Introduction

This book is intended to be an introduction to a rich and elegant area of Ramsey theory that concerns itself with coloring infinite sequences of objects and which is for this reason sometimes called infinite-dimensional Ramsey theory. Transferring basic pigeon hole principles to their higher dimensional versions to increase their applicability is thus the subject matter of this theory. In fact, this tendency in Ramsey theory could be traced back to the invention of the original Ramsey theorem, which is nothing other than a higher dimensional version of the principle that says that a finite coloring of an infinite set must involve at least one infinite monochromatic subset. Ramsey's original application of the finite-dimensional Ramsey theorem was to obtain a rough classification of relational structures on the set $\mathbb{N}$ of natural numbers that he needed for a decision procedure that would test the validity of a certain kind of logical sentence. This original application of the finite-dimensional Ramsey theorem was matched in depth only forty years later by the Brunel-Sucheston use of this theorem in showing the existence of the so-called spreading model of a given Banach space, a notion that has eventually triggered important developments in that area of mathematics. The infinite-dimensional extension was also done for utilitarian reasons. It was initiated forty years ago by Nash-Williams in the course of developing his theory of better-quasi-ordered sets that eventually led him to the proof of that trees are well-quasi-ordered under the embedability relation. The full statement of the infinite-dimensional Ramsey theorem came, however, only through the work of Galvin-Prikry, Silver, Mathias, and especially Ellentuck, who was the first to use topological notions to describe what is today generally considered the optimal form of this result. In this book we present a general procedure to transfer any other Ramsey theoretic principles to higher and especially infinite dimensions trying to match the clarity of the Ellentuck result, but going beyond his topological Ramsey theory. As seen in the prototype example of the Ramsey space of infinite sequences of words and variable words over a fixed finite alphabet, topological Ramsey theory fails to capture the situation in which the objects that generate combinatorial subspaces are not the objects that one colors. For this, one needs the new theory of Ramsey spaces in which there are no natural topologies to describe the complexity of the allowed colorings. In other words, the new theory addresses not only the challenging problem of finding the right hypothesis for the colorings but also the problem of whether such a hypothesis can be preserved under classical operations such as the Souslin operation. The topological Ramsey theory of Ellentuck relies at this point on the classical result of Nikodym asserting that the Baire property relative to an arbitrary topology is preserved under the Souslin operation, while general Ramsey space theory requires a special proof of the corresponding fact. The abstract infinite-dimensional Ramsey theorem that we prove in Chapter Four leads to many interesting examples of Ramsey spaces. We had to be quite selective when choosing which of these Ramsey spaces to present in some detail and which not, and our choices are all made on the basis of known applications of these spaces. It is expected that in the years to come many other Ramsey spaces will find similar applications explaining our main motivation for writing this book.

The book is organized as follows. The Appendix gathers some special notation and supplementary material to help the reader in following the book. Preliminaries about the Ramsey theorem are given in Chapter One. This chapter is intended to serve as an indication of the general high-dimensional Ramsey theory that will be developed from Chapter Four on. So the reader who is encountering this material for the first time is kindly asked to patiently wait for a more thorough understanding until the abstract theory is developed in later chapters. As it will be seen, however, we assume nothing more from the reader than the familiarity with the general mathematical culture. The basic pigeon hole principles used in the rest of the book are briefly presented in Chapters Two and Three. The reader is, however, advised to skip these two chapters on first reading and instead return to a particular pigeon hole principle when needed. The reason for being brief here is that all of these pigeon hole principles are well known, and have already whole monographs devoted to them and so we found no reason to reproduce more than is needed to make this book partially self-contained. The abstract Ramsey theorem is given in Chapter Four. The two chapters that follow Chapter Four are devoted to analysis of this theorem based on one of the basic principles given in Chapters Two and Three. Local Ramsey theory is presented in Chapter Seven. The optimal parametrization of the infinite-dimensional Ramsey theorem is given in Chapter Nine, a chapter which is in part based on Chapter Eight and deals with the Ramsey theory of products of finite sets. Historical notes, remarks, and suggestions for further reading and information about related developments are given at the end of each chapter.