Introduction

The purpose of this book is to develop the analytic theory of the hypoelliptic Laplacian and to establish corresponding results on the associated Ray-Singer analytic torsion. We also introduce the corresponding theory for families of hypoelliptic Laplacians, and we construct the associated analytic torsion forms. The whole setting will be equivariant with respect to the action of a compact Lie group G.

Let us put in perspective the various questions which are dealt with in this book. In [B05], one of us introduced a deformation of classical Hodge theory. Let X be a compact Riemannian manifold, let $\left(F, \nabla^F, g^F\right)$ be a complex flat Hermitian vector bundle on X. Let $\left(\Omega^{\cdot}(X,F), d^X\right)$ be the de Rham complex of smooth forms on X with coefficients in F, let d^{X*} be the formal adjoint of d^X with respect to the obvious Hermitian product on $\Omega^{\cdot}(X,F)$. Then the Laplacian $\square^X = \left[d^X, d^{X*}\right]$ is a second order nonnegative elliptic operator acting on $\Omega^{\cdot}(X,F)$. Let $\mathcal{H}^X = \ker \square^X$ be the vector space of harmonic forms. Classical Hodge theory asserts that we have a canonical isomorphism,

$$\mathcal{H}^X \simeq H^{\cdot}(X,F). \tag{0.1}$$

Let T^*X be the cotangent bundle of X, let $\left(\Omega^{\cdot}(T^*X, \pi^*F), d^{T^*X}\right)$ be the corresponding de Rham complex over T^*X. In [B05], a deformation of classical Hodge theory was constructed, which is associated to a Hamiltonian $\mathcal{H}$ on T^*X. The corresponding Laplacian is denoted by $A^2_{\phi,\mathcal{H}}$. In the case where $\mathcal{H} = \frac{|p|^2}{2}$ and $\mathcal{H}^c = c\mathcal{H}$ depends on a parameter $c = \pm 1/b^2 \in \mathbf{R}^*$, with $b \in \mathbf{R}^*_+$, an operator which is conjugate to $A^2_{\phi,\mathcal{H}^c}$, the operator $\mathfrak{A}^2_{\phi,\mathcal{H}^c}$, is given by the formula

$$\begin{aligned}\mathfrak{A}^2_{\phi,\mathcal{H}^c} = \frac{1}{4}\Bigg(&-\Delta^V + c^2|p|^2 + c\left(2\widehat{e}_i i_{\widehat{e}^i} - n\right) \\ &- \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}\Bigg) \\ - \frac{1}{2}\Bigg(&cL_{Y^{\mathcal{H}}} + \frac{c}{2}\omega\left(\nabla^F, g^F\right)\left(Y^{\mathcal{H}}\right) + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla^F_{e_i}\omega\left(\nabla^F, g^F\right)(e_j) \\ &+ \frac{1}{2}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i}\Bigg).\end{aligned} \tag{0.2}$$

In (0.2), Δ^V is is the Laplacian along the fibers of T^*X, R^{TX} is the curvature tensor of the Levi-Civita connection ∇^{TX}, the $e^i, \widehat{e}_i$ are horizontal and vertical 1-forms, which produce orthonormal bases of T^*X and TX, $Y^{\mathcal{H}}$ is the Hamiltonian vector field associated to $\mathcal{H}$, i.e., the generator of the geodesic flow, $L_{Y^{\mathcal{H}}}$ is the Lie derivative operator associated to $Y^{\mathcal{H}}$, and $\omega\left(\nabla^F, g^F\right)$ is the variation of g^F with respect to ∇^F. The differential operator which appears in the first line in the right-hand side of (0.2) is a harmonic oscillator. A fundamental feature of the operator $\mathfrak{A}^2_{\phi,\mathcal{H}^c}$ is that by a theorem of Hörmander [Hör67], $\frac{\partial}{\partial u} - \mathfrak{A}^2_{\phi,\mathcal{H}^c}$ is hypoelliptic.

In [B05], algebraic arguments were given which indicated that when b varies between 0 and $+\infty$, the Laplacian $2A^2_{\phi,\mathcal{H}^c}$ interpolates in a proper sense between the Hodge Laplacian $\square^X/2$ and the operator $|p|^2/2 - L_{Y^{\mathcal{H}}}$. Moreover, $A^2_{\phi,\mathcal{H}^c}$ was shown in [B05] to be self-adjoint with respect to a Hermitian form of signature (∞,∞).

A key motivation for the construction of the Laplacian $A^2_{\phi,\mathcal{H}^c}$ is its relation to the Witten deformation of classical Hodge theory. Let us simply recall that if $f : X \to \mathbf{R}$ is a smooth function, the associated Witten Laplacian is a one parameter deformation $\square^X_T$ of the classical Laplacian $\square^X$, which coincides with $\square^X$ for $T = 0$, which also consists of elliptic self-adjoint operators for which the Hodge theorem holds. If f is a Morse function, Witten showed that as $T \to \pm\infty$, the small eigenvalue eigenspaces localize near the critical points of f. He also conjectured that the corresponding complex of small eigenvalue eigenforms can be identified with the corresponding Thom-Smale [T49, Sm61] of the gradient field $-\nabla f$, in the case where this gradient field satisfies the Thom transversality conditions [T49]. This conjecture was proved by Helffer-Sjöstrand [HeSj85]. The Witten deformation was used in [BZ92, BZ94] to give a new proof of the Cheeger-Müller theorem [C79, Mül78] on the equality of the Reidemeister torsion and of the analytic torsion for unitary flat exact vector bundles, and more generally of the Ray-Singer metric on $\lambda = \det H^{\cdot}(X,F)$, which one defines using the Ray-Singer torsion, with the so-called Reidemeister metric [Re35], which is defined combinatorially. Let us just recall here that the Ray-Singer analytic torsion can be obtained via the derivative at $s = 0$ of the zeta functions of the Laplacian $\square^X$.

Let LX be the loop space of X, i.e., the set of smooth maps $s \in S_1 \to X$, and let E be the energy functional $E = \frac{1}{2}\int_0^1 |\dot{x}|^2\, ds$. The functional integral interpretation of the Laplacian $A^2_{\phi,\mathcal{H}^c}$ is explained in detail in [B04, B05]. In particular $2A^2_{\phi,\mathcal{H}^c}$ interpolating between $\square^X/2$ and $|p|^2/2 - L_{Y^{\mathcal{H}}}$ should be thought of as a semiclassical version of the fact that the Witten Laplacian $\square^{LX}_T$ on LX associated to the energy functional E should interpolate between the Hodge Laplacian $\square^{LX}$ and the Morse theory for E, whose critical points are precisely the closed geodesics. Incidentally, let us recall that neither $\square^{LX}$ nor its Witten deformation has ever been constructed. Let us also mention that if one follows the analogy of the deformation in [B05] with the Witten Laplacian, then $c = 1/T$, so that $T = \pm b^2$.

It was also observed in [B05] that at least formally, Fried's conjecture

[F86, F88] on the relation of the Ray-Singer torsion to dynamical Ruelle's zeta functions could be thought of as a consequence of a infinite dimensional version of the Cheeger-Müller theorem, where X is replaced by LX. This conjecture by Fried has been proved by Moscovici-Stanton [MoSta91] for symmetric spaces using Selberg's trace formula.

The present book has four main purposes:

- To develop the full Hodge theory of the Laplacian $A^2_{\phi,\mathcal{H}^c}$. This means not only proving a corresponding version of the Hodge theorem, but also studying the precise properties of its resolvent and of the corresponding heat kernel. The main difficulty is related to the fact that T^*X is noncompact, and also that the operator $A^2_{\phi,\mathcal{H}^c}$ is not classically self-adjoint.

- To develop the appropriate local index theory for the associated heat kernel.

- To adapt to such Laplacians the theory of the Ray-Singer torsion [RS71] of Ray-Singer, and of the analytic torsion forms of Bismut-Lott [BLo95].

- To give an explicit formula relating the analytic torsion objects associated to the hypoelliptic Laplacian to the classical Ray-Singer torsion for the classical Laplacian $\square^X$.

To reach these above objectives, we use the following tools:

- We refine the hypoelliptic estimates of Hörmander [Hör85] in order to control hypoellipticity at infinity in the cotangent bundle. Some of the arguments we use are similar to arguments already given by Helffer-Nier [HeN05] and Hérau-Nier [HN04] in the case where $X = \mathbf{R}^n$ in their study of the return to equilibrium for Fokker-Planck equations. It is quite striking that although we view our hypoelliptic equations as coming from a degeneration of elliptic equations on LX, we end up dealing with kinetic equations on X.

- We develop the adequate theory of semiclassical pseudodifferential operators with parameter $h = b$, combined to a computation of the resolvents as $(2, 2)$ matrices, by a method formally similar to a method we developed in the context of Quillen metrics in [BL91], in order to study the convergence as $b \to 0$ of the operator $A^2_{\phi,\mathcal{H}^c}$ to $\square^X$. One basic difference with respect to [BL91] is that our operators are no longer self-adjoint.

- We develop a hypoelliptic local index theory. This local index theory extends the well-known local index theory for the operator $d^X + d^{X*}$ [P71, Gi84, ABP73, G86]. Still, the fact that we work also with analytic torsion forms forces us to develop a very general machinery which will extend to the analysis of Dirac operators. The hypoelliptic local index theory is itself a deformation of classical elliptic local index theory.

- We study the deformation of the Ray-Singer metric and also the corresponding hypoelliptic analytic torsion forms by a method formally similar to the one used in [BL91] and later extended in [B97] to holomorphic torsion forms. At least at a formal level, even though we deal with essentially different objects, the proofs are formally very close, even in their intermediate steps.
- We develop the adequate probabilistic machinery which allows us to prove certain localization estimates, and also the Malliavin calculus [M78] corresponding to the hypoelliptic diffusion process. In particular we establish an integration by parts formula for a geometric hypoelliptic diffusion, which extends a corresponding formula established in [B84] for the classical Brownian motion.

Let us now elaborate on the functional integral interpretation of the above techniques, along the lines of [B04, B05]. For $c = 1/b^2$, the dynamics of the diffusion $(x_s, p_s) \in T^*X$ associated to the hypoelliptic Laplacian $2A^2_{\phi,\mathcal{H}^c}$ can be described by the stochastic differential equation

$$\dot{x} = p, \qquad \dot{p} = \left(-p + \dot{w}\right)/b^2, \tag{0.3}$$

where w is a standard Brownian motion. The first order differential system (0.3) can also be written as the second order differential equation on X,

$$\ddot{x} = \left(-\dot{x} + \dot{w}\right)/b^2. \tag{0.4}$$

When $b \to 0$, equation (0.4) degenerates to

$$\dot{x} = \dot{w}. \tag{0.5}$$

In (0.3), p is an Ornstein-Uhlenbeck process, whose trajectories are continuous, x is a so-called physical Brownian motion, and the trajectories of x are C^1. Incidentally observe that p is a Gaussian process with covariance $\exp\left(-\left|t-s\right|/b^2\right)/b^2$. In (0.5), x is a standard Brownian motion, and its trajectories are nowhere differentiable. Now Brownian motion is precisely the process corresponding to the Hodge Laplacian $\square^X/2$. The fact that equation (0.4) degenerates into (0.5) when $b \to 0$ is one of the arguments to justify the convergence of $2A^2_{\phi,\mathcal{H}^c}$ toward $\square^X/2$ when $b \to 0$ at a dynamical level.

The convergence argument of the trajectories in (0.4) to those in (0.5) can indeed be justified. In another form, it was already present in earlier work of Stroock and Varadhan [StV72], where another convergence scheme of the solution of a differential equation to the solution of a stochastic differential equation was given. Such convergence arguments provide the critical link between classical differential calculus and the Itô calculus.

But as explained in [B05], we are asking much more, since we want to understand the functional analytic behavior of the Laplacian $A^2_{\phi,\mathcal{H}^c}$ when $b \to 0$, and this in every degree. Arguments in favor of such a possibility were given in [B05], writing the operator $A^2_{\phi,\mathcal{H}^c}$ as a $(2,2)$ matrix with respect to to a natural splitting of a corresponding Hilbert space.

Our proof of the convergence of $2A^2_{\phi,\mathcal{H}^c}$ to $\square^X/2$ can be thought of as a functional analytic version of the Itô calculus. The analytic difficulties are in part revealing the tormented path connecting a C^1 dynamics for $b > 0$ to a nowhere differentiable dynamics for $b = 0$.

Let us still elaborate on this point from a formal point of view, along the lines of [B04, B05]. Indeed for $b > 0$, the path integral representation for the supertrace $\mathrm{Tr_s}\left[\exp\left(-tA^2_{\phi,\mathcal{H}^c}\right)\right]$ is given by

$$\mathrm{Tr_s}\left[\exp\left(-tA^2_{\phi,\mathcal{H}^c}\right)\right] = \int_{LX} \exp\left(-\frac{1}{2t}\int_0^1 |\dot{x}|^2\, ds - \frac{b^4}{2t^3}\int_0^1 |\ddot{x}|^2\, ds + \dots\right). \tag{0.6}$$

In (0.6), ... represents the fermionic part of the integral. One should be aware of the fact that the process x in (0.4) which corresponds to (0.6) is such that $\frac{1}{2}\int_0^1 |\ddot{x}|^2\, ds = +\infty$.

In (0.6), if we make $b = 0$, in the right-hand side, we recover the standard representation of the Brownian measure, for which $\frac{1}{2}\int_0^1 |\dot{x}|^2\, ds = +\infty$. Making $b = 0$ seems to be an innocuous operation in (0.6), which could be Taylor expanded. The opposite is true. First of all the H^1 norm of $\dot{x}$ is much "bigger" than its H^0 norm. Any perturbative expansion of (0.6) to $b = 0$ will lead to inconsistent divergences. The rigorous process through which one shows the convergence of (0.6) to the corresponding expression with $b = 0$ is much subtler and involves functional analytic arguments, which we now describe in more detail.

The arguments in [B05] show that the convergence of $A^2_{\phi,\mathcal{H}^c}$ to $\square^X/4$ should be obtained by inverting the harmonic oscillator fiberwise. However, this picture provides only the limit view, in which b has already been made equal to 0. Namely, the inverse of the harmonic oscillator should be viewed as a fiberwise pseudodifferential operator, supported on the diagonal of X. For $b > 0$ close to 0, the inverse of the relevant operator is no longer supported over the diagonal of X. A suitably defined version of this inverse can be viewed as a semiclassical pseudodifferential operator on X with semiclassical parameter $h = b$. This semiclassical description is valid only to describe the more and more chaotic behavior of the component $p \in T^*X$ as $b \to 0$ in (0.3). As explained in (0.4), (0.5), as $b \to 0$, the dynamics of x converges to a Brownian motion on X. The obvious implication is that the relevant calculus on operators which will give a precise account of the transition from the dynamics in (0.3) to the Brownian dynamics (0.5) will necessarily have two scales, a semiclassical scale with parameter $h = b$ and an ordinary scale.

Let us point out that we also study the transition from the small time asymptotics of the heat kernel in (0.3) to the corresponding small time asymptotics for the standard heat kernel corresponding to (0.5). This requires proving the required uniform localization in b as $t \to 0$, and also using a two scale pseudodifferential calculus, with semiclassical parameters t, b.

No attempt is made in this book to study the limit $b \to +\infty$, which should concentrate the analysis near the closed geodesics.

We now present three key results which are established in this book. Let $\lambda = \det H^{\cdot}(X, F)$ be the determinant of the cohomology of F, so that λ is a complex line. By proceeding as in [B05], for $b \in \mathbf{R}_+^*, c = 1/b^2$, we construct a generalized metric $\| \|_{\lambda,b}^2$ on the line λ, using in particular the Ray-Singer torsion for $A^2_{\phi,\mathcal{H}^c}$ in the sense of [RS71]. A generalized metric differs from a usual metric in the sense it may have a sign. Let $\|\|_{\lambda,0}^2$ be the corresponding classical Ray-Singer metric, associated to the analytic torsion for $\square^X$. The following result is established in Theorem 9.0.1.

Theorem 0.0.1. *Given $b > 0, c = 1/b^2$, we have the identity*

$$\| \|_{\lambda,b}^2 = \| \|_{\lambda,0}^2 . \tag{0.7}$$

More generally, if G is a compact Lie group acting isometrically on the above geometric objects, along the lines of [B95], we can define the logarithm of an equivariant Ray-Singer metric $\log \left(\|\|_{\lambda,b}^2 \right)$, which one should compare with the equivariant Ray-Singer metric $\log \left(\| \|_{\lambda,0}^2 \right)$. Take $g \in G$ and let $X_g \subset X$ be the fixed point manifold of X. Let $\zeta(\theta, s) = \sum_{n=1}^{+\infty} \frac{\cos(n\theta)}{n^s}, \eta(\theta, s) = \sum_{n=1}^{+\infty} \frac{\sin(n\theta)}{n^s}$ be the real and imaginary parts of the Lerch function [Le88]. Set

$$^0J(\theta) = \frac{1}{2} \left(\frac{\partial \zeta}{\partial s}(\theta, 0) - \frac{\partial \zeta}{\partial s}(0, 0) \right). \tag{0.8}$$

We denote by $e(TX_g)$ the Euler class of TX_g, and by $^0J_g\left(TX|_{X_g}\right)$ the locally constant function on X_g which is associated to the splitting of $TX|_{X_g}$ using the locally constant eigenvalues of g acting on $TX|_{X_g}$. In Theorem 9.0.1, we also establish the following extension of Theorem 0.0.1.

Theorem 0.0.2. *For $g \in G, b > 0, c = 1/b^2$, we have the identity*

$$\log \left(\frac{\| \|_{\lambda,b}^2}{\| \|_{\lambda,0}^2} \right)(g) = 2 \int_{X_g} e(TX_g)\, {}^0J_g\left(TX|_{X_g}\right) \operatorname{Tr}^F[g] . \tag{0.9}$$

A more general result is for the torsion forms $\mathcal{T}_{\mathrm{ch},g,b_0}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ which we define in chapter 6 as analogues in the hypoelliptic case of the analytic torsion forms of Bismut and Lott [BLo95] $\mathcal{T}_{\mathrm{ch},g,0}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$, normalized as in [BG01], which were obtained in the context of standard elliptic theory. The torsion forms $\mathcal{T}_{\mathrm{ch},g,0}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ are secondary invariants which refine the theorem of Riemann-Roch-Grothendieck for flat vector bundles established in [BLo95] at the level of differential forms. They were constructed using the superconnection formalism of Quillen [Q85b]. We make here a similar construction to obtain the hypoelliptic torsion forms $\mathcal{T}_{\mathrm{ch},g,b_0}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$.

Let us now explain our results on hypoelliptic torsion forms in more detail. We consider indeed a projection $p : M \to S$ with compact fiber X, the flat Hermitian vector bundle $\left(F, \nabla^F, g^F\right)$ is now defined on M, and $T^HM \subset TM$ is a horizontal vector bundle on M. The Lie group G acts along the fibers

X. The equivariant analytic torsion forms $\mathcal{T}_{\mathrm{ch},g,b_0}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ are smooth even forms on S.

Put

$$J(\theta, x) = \frac{1}{2}\left[\sum_{\substack{p\in\mathbf{N}\\ p\,\mathrm{even}}} \frac{\partial\zeta}{\partial s}(\theta, -p)\frac{x^p}{p!} + i\sum_{\substack{p\in\mathbf{N}\\ p\,\mathrm{odd}}} \frac{\partial\eta}{\partial s}(\theta, -p)\frac{x^p}{p!}\right], \tag{0.10}$$

$$^0J(\theta, x) = J(\theta, x) - J(0,0).$$

The functions $J(\theta, x)$ and $^0J(\theta, x)$ were introduced in [BG01, Definitions 4.21 and 4.25, Theorem 4.35, and Definition 7.3].

Take $g \in G$. Here $^0J_g\left(TX|_{X_g}\right)$ is now a cohomology class on $M_g \subset M$. The class $\widetilde{\mathrm{ch}}_g^{\circ}\left(\nabla^{\mathfrak{H}^{\cdot}(X,F)}, \mathfrak{h}_0^{\mathfrak{H}^{\cdot}(X,F)}, \mathfrak{h}_{b_0}^{\mathfrak{H}^{\cdot}(X,F)}\right) \in \Omega^{\cdot}(S)/d\Omega^{\cdot}(S)$ is defined in equation (8.1.1). It is a secondary class attached to a couple of generalized metrics on $\mathfrak{H}^{\cdot}(X, F) \simeq H^{\cdot}(X, F)$.

We now state a formula comparing the elliptic and the hypoelliptic torsion forms, which is established in Theorem 8.2.1.

Theorem 0.0.3. *For $b_0 > 0, c = 1/b_0^2$ and b_0 small enough, the following identity holds:*

$$\begin{aligned} &-\mathcal{T}_{\mathrm{ch},g,b_0}\left(T^H M, g^{TX}, \nabla^F, g^F\right) + \mathcal{T}_{\mathrm{ch},g,0}\left(T^H M, g^{TX}, \nabla^F, g^F\right) \\ &- \widetilde{\mathrm{ch}}_g^{\circ}\left(\nabla^{\mathfrak{H}^{\cdot}(X,F)}, \mathfrak{h}_0^{\mathfrak{H}^{\cdot}(X,F)}, \mathfrak{h}_{b_0}^{\mathfrak{H}^{\cdot}(X,F)}\right) + \int_{X_g} e(TX_g)\,{}^0J_g\left(TX|_{M_g}\right)\mathrm{Tr}^F[g] = 0 \\ &\qquad \text{in } \Omega^{\cdot}(S)/d\Omega^{\cdot}(S). \end{aligned} \tag{0.11}$$

Note that except for the restriction that b_0 has to be small, Theorems 0.0.1 and 0.0.2 follow from Theorem 0.0.3.

Let us also point out that in [BL91], given an embedding of compact complex Kähler manifolds $i : Y \to X$, and a resolution of a holomorphic vector bundle η on Y by a holomorphic complex of vector bundles (ξ, v) on X, we gave a local formula for the ratio of the Quillen metrics on the line $\det H^{0,\cdot}(Y, \eta) \simeq \det H^{0,\cdot}(X, \xi)$. This problem seems to be of a completely different nature from the one which is being considered here. In particular all the operators considered in [BL91] are self-adjoint. Still from a certain point of view, the structures of the proofs are very similar, probably because of the underlying path integrals, which are very similar in both cases.

The book is organized as follows. In chapter 1, we describe the results obtained by Bismut and Lott [BLo95] and Bismut and Goette [BG01] in the context of classical Hodge theory. In particular we recall the construction in [BLo95] of the analytic elliptic torsion forms, which are obtained by transgression of certain elliptic odd Chern forms, and we describe various properties of Ray-Singer metrics on the line $\det H^{\cdot}(X, F)$.

In chapter 2, we recall the construction given in [B05] of a deformation of classical Hodge theory on a Riemannian manifold X, whose Laplacian $A^2_{\phi,\mathcal{H}^c}$ is a hypoelliptic operator on T^*X, this theory being also developed in the context of families. Also we give the general set up which will ultimately permit us to establish the above three results.

In chapter 3, given $b > 0$, we discuss the Hodge theory for the hypoelliptic Laplacian, and we summarize the main properties of its heat kernel. We discuss in detail the spectral theory of $A^2_{\phi,\mathcal{H}^c}$ and the behavior of the spectrum as $b \to 0$. We show that for $b > 0$, the spectrum is discrete and conjugation-invariant. We prove that for $b > 0$ small enough, the results of classical Hodge theory still hold, and also that except for the 0 eigenvalue, the other eigenvalues have a positive real part and remain real at finite distance. Also we prove that the set of $b > 0$ such that the Hodge theorem does not hold is discrete. The bulk of the analytic arguments used in this chapter is taken from the key chapters 15 and 17.

In chapter 4, we construct hypoelliptic odd Chern forms, which depend on two parameters, $b > 0, t > 0$, with $c = \pm 1/b^2$. Also we show that their asymptotics as $t \to 0$ coincide with the asymptotics of the corresponding elliptic odd Chern forms. These results are obtained using a new version of the Getzler rescaling of Clifford variables [G86] in the context of hypoelliptic operators. The arguments of localization are obtained using probabilistic methods and arguments from chapter 14. Let us also point out that in [L05], one of us has studied in detail the asymptotics of the hypoelliptic heat kernel on functions, also outside the diagonal, and obtained a corresponding large deviation principle, in which the action considered in the formal representation (0.6) ultimately appears in an exponentially small term as $t \to 0$. Alternative localization techniques are given in chapters 15 and 17. These techniques will play an essential role when studying the combined asymptotics for the heat kernel as $b \to 0, t \to 0$.

In chapter 5, we study the behavior of the hypoelliptic odd Chern forms when $t \to +\infty$ or $b \to 0$. We study in particular the uniformity of the convergence.

In chapter 6, using the results of chapters 4 and 5, for $b > 0$ small enough, we construct the corresponding analytic hypoelliptic torsion forms, which are obtained by transgression of the hypoelliptic odd Chern forms, and we construct corresponding hypoelliptic Ray-Singer metrics for any b. The elliptic and hypoelliptic torsion forms verify similar transgression equations, which makes plausible Theorem 0.0.3, which asserts essentially that their difference is topological. Also we show that the hypoelliptic Ray-Singer metrics does not depend on b.

In chapter 7, we compute the hypoelliptic torsion forms which are attached to a vector bundle. This chapter is based on explicit computations involving the harmonic oscillator and Clifford variables. This computation plays a key role in the proof of our final formula.

In chapter 8, we establish our main result, which was stated as Theorem 0.0.3, where we give a formula comparing the hypoelliptic to the elliptic torsion forms. The proof is based on a series of intermediate results, whose proofs are themselves deferred to chapters 10-13.

In chapter 9, we prove Theorems 0.0.1 and 0.0.2, i.e., we give a formula comparing the elliptic and hypoelliptic Ray-Singer metrics.

In chapter 10, given a cohomology class, we calculate the asymptotic expansion of the corresponding suitably rescaled harmonic forms as $b \to 0$.

In chapter 11, we give the proof of an intermediate result associated with the smooth kernel for $\exp\left(-tA^2_{\phi,\mathcal{H}^c}\right)$ when $b \simeq \sqrt{t}$.

In chapter 12, we get uniform bounds on the heat kernel when $b \in \left[\sqrt{t}, b_0\right]$, with $t \in]0,1]$, and $b_0 > 0$.

In chapter 13, we study the heat kernel for $A^2_{\phi,\mathcal{H}^c}$ in the range $b \in]0, \sqrt{t}]$, $t \in]0,1]$. Note here that local index methods are also developed in chapters 12 and 13.

In chapter 14, we establish an integration by parts formula for the hypoelliptic diffusion, in the context of the Malliavin calculus [M78]. Some of the objects which appear there are the concrete manifestation of the dreams described in [B05].

Chapters 15-17 contain most of the analytic machine used in the book.

In chapter 15, given a fixed $b > 0$, we develop the hypoelliptic estimates for the operator $A^2_{\phi,\mathcal{H}^c}$. The noncompactness of T^*X introduces extra difficulties with respect to Hörmander [Hör67, Hör85]. These are handled using a Littlewood-Paley decomposition of the chapters of the given vector bundles on annuli. We show that the spectrum of $A^2_{\phi,\mathcal{H}^c}$ is included in a region of $\mathbf{C}$ which is limited by a cusplike boundary. Also we study the trace class properties of adequate powers of the resolvent.

In chapter 16, we develop some of the key tools which are needed to study the limit $b \to 0$. Indeed when microlocalizing this asymptotics, we are essentially back to the case of a flat manifold. In the case of flat tori, it was shown in [B05, subsection 3.10] that the hypoelliptic operator $A^2_{\phi,\mathcal{H}^c}$ is essentially isospectral to $\square^X/4$. In particular the spectrum of $A^2_{\phi,\mathcal{H}^c}$ is real. Still the method used in chapter 17 to study the limit $b \to 0$ consists in writing our operator as a $(2,2)$ matrix. Even in the case of the torus, this method is nontrivial. The asymptotics as $b \to 0$ of the matrix component are determined by a function $J_0(y,\lambda)$, $(y,\lambda) \in \mathbf{R} \times \mathbf{C}$, whose behavior is studied in detail. The Bargman representation of the harmonic oscillator in terms of bosonic creation and annihilation operators plays a key role in the analysis.

Finally, in chapter 17, we study the asymptotics of the resolvent of the operator $A^2_{\phi,\mathcal{H}^c}$ as $b \to 0$. This chapter is technically difficult. Its purpose is to give a detailed analysis of the behavior of the resolvent of $A^2_{\phi,\mathcal{H}^c}$ as $b \to 0$. This means that the hypoelliptic estimates of chapter 15 have to be combined with the computation of the resolvent as a $(2,2)$ matrix. Here, in the hypoelliptic analysis, as in chapter 15, we use Kohn's method of proof [Ko73] of Hörmander's theorem [Hör67] to get a global estimate with a gain of $1/4$ derivative, and a parametrix construction in which we use a subelliptic estimate with a gain of $2/3$ derivatives in appropriate function spaces. One should observe here that this subelliptic estimate is not optimal for large $|p|$, but that in the $(2,2)$ matrix calculus, projection on the kernel of the fiberwise harmonic oscillator which appears in (0.2) compensates for that. Optimal hypoelliptic estimates have been obtained by one of us in [L06]. In chapter 17, we also study the behavior of the heat kernel when $b \to 0, t \to 0$.

In the text, to make the book more readable, we often use results of chapters 15-17, referring to those chapters for the complete proofs. This is the case in particular in chapter 13. In principle, except for notation, the various chapters in the book can be read independently, with the help of the index of notation which is given at the end of the book.

In the whole book, the positive constants C which appear in our estimates can vary from line to line, even when the same notation is used for them. Also in many cases, when dependence on parameters is crucial, the parameters on which they depend are noted as subscripts.

The results contained in this book were announced in [BL05].

In the whole book, if $\mathcal{A}$ is a $\mathbf{Z}_2$-graded algebra, if $a, a' \in \mathcal{A}$, we denote by $[a, a']$ their supercommutator.

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