Introduction

WHAT IS VALIDATED NUMERICS?

Validated numerics is the field aiming at bridging the gap between scientific computing and pure mathematics – between speed and reliability.

One strain of the field aims at pushing the frontiers of computer–aided proofs in mathematical analysis. This area of research deals with problems that cannot be solved by traditional mathematical methods. Typically, such problems have a global component (e.g. a large search space) as well as a non-linear ingredient (the output is not proportional to the input). These problems have traditionally been studied through numerical simulations, and therefore our knowledge of these lack the rigour demanded by a formal proof.

Validated numerics aims to bridge the gap between a numerically observed phenomenon, and its mathematical counter-part. This is achieved by developing a means of computing numerically yet with mathematical rigour. Validated numerics merges pure mathematics with scientific computing in a novel way: instead of computing *approximations* to sought quantities, the aim is to compute *enclosures* of the same. The width of such an enclosure gives a direct quality measurement of the computation, and can be used to adaptively improve the calculations at run-time. This fundamental change of focus results in efficient and extremely fault-tolerant numerical methods, ideal for the systematic study of complex systems. As such, validated numerics can play an instrumental role as it is the only way to certify that a numerical computation meets required error tolerances. As computer simulations are gradually replacing physical experiments, this type of certification is of utmost importance.

Many challenging problems share the same type of inaccessibility due to nonlinearities affecting the global behaviour of the systems. Neither tools from pure mathematics nor scientific computation alone have been successful in establishing quantitative information for such complex systems. The field of validated numerics aims not only to develop the mathematical foundation needed to overcome these obstacles, but also to produce concrete numerical methods able to provide mathematical statements about such systems.

THE SCOPE AND AIM OF THIS BOOK

The main goal of this text is to introduce the reader to the field of validated numerics by providing a theoretical foundation supplemented with illuminating examples.

The target audience is undergraduate or graduate students new to the field, and who are not necessarily trained mathematicians. Some knowledge of programming is useful, but not strictly necessary. The restriction to the one-dimensional setting is a conscious one. It allows us to focus exclusively on simple, but yet interesting problems, without too much mathematical framework. The computer exercises are intended to entice the reader into actually discovering how simple it is to implement the methods, and to solve numerical problems with rigor.

FURTHER READING

There are several books that treat the topic of validated numerics, all with different scopes. Some of my favourites include the classic by Moore [Mo66], which has now been superseded by [Mo79, MK09], together with the implementation–geared [KM81] by Kulish and Miranker, and the comprehensive [AH83] by Alefeld and Herzberger. The two books by Neumaier [Ne90, Ne01] are a must, and the ones by Aberth [Ab88, Ab98] offer some interesting reading too. Complex–valued problems are considered in [PP98]. Some more recent good books are [WH03], by Walster and Hansen, aimed at global optimization (as is Kearfott's book [Ke96]), and [JK01], by Jaulin et. al., which focuses on constraint propagation. Of course, nowadays the World Wide Web has a lot to offer, and the interval community has its place there too, see http://www.cs.utep.edu/interval-comp/main.html for a vast collection of references, software, upcoming conferences, research groups, mailing lists, etc.

ACKNOWLEDGMENTS

I would like to thank all students who have helped me transform my somewhat erratic lecture notes into a more self-contained, polished manuscript. This has been a lengthy process, and without the enthusiasm of the participants of my course on validated numerics, I would not have come this far.

Warwick Tucker
Uppsala
April 27, 2010