wide opens (they are closely related to basic curves considered here), such that the primitives are defined up to a constant. Namely, given a branch of the logarithm  $\log^\lambda(T)$ , he constructed for every such curve  $X$  an  $\mathcal{O}(X)$ -algebra  $A(X)$  of locally analytic functions filtered by free  $\mathcal{O}(X)$ -modules of finite rank  $A^0(X) \subset A^1(X) \subset \dots$  with  $dA^i(X) \subset A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$  and such that

- (a)  $A^0(X) = \mathcal{O}(X)$ ;
- (b) every function from  $A(X)$  with zero differential is a constant;
- (c)  $A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X) \subset dA^{i+1}(X)$ ;
- (d)  $A^{i+1}(X)$  is generated over  $\mathcal{O}(X)$  by primitives of one-forms from  $A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ ;
- (e) if  $X$  is an open annulus with center at zero, then  $\log^\lambda(T) \in A^1(X)$ ;
- (f) for a morphism  $X' \rightarrow X$  and a function  $f \in A^i(X)$ , one has  $\varphi^*(f) \in A^i(X')$ .

Moreover, if a function in  $A(X)$  is equal to zero on a nonempty open subset of  $X(k)$ , it is equal to zero everywhere. Thus, if  $\omega$  is a one-form in  $A(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ , then for any pair of points  $x, y \in X(k)$  one can define an integral  $\int_x^y \omega \in k$ . It follows also that, given a coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with a unipotent integrable connection, the kernel of the induced connection on the  $\mathcal{O}(X)$ -module  $\mathcal{F}(X) \otimes_{\mathcal{O}(X)} A(X)$  is a vector space of dimension equal to the rank of  $\mathcal{F}$  and, therefore, for any pair of points  $x, y \in X(k)$ , one can define a parallel transport  $T_{x,y} : \mathcal{F}_x^\vee \xrightarrow{\sim} \mathcal{F}_y^\vee$ .

Since then there have been several attempts to extend Coleman's work to higher dimensions. Coleman himself ([Col2]) constructed an integral  $\int_x^y \omega$  of a closed analytic one-form  $\omega$  on the analytification  $\mathcal{X}^{\text{an}}$  of a connected projective scheme  $\mathcal{X}$  with good reduction. Yu. Zarhin ([Zar]) and P. Colmez ([Colm]) constructed similar integrals for arbitrary connected smooth  $\mathcal{X}$  (see Remark 9.1.3(ii)). A. Besser ([Bes]) constructed iterated integrals on the generic fiber  $\mathfrak{X}_\eta$  of a connected smooth formal scheme  $\mathfrak{X}$ , which is an open subscheme of a formal scheme  $\mathfrak{Y}$  of finite type over  $k^\circ$  such that the Zariski closure of the closed fiber  $\mathfrak{X}_s$  in  $\mathfrak{Y}_s$  is proper (see Remark 8.1.5). V. Vologodsky ([Vol]) constructed a parallel transport  $T_{x,y}$  on  $\mathcal{X}^{\text{an}}$  for arbitrary connected smooth  $\mathcal{X}$  (see Remark 9.4.4). All of these constructions gave additional evidence for a certain phenomenon of local analytic nature which was already present in the work of Coleman and is described as follows.

Let us call a naive analytic function on a smooth  $k$ -analytic space  $X$  a map that associates to every point  $x \in X_{\text{st}}$  an element  $f(x) \in \mathcal{H}(x)$  such that there is an

analytic function  $g$  defined at an open neighborhood  $U$  of  $x$  with  $f(y) = g(y)$  for all  $y \in U_{\text{st}}$ . This class of functions is better than that of locally analytic ones since, for every closed subfield  $k \subset k' \subset \mathbf{C}_p$  and every naive analytic function  $f$  on  $X$ , one can define the pullback of  $f$  on  $X \widehat{\otimes}_k k'$ . For example, the locally analytic function  $\log^\lambda(T)$  is the restriction to  $k^*$  of a natural naive analytic function  $\text{Log}^\lambda(T)$  on the multiplicative group  $\mathbf{G}_m$ , and in fact for a basic curve  $X$  all locally analytic functions in Coleman's algebra  $A(X)$  are restrictions to  $X(k)$  of natural naive analytic functions on  $X$ . If now  $\mathfrak{n}(X)$  denotes the space of naive analytic functions on  $X$ , the correspondence  $U \mapsto \mathfrak{n}(U)$  is a sheaf of  $\mathcal{O}_X$ -algebras denoted by  $\mathfrak{n}_X$ . Coleman's work was actually evidence for the fact that, for a fixed branch of the logarithm, every smooth  $k$ -analytic space  $X$  is provided with an  $\mathcal{O}_X$ -subalgebra of  $\mathfrak{n}_X$  whose associated de Rham complex is exact and in which the kernel of the first differential coincides with the sheaf of constant analytic functions  $\mathfrak{c}_X = \text{Ker}(\mathcal{O}_X \xrightarrow{d} \Omega_X^1)$ .

The purpose of this work is to show that such an  $\mathcal{O}_X$ -subalgebra  $\mathcal{S}_X$  of  $\mathfrak{n}_X$  exists and is unique with respect to certain very natural properties. More precisely,  $\mathcal{S}_X$  is a filtered  $\mathcal{O}_X$ -algebra with  $d\mathcal{S}_X^i \subset \mathcal{S}_X^i \otimes_{\mathcal{O}_X} \Omega_X^1$  for all  $i \geq 0$ , and the properties are similar to (a)–(f) from above. Although we are not yet able to prove the exactness of the whole de Rham complex for  $\mathcal{S}_X$ , we show that the de Rham complex is exact at  $\Omega^1$  and the kernel of the first differential coincides with  $\mathfrak{c}_X$ . In particular, under a certain natural assumption (which is automatically satisfied if  $k = \mathbf{C}_p$ ) one can define an integral  $\int_\gamma \omega$  of a closed one-form  $\omega \in (\mathcal{S}_X \otimes_{\mathcal{O}_X} \Omega_X^1)(X)$  along a path  $\gamma : [0, 1] \rightarrow X$  with ends in  $X(k)$ . Furthermore, the extended class of functions contains a full set of local solutions of all unipotent differential equations. In fact, a coherent  $\mathcal{O}_X$ -module with an integrable connection has a full set of local horizontal sections in the étale topology if and only if it is locally unipotent in the étale topology (such a module is called here locally quasi-unipotent). As a consequence, we construct parallel transport along a path and an étale path of local horizontal sections of locally unipotent and locally quasi-unipotent modules, respectively. In comparison with the constructions mentioned above, both integral and parallel transport depend nontrivially on the homotopy class of a path and not only on its ends.

The filtered  $\mathcal{O}(X)$ -algebra  $A(X)$ , constructed by Coleman for a basic curve  $X$ , appears here in the following way. One can show that for such  $X$  the group  $H^1(X, \mathfrak{c}_X)$  is zero and, therefore, every one-form  $\omega \in (\mathcal{S}_X^i \otimes_{\mathcal{O}_X} \Omega_X^1)(X)$  has a primitive in  $\mathcal{S}^{i+1}(X)$ . Then  $A^0(X) = \mathcal{O}(X)$  and, for  $i \geq 0$ ,  $A^{i+1}(X)$  is generated over  $\mathcal{O}(X)$  by the primitives of all one-forms  $\omega \in A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$  in  $\mathcal{S}^{i+1}(X)$ . The algebra  $\mathcal{S}(X)$  is in fact much bigger than  $A(X)$ . For example, if  $X$  is an open annulus, then  $A(X) = \mathcal{O}(X)[\log^\lambda(T)]$ , but the  $\mathcal{O}(X)$ -modules  $\mathcal{S}^i(X)/\mathcal{S}^{i-1}(X)$  are of infinite rank for all  $i \geq 1$ . By the way, the latter is even true for the projective line  $\mathbf{P}^1$  (see Lemma 8.5.2).

In fact the sheaves  $\mathcal{S}_X$  are constructed in a more general setting. The reason for doing so is as follows. Let  $X$  be the Tate elliptic curve which is the quotient of  $\mathbf{G}_m$  by the discrete subgroup generated by an element  $q \in k^*$  with  $|q| \neq 1$ , and let  $\omega$  be the invariant one-form on  $X$  whose preimage on  $\mathbf{G}_m$  is  $\frac{dT}{T}$ . The curve  $X$  is homotopy equivalent to a circle, and the only reasonable value for the integral of  $\omega$  along a loop  $\gamma : [0, 1] \rightarrow X$  with end in  $X(k)$ , whose class generates the

fundamental group of  $X$ , should be  $\text{Log}^\lambda(q)$  (up to a sign). But for every  $q \in k^*$  with  $|q| \neq 1$  there exist  $\lambda$ 's in  $k$  with  $\text{Log}^\lambda(q) \neq 0$  as well as those with  $\text{Log}^\lambda(q) = 0$ .

A natural way to resolve the above problem is to consider the universal logarithm, i.e., the one whose value at  $p$  is a variable. Such a universal logarithm was already used in the work of P. Colmez and V. Vologodsky mentioned above, and it can be specialized to any of the branches of the logarithm whose values at  $p$  are elements of  $k$ . But the properties of the sheaves  $\mathcal{S}_X$ , whose construction is based on the universal logarithm, do not seem to easily imply the same properties of the similar sheaves whose construction is based on a classical branch of the logarithm. Thus, to consider all possible branches of the logarithm simultaneously, we proceed as follows. Fix a filtered  $k$ -algebra  $K$ , i.e., a commutative  $k$ -algebra provided with an exhausting filtration by  $k$ -vector spaces  $K^0 \subset K^1 \subset \dots$  with  $K^i \cdot K^j \subset K^{i+j}$ . Furthermore, define a filtered  $\mathcal{O}_X$ -algebra of naive analytic functions  $\mathfrak{N}_X^K$  in the same way as  $\mathfrak{n}_X$  but starting with the filtered algebra  $\mathcal{O}_X^K = \mathcal{O}_X \otimes_k K$  instead of  $\mathcal{O}_X$  and, for an element  $\lambda \in K^1$ , define a logarithmic function  $\text{Log}^\lambda(T)$ , which is an element of  $\mathfrak{N}_X^{K,1}(\mathbf{G}_m)$ . In a similar way we define the  $\mathcal{O}_X$ -algebras of naive analytic  $q$ -forms  $\Omega_{\mathfrak{N}_X^K}^q$ . One has  $\mathfrak{N}_X^K \otimes_{\mathcal{O}_X} \Omega_X^q \xrightarrow{\sim} \Omega_{\mathfrak{N}_X^K}^q$  and, in particular,  $\mathfrak{N}_X^K$  is a filtered  $\mathcal{D}_X$ -algebra (the latter notion is defined in §1.3). For a  $\mathcal{D}_X$ -submodule  $\mathcal{F}$  of  $\mathfrak{N}_X^K$ , let  $\Omega_{\mathcal{F},X}^q$  denote the image of the canonical injective homomorphism  $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^q \rightarrow \Omega_{\mathfrak{N}_X^K}^q$ .

The main result (Theorem 1.6.1) states that, given  $K$  and  $\lambda \in K^1$ , there is a unique way to provide every smooth  $k$ -analytic space  $X$  with a filtered  $\mathcal{D}_X$ -subalgebra  $\mathcal{S}_X^\lambda \subset \mathfrak{N}_X^K$  so that the following are true:

- (a)  $\mathcal{S}_X^{\lambda,0} = \mathcal{O}_X \otimes_k K^0$ ;
- (b)  $\text{Ker}(\mathcal{S}_X^{\lambda,i} \xrightarrow{d} \Omega_{\mathcal{S}_X^{\lambda,i},X}^1) = \mathcal{c}_X \otimes_k K^i$ ;
- (c)  $\text{Ker}(\Omega_{\mathcal{S}_X^{\lambda,i},X}^1 \xrightarrow{d} \Omega_{\mathcal{S}_X^{\lambda,i},X}^2) \subset d\mathcal{S}_X^{\lambda,i+1}$ ;
- (d)  $\mathcal{S}_X^{\lambda,i+1}$  is generated by the local sections  $f$  for which  $df$  is a local section of  $\Omega_{\mathcal{S}_X^{\lambda,i},X}^1$ ;
- (e)  $\text{Log}^\lambda(T) \in \mathcal{S}_X^{\lambda,1}(\mathbf{G}_m)$ ;
- (f) for a morphism  $\varphi : X' \rightarrow X$  and a function  $f \in \mathcal{S}_X^{\lambda,i}(X)$ , one has  $\varphi^*(f) \in \mathcal{S}_X^{\lambda,i}(X')$ .

In Theorem 1.6.2 we list several properties of the sheaves  $\mathcal{S}_X^\lambda$ . Among them is the uniqueness property, which tells that, if  $X$  is connected, then for any nonempty open subset  $\mathcal{U} \subset X$  the restriction map  $\mathcal{S}^\lambda(X) \rightarrow \mathcal{S}^\lambda(\mathcal{U})$  is injective. The sheaves  $\mathcal{S}_X^\lambda$  are functorial in the best possible sense, involving an embedding of the ground fields  $k \hookrightarrow k'$ , a morphism  $X' \rightarrow X$ , and a homomorphism of filtered algebras  $K \rightarrow K'$  over that embedding. If one is given only a homomorphism of filtered  $k$ -algebras  $K \rightarrow K' : \lambda \mapsto \lambda'$ , there is a canonical isomorphism  $\mathcal{S}_X^\lambda \otimes_K K' \xrightarrow{\sim} \mathcal{S}_{X'}^{\lambda'}$ . In particular, if  $\mathcal{S}_X$  denotes the sheaf constructed for the universal logarithm, then the canonical homomorphism  $k[\text{Log}(p)] \rightarrow K : \text{Log}(p) \mapsto \lambda$  gives rise to an isomorphism  $\mathcal{S}_X \otimes_{k[\text{Log}(p)]} K \xrightarrow{\sim} \mathcal{S}_X^\lambda$ .

Theorems 1.6.1 and 1.6.2 are proven in §7. The proof is based on preliminary results obtained in §§1–6 and having an independent interest. Since the formulation

of the main result actually makes sense for an arbitrary non-Archimedean field of characteristic zero, the preparatory part of the proof in §§1–5 is done over fields as general as possible, and the assumption that  $k$  is a closed subfield of  $\mathbf{C}_p$  is only made beginning with §6. In §8, further properties of the sheaves  $\mathcal{S}_X^\lambda$  are established and, in §9, they are used for a construction of the integral and parallel transport along a path. A detailed summary of each section is given at its beginning.

There are many natural questions one may ask on the sheaves  $\mathcal{S}_X^\lambda$ . Here are some of them.

- (1) Is the de Rham complex associated to  $\mathcal{S}_X^\lambda$  exact? We believe this is true.
- (2) Does the extended class of functions contain local primitives of relative closed one-forms with respect to an arbitrary smooth morphism  $\varphi : Y \rightarrow X$ ? Again, we believe this is true. It is in fact enough to consider morphisms of dimension one, and the positive answer to this question would imply a positive answer to (1) and to the relative version of (1).
- (3) Are the sheaves of rings  $\mathcal{S}_K^\lambda$  coherent for reasonable  $K$  (e.g.,  $K = k[\text{Log}(p)]$  or  $K = k$ )? Like (1) and (2), we believe this is true. Notice that, for a point  $x \in X$  whose field  $\widetilde{\mathcal{H}(x)}$  is transcendental over  $\widetilde{k}$ , the stalk  $\mathcal{S}_{X,x}^\lambda$  is a non-Noetherian ring.
- (4) What are the cohomology groups  $H^q(X, \mathcal{S}_X^{\lambda,n})$  and  $H^q(X, \mathcal{S}_X^\lambda)$ ?
- (5) Assume that  $k$  is finite over  $\mathbf{Q}_p$ , a  $p$ -adic group  $G$  acts continuously on a smooth  $k$ -analytic space  $X$  (e.g.,  $G = \text{PGL}_d(k)$ , and  $X$  is the projective space  $\mathbf{P}^{d-1}$  or the Drinfeld half-plane  $\Omega^d \subset \mathbf{P}^{d-1}$ ). What are the representations of  $G$  on the space of global sections  $\mathcal{S}^\lambda(X)$ ?
- (6) Let  $X$  and  $Y$  be smooth  $k$ -analytic spaces, and assume there is a morphism of germs of analytic spaces  $\varphi : (Y, Y_{\text{st}}) \rightarrow (X, X_{\text{st}})$  (see [Ber2, §3.4]) which takes local sections of  $\mathcal{S}_X^\lambda$  to those of  $\mathcal{S}_Y^\lambda$ . Is it true that  $\varphi$  is induced by a morphism of analytic spaces  $Y \rightarrow X$ ?

The answer to (6) would shed light on the following philosophical question. What does the existence of the sheaves  $\mathcal{S}_X^\lambda$  mean? A negative answer to (6) would mean that smooth  $p$ -adic analytic spaces can be considered as objects of a category with bigger sets of morphisms in the same way as complex analytic spaces can be considered as real analytic or differentiable manifolds. On the other hand, a positive answer to (6) could mean that complex analytic functions have at least two  $p$ -adic analogs, namely, genuine analytic ones and functions from the broader class provided by the sheaves  $\mathcal{S}_X^\lambda$ . This reminds us of the similar phenomenon with the topology of a complex analytic space whose  $p$ -adic reincarnation is two-faced. It appears as the usual topology of a  $p$ -adic analytic space as well as the stronger étale topology of the space. In addition, the existence of the sheaves  $\mathcal{S}_X^\lambda$  is somehow related to the fact that smooth  $p$ -adic analytic spaces are not locally simply connected in the étale topology. In any case, we hope what is done in this book will be useful for understanding the  $p$ -adic Hodge theory in terms of  $p$ -adic analytic geometry.

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As is clear from the above and the text which follows, this book is motivated by and based on the work and ideas of R. Coleman. In its very first version I constructed local primitives of closed analytic one-forms (or, equivalently, the sheaves  $\mathcal{S}_X^{\lambda,1}$ ) on smooth analytic spaces defined over finite extensions of  $\mathbf{Q}_p$ . The construction of all of the sheaves  $\mathcal{S}_X^{\lambda,n}$  was done after I borrowed the idea of using unipotent isocrystals from A. Besser's work [Bes]. Finally, I am very grateful to O. Gabber for providing a key fact (Lemma 5.5.1) which allowed me to extend the whole theory for arbitrary closed subfields of  $\mathbf{C}_p$ .

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