Introduction

I have long been fascinated by the wonders of nature encountered on a simple walk, even in residential areas, as long as there is a view of the sky, roads lined with trees, and bodies of water within easy walking distance. Such is the case where I live in Norfolk, Virginia, and while most of my nature walks are local in character, I have been fortunate to have had the opportunity to travel farther afield at times, both within the United States and beyond its borders. Many of the topics in this book are consequences of such opportunities. And just as some walks are fairly short, and some are longer—veritable hikes, indeed—so are the chapter sections correspondingly varied.

Needless to say, the questions posed, and answered, are based on *many* walks, sometimes only into the back garden, or across the street, to gain a better view. I doubt that more than a few of these topics could be encompassed in a single walk, no matter how long! Mathematically, some are more involved than others; some are merely short forays into the topic, thinking on paper, as it were. Indeed, the shorter answers are perhaps best seen as "toy" models—merely intended to whet the appetite, to stimulate and encourage the reader to pursue the topic further on another occasion. Indeed, frequently, in my experience, at least, thinking about a question may lead nowhere initially, and it has to be put aside for a time. Later, however, it may yield some fascinating and valuable insights, perhaps of pedagogic value. The reader is encouraged always to take such "answers" and use them as a starting point to develop a fuller appreciation for the topic.

There is one question that should be addressed at the outset in a book like this, however. It is *Can you really use mathematics to describe that?* Regardless of whatever "that" may be, this is a question with which most people who teach the subject can identify, I suspect. Mathematically self-effacing, but otherwise educated people are frequently surprised when, in casual conversation, the subject of mathematical modeling comes up, often, in my experience, in connection with biological problems. Most people are at least vaguely aware of the importance of mathematics in physics, chemistry, or astronomy, for example, perhaps through unfortunate encounters with those subjects in earlier years. However, much less common is the notion that other subjects are also amenable to mathematical modeling in this way, and so part of the motivation for this book has been to address questions of this type by identifying an eclectic collection of "how to model [put your favorite topic here]" chapters. The unifying themes are these: *mathematical modeling* and *nature*.

What, then, *is* a mathematical model? Under what kinds of assumptions can we formulate such a model, and will it be realistic? Let us try and illustrate the answer in two stages; first, by drawing on a simple model discussed elsewhere (Adam 2006) in answer to the question: *Half the mass of a snowball melts in an hour. How long will it take for the remainder to melt?* Following that, a more general account of modeling is presented. This type of problem is often posed in first-year calculus textbooks, and as such requires only a little basic mathematical material, e.g., the chain rule and elementary integration. However, it's what we do with all this that makes it an interesting and informative exercise in mathematical modeling. There are several reasonable assumptions that can be made in order to formulate a model of snowball melting; however, unjustifiable assumptions are also a possibility! The reader may consider some or all of these to be in the latter category, but ultimately the test of a model is how well it fits known data and predicts new phenomena. The model here is less ambitious (and not a particularly good one either), for we merely wish to illustrate how one might approach the problem. It can lead to a good discussion in the classroom setting, especially during the winter. Some plausible assumptions might be as follows.

(i) Assume the snowball is a sphere of radius $r(t)$ at all times. This is almost certainly never the case, but the question becomes one of simplicity: is the snowball roughly spherical initially? Subsequently? Is there likely to be preferential warming and melting on one side even if it starts life as a sphere? The answer to this last question is yes: preferential melting will probably occur in the direction of direct sunlight unless the snowball is in the shade or the sky is uniformly overcast. If we can make this assumption, then the resulting surface area and volume considerations involve only the one spatial variable r.

(ii) Assume that the density of the snow/ice mixture is constant thoughout the snowball, so there are no differences in "snow-packing." This may be reasonable for small snowballs (i.e., hand-sized ones) but large ones formed by rolling will probably become more densely packed as their weight increases. A major advantage of the constant density assumption is that the mass (and weight) of the snowball is then directly proportional to its volume.

(iii) Assume the mass of the snowball decreases at a rate proportional to its surface area, and only this. This appears to make sense since it is the outside surface of the snowball that is in contact with the warmer air, which induces melting. In other words, the transfer of heat occurs at the surface. This assumption in particular will be examined in the light of the model's prediction. But even if it is a good assumption to make, is the "constant" of proportionality really constant? Might it not depend on the humidity of the air, the angle of incidence and intensity of sunlight, the external temperature, and so on?

(iv) Assume that no external factors change during the "lifetime" of the snowball. This is related to assumption (iii) above, and is probably the weakest of them all. Unless the melting time is very much less than a day it is safe to say that external factors will vary! Obviously, the angle and intensity of sunlight will change over time, and also possibly other factors as noted above.

Let's proceed on the basis of these four assumptions and formulate a model by examining some of their mathematical consequences. We may do so by asking further questions. For example,

(i) What are expressions for the mass, volume, and surface area of the snowball?
(ii) How do we formulate the governing equations? What are the appropriate initial and/or boundary conditions? How do we incorporate the information provided?
(iii) Can we obtain a solution (analytic, approximate, or numerical) of the equations?
(iv) What is the physical interpretation of the solution and does it make sense? That is, is it consistent with the information provided and are the predictions from the model reasonable?

Let $r(t)$ be the radius of the snowball at time t hours after the start of our "experiment," and let the initial radius of the snowball be $r(0) = R$. The surface area of a sphere of radius r is $4\pi r^2$ and its volume is $4\pi r^3/3$. If we denote the uniform density of the snowball by ρ, then the mass of the snowball at any time t is

$$M(t) = \tfrac{4}{3}\pi\rho r^3(t).\psi \qquad \text{(I.1)}$$

The instantaneous rate of change of the mass of the snowball (the derivative of $M(t)$ with respect to t) is then

$$\frac{dM}{dt} = 4\pi\rho r^2 \frac{dr}{dt}.\psi \qquad \text{(I.2)}$$

By assumption (iii), dM/dt is proportional to $S(t)$, the surface area at time t:

$$\frac{dM}{dt} = -4\pi r^2 k, \tag{I.3}$$

where k is the (positive) constant of proportionality, the negative sign implying that the mass is *decreasing* with time. By equating the last two expressions it follows that

$$\frac{dr}{dt} = -\frac{k}{\rho} = -\alpha, \text{ say.} \tag{I.4}$$

Note that this implies that according to this model, the radius of the snowball decreases uniformly with time. This means that the radius $r(t)$ is a linear function of t with slope $-\alpha$; since the initial radius is R, we must have

$$r(t) = R - \alpha t = R\left(1 - \frac{t}{t_{\mathrm{m}}}\right) = 0 \text{ when } t = \frac{R}{\alpha} = t_{\mathrm{m}}, \tag{I.5}$$

where t_{m} is the time for the original snowball to melt, which occurs when its radius is zero. However, we do not know the value of α since that information was not provided, but we *are* informed that after one hour, half the snowball has melted, so we have from equation (I.5) that $r(1) = R - \alpha$. A sketch of the linear equation in (I.5) and use of similar triangles (figure I.1) shows that

$$t_{\mathrm{m}} = \frac{R}{R - r(1)}, \tag{I.6}$$

and furthermore

$$\frac{V(1)}{V(0)} = \frac{1}{2} = \frac{r^3(1)}{R^3}, \tag{I.7}$$

so that

$$r(1) = 2^{-1/3} R \approx 0.79R.$$

Hence, $t_{\mathrm{m}} \approx 4.8$ hours, so that according to this model the snowball will take a little less than 4 more hours to melt away completely. This is a rather long time, and certainly the sun's position will have changed during that time

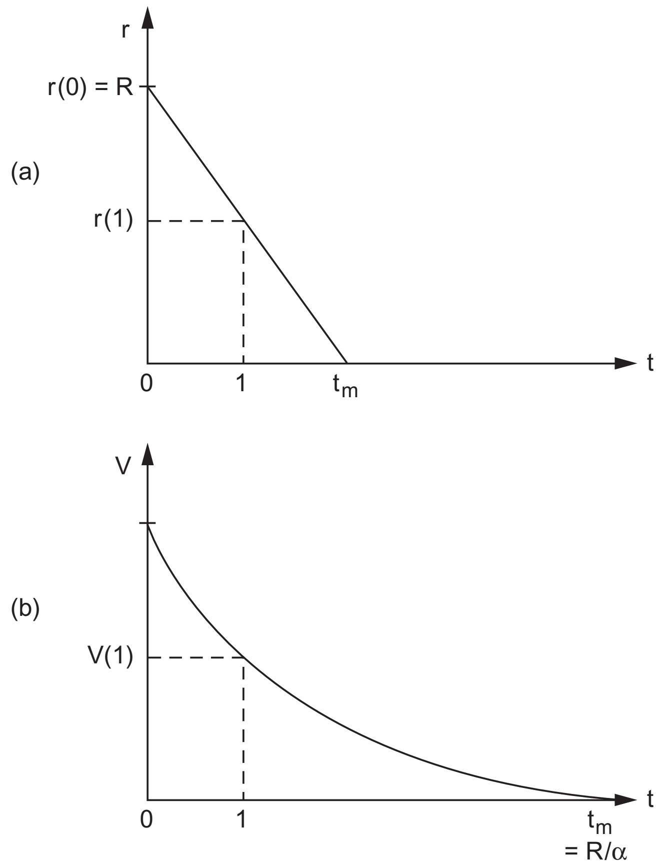

Figure I.1. Snowball radius *r*(*t*) and volume *V*(*t*)

(through an arc of about 70°), so in retrospect assumption (iv) is not really justified. A further implication of equation (I.5) is that the volume (and also mass of the snowball by assumption (ii)) decreases like a cubic polynomial in t, i.e.,

$$V(t) = V(0)\left(1 - \frac{t}{t_m}\right)^3 . \psi \qquad \text{(I.8)}$$

Note that $V'(t) < 0$ as required, and $V'(t_m) = 0$. Since $V''(t) > 0$, it is clear that the snowball melts more quickly at first, when $|V'|$ is larger, than at later times, as figure I.1(b) attests. I recall being told as a child by my mother that "snow waits around for more" but this model is hardly a proof of that, despite further revelations below! It may be adequate under some circumstances but there are obvious deficiencies given the initial data (which I invented). What other factors have been ignored here? Here are some.

We are all familiar with the fact that the consistency of snow varies depending on whether it is "wet" or "dry"; snowballs are more easily made with the former. Wet snow can be packed more easily and a layer of ice may be formed on the outside. This can in turn cool a thin layer of air around the surface, which will insulate (somewhat) the snowball from the warmer air beyond that. A nice clean snowball, as opposed to one made with dirty snow, may be highly reflective of sunlight (i.e., it has a high *albedo*) and this will reduce the rate of melting further. There are no doubt several other factors missing.

Some other aspects of the model are more readily appreciated if we generalize the original problem by suggesting instead that a fraction β of a snowball melts in h hours. The melting time is then found to be

$$t_{\mathrm{m}} = \frac{h}{1 - \sqrt[3]{1-\beta}}, \tag{I.9}$$

which depends linearly on h and in a monotonically decreasing manner on β. The dependence on h is not surprising; if a given fraction β melts in half the time, the total melting time is also halved. For a given value of h, the dependence on β is also plausible: the larger the fraction that melts in time h, the shorter the melting time.

Leaving the snowball to melt, we have seen in an elementary way from this example that certain fundamental steps are necessary in developing a mathematical model (see figure I.2): formulating a real world problem in mathematical terms using whatever appropriate simplifying assumptions may be necessary; solving the problem thus posed, or at least extracting sufficient information from it; and finally interpreting the solution in the context of the original problem. Thus, the "art" of good modeling relies on (i) a sound understanding and appreciation of the problem, such as what factors affect the melting rate of the snowball; (ii) a realistic, but not unnecessarily mathematical representation of the important phenomena; (iii) finding useful solutions, preferably quantitative ones, and (iv) interpretation of the mathematical results—yielding insights, predictions, such as when the snowball will melt away completely, and so on. Sometimes the mathematics used can be very simple, as above; indeed, the usefulness of a mathematical model should not be judged by the sophistication of the mathematics, but by its predictive capability, among other factors. Mathematical models are not necessarily "right" (though they may be wrong as a result of ignoring result of ignoring fundamental processes). One model may be better than another in that it has

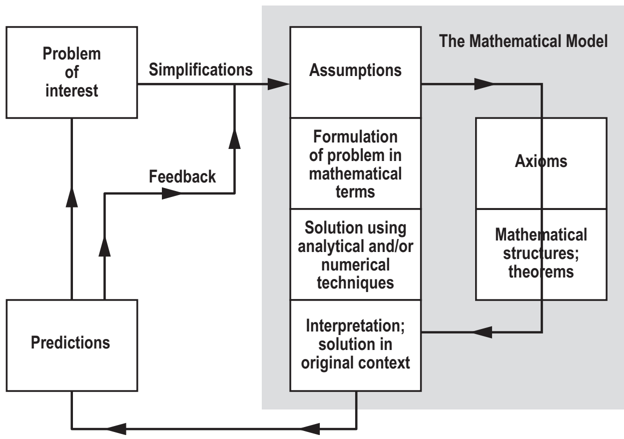

Figure I.2. Steps associated with developing a mathematical model

better explanatory features, or more specific predictions can be made, which are subsequently confirmed, at least to some degree. Sometimes models can be controversial: this is a good thing, for it generates scientific and mathematical discussion. Indeed, I venture to go further and suggest that all mathematical models are flawed to some extent: many by virtue of inappropriate assumptions made in formulating the model, or (which may amount to the same thing) by the omission of certain terms in the governing equations, or even by misinterpretation of the mathematical conclusions in the original context of the problem. Occasionally, models may be incorrect because of errors in the mathematical analysis, even if the underlying assumptions are valid. And paradoxically it can happen that even a less accurate model is preferable to a more mathematically sophisticated one; it was the mathematical statistician John Tukey who stated that "it is better to have an approximate answer to the right question than an exact answer to the wrong one." But another comment must be made about modeling, and mathematical problems in general.

Inverse problems: what is a question to which the answer is . . . ?

> Usually in mathematics you have an equation and you want to find a solution. Here you were given a solution and you had to find the equation. I liked that.
> —Julia Robinson (as quoted by C. W. Groetsch)

Many problems can be posed in formal terms, such as $A \Longrightarrow ?$, meaning *what does A imply*? Thus, we might ask, *What is 47 times 59?* or *Given a bird's egg, is it possible to describe its shape mathematically?* (see Q.62–Q.67) or, as posed above, *How long will it take a snowball to melt?* Most people encounter only these so-called *direct problems* in their mathematical education. But there is another side to all this, indeed, a whole new universe of potential questions in the form of *What implies A*? Life is full of such *inverse problems*, as they are called, though we may be forgiven for not recognizing it. Given a "spot" on an X-ray, diagnosticians seek to determine what has caused it. An ultrasound examination is used to determine whether the baby in the womb is a girl or a boy. Radar (or sonar) is used to infer "what is out there," or more precisely, what object is causing this particular pattern of electromagnetic (or acoustic) wave reflection. I've long been fascinated by the concept of inverse vs. direct problems. Let me try to explain more precisely what I mean by these descriptors. There are many levels on which we can think about this, the simplest perhaps being with regard to multiplication: as above, let's ask, *What is the product of 47 and 59? 2773*, you instantly reply. Now, find a pair of factors for 9831. It's not quite such an easy task to find the factors 87 and 113 (and there are other possible pairs because 87 is 3 times 29, i.e., the prime factors are 3, 29, and 113, so clearly the decomposition is not unique).

And, incidentally, this illustrates another point: as with many direct problems, answers to inverse problems may not be unique. In a fascinating article on these problems, Joseph Keller introduces this point with three inverse word problems, one of which is *What is a question to which the answer is 9W*? In a classroom setting this invariably produces the standard question, *What is 9 times W*?, dismissed immediately (though correct) as being far too pedestrian! I try to encourage the students to think a little more "outside the box," so we do get questions like *What's your shoe size*? (so perhaps it's a shoebox) and *What route gets you from "A" to "B"*?, and these are quite satisfactory. Keller's question, wonderfully creative (and, in my view, very funny), is, *Do you spell your name with a "V", Herr Wagner?* Herr Wagner's answer is, in fact, *Nein, W*.

Another elementary example, this time from ballistics, is valuable, especially if we can neglect air resistance and the curvature and the rotation of the earth, so the only force to consider is that of gravity. The direct problem is then to determine the range of the projectile (a cannonball, say) given its initial speed and the angle of elevation of the barrel. There is a readily found unique solution to this problem, using Newton's second law of motion: neglecting air resistance, the distance d traveled by a projectile with angle of elevation α $(0 < \alpha < \pi/2)$ and initial speed u_0 is given by the formula

$$d = \frac{u_0^2}{g}\sin 2\alpha. \tag{I.10}$$

Therefore, the direct problem (given u_0 and α, find d) has a unique solution. On the other hand, given the range of the cannonball with u_0 fixed, there may be zero, one, or two possible solutions for the inverse problem: if $d < u_0^2/g$ is specified, then these solutions can be seen to exist from the graph of equation (I.10), and they are symmetrically placed about the line $\alpha = \pi/4$, the angle for maximum range d. Analytically, the solutions are

$$\alpha_1 = \frac{1}{2}\arcsin\left(\frac{gd}{u_0^2}\right), \quad \alpha_2 = \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{gd}{u_0^2}\right), \tag{I.11}$$

as is easily verified.

Exercise: Establish the results (I.10) and (I.11).

As pointed out by Groetsch in the book cited below, many of the major historical breakthoughs in science are a result of solving, in essence, an inverse problem. Thus, the curved shadow of the earth on the surface of the moon during a lunar eclipse enabled Plato's student Aristotle (384–322 BC) to infer (among other arguments) that the earth was spherical. This indirect form of reasoning was used, still within an astronomical context, by Newton to derive the inverse-square law of gravitation from Kepler's laws of planetary motion, which were themselves inferred from Tycho Brahe's observations. Furthermore, the young English mathematician John Couch Adams (1819–1892) and the French mathematician Urbain LeVerrier (1811–1877) independently used deviations in the observed position of the outermost of the planets known at the time, Uranus, to infer the existence and position of a perturbing body beyond Uranus: the planet Neptune. When telescopes were turned to the region of sky predicted by the mathematics, the inverse problem and its solution were validated—there was the new planet!

These simple models illustrate much of the fundamental approach I have taken in the "walks" discussed in this book. None are exhaustive or complete; and therefore all are open to improvement. Some are very basic and simplistic (of the "back-of-the-envelope" variety), while others are more sophisticated. Some are direct problems and some are inverse problems (try to determine which). So from my backyard to the Rockies and beyond, allow me to invite you to accompany me on my mathematical nature walks.

References

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