Introduction

The purpose of this book is to use the hypoelliptic Laplacian to evaluate semisimple orbital integrals, in a formalism that unifies index theory and the trace formula.

0.1 The trace formula as a Lefschetz formula

Let us explain how to think formally of such a unified treatment, while allowing ourselves a temporarily unbridled use of mathematical analogies. Let X be a compact Riemannian manifold, and let Δ^X be the corresponding Laplace-Beltrami operator. For $t > 0$, consider the trace of the heat kernel $\mathrm{Tr}\left[\exp\left(t\Delta^X/2\right)\right]$. If L_2^X is the Hilbert space of square-integrable functions on X, $\mathrm{Tr}\left[\exp\left(t\Delta^X/2\right)\right]$ is the trace of the 'group element' $\exp\left(t\Delta^X/2\right)$ acting on L_2^X.

Suppose that L_2^X is the cohomology of an acyclic complex R on which Δ^X acts. Then $\mathrm{Tr}\left[\exp\left(t\Delta^X/2\right)\right]$ can be viewed as the evaluation of a Lefschetz trace, so that cohomological methods can be applied to the evaluation of this trace. In our case, R will be the fibrewise de Rham complex of the total space $\widehat{\mathcal{X}}$ of a flat vector bundle over X, which contains TX as a subbundle. The Lefschetz fixed point formulas of Atiyah-Bott [ABo67, ABo68] provide a model for the evaluation of such cohomological traces.

The McKean-Singer formula [McKS67] indicates that if $\square^R$ is a Hodge like Laplacian operator acting on R and commuting with Δ^X, for any $b > 0$,

$$\mathrm{Tr}^{L_2^X}\left[\exp\left(t\Delta^X/2\right)\right] = \mathrm{Tr_s}^R\left[\exp\left(t\Delta^X/2 - t\square^R/2b^2\right)\right]. \qquad (0.1)$$

In (0.1), $\mathrm{Tr_s}$ is our notation for the supertrace. Note that the formula involves two parameters: t is a parameter in a Lie algebra, and $1/b^2$ is a genuine time parameter. For $b \to 0$, the right-hand side of (0.1) obviously converges to the left-hand side.

To establish the Atiyah-Bott formulas, the heat equation method of Gilkey [Gi73, Gi84] and Atiyah-Bott-Patodi [ABoP73] consists in making $b \to +\infty$ in (0.1), and to show that the local supertrace in the right-hand side of (0.1) localizes on the fixed point set of the isometry $\exp\left(t\Delta^X/2\right)$, while exhibiting the nontrivial local cancellations anticipated by McKean-Singer [McKS67]. One should obtain formulas this way that are analogous to the fixed point formulas of [ABo67, ABo68].

The present book is an attempt to make sense of the above, in the case where X is a compact locally symmetric space of noncompact type. In this case, the Selberg trace formula should be thought of as the evaluation of a Lefschetz trace. Contrary to what happens in Atiyah-Bott [ABo67, ABo68], the operator $\mathcal{L}_b^X = \Delta^X/2 - \Box^R/2b^2$ is not elliptic, but just hypoelliptic.

0.2 A short history of the hypoelliptic Laplacian

Let us now give the proper rigorous background to the present work. Let X be a compact Riemannian manifold, let $\mathcal{X}, \mathcal{X}^*$ be the total spaces of its tangent and cotangent bundle. In [B05], we introduced a deformation of the classical Hodge theory of X. The corresponding Laplacian $L_b^X, b > 0$ is a hypoelliptic operator acting over $\mathcal{X}^*$. It is essentially the weighted sum of the harmonic oscillator along the fibre and of the generator of the geodesic flow. Arguments given in [B05] showed that as $b \to 0$, the operator L_b^X should converge in the proper sense to the Hodge Laplacian $\Box^X/2$ of X via a collapsing mechanism, and that as $b \to +\infty$, L_b^X converges to the generator of the geodesic flow.

The program outlined in [B05] was carried out in Bismut-Lebeau [BL08], at least for bounded values of b. In [BL08], it was shown that in a very precise way, L_b^X converges to $\Box^X/2$. A consequence of the results of [BL08] is that given $t > 0$, $\exp\left(-tL_b^X\right)$ is trace class, and that as $b \to 0$, its trace converges to the trace of $\exp\left(-t\Box^X/2\right)$. The spectral theory of L_b^X was also studied in [BL08], as well as its local index theory. An important result of [BL08] is that if F is a flat vector bundle on X, the Ray-Singer metric on $\det H^{\cdot}(X, F)$ one can attach to L_b^X coincides with the classical elliptic Ray-Singer metric [RS71, BZ92]. This paves the way to a possible proof using the hypoelliptic Laplacian of the Fried conjecture [Fri86, Fri88] concerning the relation of the Ray-Singer torsion to special values of the dynamical zeta function of the geodesic flow.

In [B08a], we gave a deformation of the classical Dirac operator, the deformed Dirac operator acting over $\mathcal{X}$, its square still being hypoelliptic, and having the same analytic structure as the operator L_b^X. Results similar to the ones in [BL08] were established in [B08a] for Quillen metrics.

As a warm-up to the present book, if G is a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$, we produced in [B08b] a deformation of the Casimir operator of G to a hypoelliptic Laplacian over $G \times \mathfrak{g}$ acting on smooth sections of $\Lambda^{\cdot}(\mathfrak{g}^*)$, and we showed that the supertrace of its heat kernel coincides with the trace of the scalar heat kernel of G. In particular, the spectrum of the Casimir operator is embedded as a fixed part of the spectrum of the hypoelliptic Laplacian. By making $b \to +\infty$, we recovered known formulas [Fr84] expressing the heat kernel of G as Poisson sums over its coroot lattice. The deformation of the Casimir operator in [B08b] was obtained via a deformation of the Dirac operator of Kostant [Ko76, Ko97], whose square coincides, up to a constant, with the Casimir operator.

The question arises of knowing whether, in the case of locally symmetric spaces of noncompact type, a deformation of the Casimir operator to a hypoelliptic operator is possible, which would have the same interpolation properties as before, and would produce a version of the Selberg trace formula. Such a construction would provide a justification for the formal considerations we made in section 0.1. The purpose of this book is to show that this question has a positive answer.

In this book, the spectral side of the trace formula will be essentially ignored. Our main result, which is given in chapter 6, is an explicit local formula for certain semisimple orbital integrals in a reductive group. The Selberg trace formula expresses the trace of certain trace class operators as a sum of orbital integrals. What is done in the book is to evaluate these orbital integrals individually, by a method that is inspired by index theory.

0.3 The hypoelliptic Laplacian on a symmetric space

First, we will explain the construction of the hypoelliptic Laplacian that is carried out in the present book.

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$, let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, let B be an invariant nondegenerate bilinear form on $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$. Let $U(\mathfrak{g})$ be the enveloping algebra. Let $C^{\mathfrak{g}} \in U(\mathfrak{g})$ be the Casimir element.

Set $X = G/K$. Then X is contractible. Moreover, if $\rho^E : K \to \mathrm{Aut}(E)$ is a unitary representation of K, E descends to a Hermitian vector bundle F on X.

Then $\mathfrak{p}, \mathfrak{k}$ descend to vector bundles TX, N, TX being the tangent bundle of X. Moreover, $TX \oplus N$ can be canonically identified with the trivial vector bundle $\mathfrak{g}$ over X, and so it is equipped with a canonical flat connection. Let $\pi : \mathcal{X} \to X, \pi : \mathcal{X} \to X$ be the total spaces of $TX, TX \oplus N$.

Let $c(\mathfrak{g})$ be the Clifford algebra of $(\mathfrak{g}, -B)$. Following Kostant [Ko76, Ko97], in Definition 2.7.1, we introduce the Dirac operator $D^{\mathfrak{g}} \in c(\mathfrak{g}) \otimes U(\mathfrak{g})$, whose square coincides, up to a constant, with the negative of the Casimir.

In Definition 2.13.1, from $-\frac{1}{2} D^{X,2}$, we obtain the elliptic operator $\mathcal{L}^X$ acting on $C^\infty(X, F)$. Up to a constant, $\mathcal{L}^X$ coincides with the action of $\frac{1}{2} C^{\mathfrak{g}}$ on $C^\infty(X, F)$, so that $\mathcal{L}^X$ is an elliptic operator.

Also, the Dirac operator $D^{\mathfrak{g}}$ descends to an operator $D^{\mathfrak{g},X}$ acting on

$$C^\infty\left(\mathcal{X}, \pi^* \left(\Lambda^\cdot (T^*X \oplus N^*) \otimes F\right)\right).$$

In Definition 2.9.1 and in section 2.12, for $b > 0$, we introduce the operator $\mathfrak{D}_b^X$ that acts on the above vector space. It is given by the formula

$$\mathfrak{D}_b^X = D^{\mathfrak{g},X} + ic\left(\left[Y^N, Y^{TX}\right]\right) + \frac{1}{b}\left(\mathcal{D}^{TX} + \mathcal{E}^{TX} - i\mathcal{D}^N + i\mathcal{E}^N\right). \quad (0.1)$$

In (0.1), c denotes the natural action of the Clifford algebra of $(TX \oplus N, B)$ on $\Lambda^\cdot (T^*X \oplus N^*)$. Also the operator $\mathcal{D}^{TX} + \mathcal{E}^{TX} - i\mathcal{D}^N + i\mathcal{E}^N$ is some version of the standard Dirac operator along the Euclidean fibre $TX \oplus N$ introduced by Witten [Wi82].

In Definition 2.13.1, we define the operator $\mathcal{L}_b^X$, which is given by

$$\mathcal{L}_b^X = -\frac{1}{2} D^{\mathfrak{g},X,2} + \frac{1}{2} \mathfrak{D}_b^{X,2}. \tag{0.2}$$

As an aside, we suggest the reader compare the operator that appears in the right-hand side of (0.1) with the right-hand side of (0.2).

The following formula is established in section 2.13,

$$\begin{aligned}\mathcal{L}_b^X = \frac{1}{2} \left| \left[Y^N, Y^{TX} \right] \right|^2 + \frac{1}{2b^2} \left(-\Delta^{TX \oplus N} + |Y|^2 - m - n \right) + \frac{N^{\Lambda^{\cdot}(T^*X \oplus N^*)}}{b^2} \\ + \frac{1}{b} \Bigg(\nabla_{Y^{TX}}^{C^\infty\left(TX \oplus N, \pi^*\left(\Lambda^{\cdot}(T^*X \oplus N^*) \widehat{\otimes} F\right)\right)} + c\left(\mathrm{ad}\left(Y^{TX}\right)\right) \\ - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta \mathrm{ad}\left(Y^N\right)\right) - i\rho^E\left(Y^N\right) \Bigg). \end{aligned} \tag{0.3}$$

Let us just say that in (0.3), $\Delta^{TX \oplus N}$ is the Euclidean Laplacian along the fibre $TX \oplus N$. Also observe that in $\mathcal{L}_b^X$, the harmonic oscillator along the fibres of the Euclidean vector bundle $TX \oplus N$ appears with the factor $1/b^2$. By results of Hörmander [Hö67], the operator $\frac{\partial}{\partial t} + \mathcal{L}_b^X$ is hypoelliptic. Its structure is very close to the structure of the hypoelliptic Laplacian in [B05, BL08, B08a].

By adding a trivial matrix operator A to $\mathcal{L}^X, \mathcal{L}_b^X$, we obtain operators $\mathcal{L}_A^X, \mathcal{L}_{A,b}^X$. In the context of the present book, the operator $\mathcal{L}_{A,b}^X$ will be called a hypoelliptic Laplacian.

0.4 The hypoelliptic Laplacian and its heat kernel

In chapter 11, the proper functional analytic machinery is developed, in order to obtain a chain of Sobolev spaces on which the hypoelliptic Laplacian acts as an unbounded operator. This is done by inspiring ourselves from our previous work with Lebeau [BL08, chapter 15], which is valid for the case where the base manifold is compact. Also regularizing properties of its resolvent and of its heat operator are obtained. The heat operator is shown to be given by a smooth kernel.

A probabilistic method is also given to construct the heat operator on $\mathcal{X}$ or $\mathcal{X}$. The fact that the functional analytic and probabilistic constructions coincide is proved using the Itô calculus. The probabilistic construction of the heat kernel is relatively easy, but does not give the refined properties on the resolvent that one obtains by the functional analytic machinery.

In the remainder of the book, most of the hard analysis is done via the probabilistic construction of the heat kernel, while the functional analytic estimates do not play a significant role. This is because contrary to the situation in [BL08], where it was essential to obtain proper understanding of the spectral properties of the hypoelliptic Laplacian, here, this aspect can be essentially disregarded.

0.5 Elliptic and hypoelliptic orbital integrals

Let $\gamma \in G$ be semisimple. In chapter 4, we introduce the heat operators $\exp\left(-t\mathcal{L}_A^X\right)$, $\exp\left(-t\mathcal{L}_{A,b}^X\right)$, and we define the corresponding orbital integrals $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_A^X\right)\right], \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_{A,b}^X\right)\right]$. These orbital integrals are said to be respectively elliptic and hypoelliptic. As the notation suggests, the elliptic orbital integrals are generalized traces, while the hypoelliptic orbital integrals are generalized supertraces. While the existence of the elliptic orbital integrals follows from standard Gaussian estimates for the heat kernel for $\exp\left(-t\mathcal{L}_A^X\right)$ on X, the existence of the orbital integrals for $\exp\left(-t\mathcal{L}_{A,b}^X\right)$ relies on a nontrivial estimate on the hypoelliptic heat kernel $q_{b,t}^X\left((x,Y),(x',Y')\right)$. This estimate is stated in Theorem 4.5.2. It says that given $\epsilon > 0, M > 0, \epsilon \leq M$, there exist $C > 0, C' > 0$ such that for $0 < b \leq M, \epsilon \leq t \leq M$, and $(x,Y),(x',Y') \in \mathcal{X}$, then

$$q_{b,t}^X\left((x,Y),(x',Y')\right) \leq C\exp\left(-C'\left(d^2(x,x') + |Y|^2 + |Y'|^2\right)\right). \tag{0.1}$$

0.6 The limit as $b \to 0$

In Theorem 4.6.1, we prove that for any $b > 0, t > 0$,

$$\mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_{A,b}^X\right)\right] = \mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_A^X\right)\right]. \tag{0.1}$$

Equation (0.1) is closely related to a corresponding identity established for ordinary traces over a compact Lie group G in [B08b].

The proof of (0.1) consists of two steps. The fact that the left-hand side of (0.1) does not depend on $b > 0$ is proved by a method very closely related to the proof of the McKean-Singer formula [McKS67] in index theory. It is in this sense that the book unifies index theory and the evaluation of orbital integrals.

The proof of (0.1) is then reduced to showing that as $b \to 0$, the left-hand side converges to the right-hand side. Proving this fact is obtained by a nontrivial analysis of the heat kernel for $\exp\left(-t\mathcal{L}_{A,b}^X\right)$. The uniform estimate (0.1) plays a crucial role in the proof. In section 0.8, we will give more details on the analytic arguments used in the book.

0.7 The limit as $b \to +\infty$: an explicit formula for the orbital integrals

Our final formula is obtained by making $b \to +\infty$ in (0.1). Let $d_\gamma(x) = d(x, \gamma x)$ be the displacement function associated with γ [BaGSc85], which is known to be a convex function. Let $X(\gamma) \subset X$ be its critical set, which is a totally geodesic submanifold of X. Then $X(\gamma)$ is the symmetric space associated with the centralizer $Z(\gamma) \subset G$ of γ. As $b \to +\infty$, the analysis of the hypoelliptic orbital integral $\mathrm{Tr_s}^{[\gamma]} \exp -t\mathcal{L}^X_{A,b}$ localizes near $X(\gamma)$. More precisely, in chapter 9, to obtain the asymptotics of $\mathrm{Tr_s}^{[\gamma]} \exp -t\mathcal{L}^X_{A,b}$, we choose $x \in X(\gamma)$, and we take the suitable expansion of $\mathcal{L}^X_{A,b}$ near the geodesic connecting x and γx. This expansion involves a rescaling of coordinates, and also a corresponding Getzler rescaling [Ge86] of the Clifford variables c.

Ultimately, the limit operator is a hypoelliptic operator acting on $\mathfrak{p} \times \mathfrak{g}$. The existence of a canonical flat connection over $TX \oplus N$ plays a critical role in the computations. Combining the existence of this flat connection with the existence of the central Casimir operator introduces two major differences with respect to what was done in [B05, BL08, B08a].

After conjugation, we may assume that $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K$, and $\mathrm{Ad}(k)a = a$. Let $\mathfrak{k}(\gamma) \subset \mathfrak{k}$ be the Lie algebra of the centralizer of γ in K. Our explicit formula for $\mathrm{Tr}^{[\gamma]} \exp -t\mathcal{L}^X_{A,b}$ is stated in Theorem 6.1.1. It is given by an explicit integral over $\mathfrak{k}(\gamma)$. In chapter 6, we show how to derive corresponding formulas for arbitrary kernels, which include the wave kernel. This is all the more remarkable, since, contrary to $\mathcal{L}^X_A$, $\mathcal{L}^X_{A,b}$ does not have a wave kernel.

0.8 The analysis of the hypoelliptic orbital integrals

In the analysis of the hypoelliptic orbital integrals, there is some overlap with the analysis of the hypoelliptic Laplacian in [BL08]. In [BL08], the Riemannian manifold X was assumed to be compact, and genuine traces or supertraces were considered. Here X is noncompact, and the orbital integrals that appear in (0.1) are defined using explicit properties of the corresponding heat kernels like the estimate in (0.1). Such estimates do not follow from [BL08].

In [BL08], the limit as $b \to 0$ of hypoelliptic supertraces was studied by functional analytic methods involving semiclassical pseudodifferential operators. Chapter 17 in [BL08] is entirely devoted to this question. Since here X is noncompact, and since we deal explicitly with the kernels of the considered operators, the results of [BL08] cannot be used as such.

Finally, the limit as $b \to +\infty$ involves questions that were not addressed in [BL08]. Again uniform estimates are needed.

In the present book, these analytic questions are dealt with by a combination of probabilistic and analytic methods. In probability, we use the Itô calculus, and also the stochastic calculus of variations, or Malliavin calculus [M78, St81b, B81a, M97].

0.9 The heat kernel for bounded b and the Malliavin calculus

Estimates like (0.1) are essentially obtained in three steps:

1. In chapter 12, we obtain rough estimates on scalar heat kernels associated with a scalar version of $\mathcal{L}_b^X$. By rough estimates, we mean uniform bounds on the kernels and their derivatives of arbitrary order. Such bounds are obtained using the Malliavin calculus. Also we study the limit as $b \to 0$ of the scalar hypoelliptic heat kernel.

2. In chapter 13, using the semigroup property of the scalar heat kernel combined with the rough bounds, we establish decay estimates similar to (0.1) for the scalar heat kernel.

3. In chapter 14, the estimates for the scalar heat kernel are transferred to the kernel $q_{b,t}^X$.

We will briefly explain why probabilistic methods are relevant for step (1). Note that the geodesic flow on X is a differential equation. Also, by (0.3), $\mathcal{L}_{A,b}^X$ is a differential operator of order 1 in the variables in X, while being of order 2 in the variables in $TX \oplus N$. To the scalar part of $\mathcal{L}_{A,b}^X$, one can associate a stochastic differential equation on $\mathcal{X}$, which projects to a differential equation on X. This differential equation is a perturbation of the geodesic flow. The heat equation semigroup for the scalar part of $\mathcal{L}_{A,b}^X$ describes the probability law in $\mathcal{X}$ of the corresponding diffusion process at a given time t.

The Malliavin calculus consists in exploiting the structure of the stochastic differential equation. More precisely, the properties of the heat kernel are obtained by using the fact that the scalar heat kernel is the image by the stochastic differential equation map Φ of a classical Brownian measure. Integration by parts on Wiener space can then be used to control the derivatives of the heat kernel. Estimates on heat kernels are ultimately obtained via the estimation of the Malliavin covariance matrix $\Phi'\Phi'^*$.

For bounded b, estimating the covariance matrix is essentially equivalent to the proper uniform control of an action functional depending on $b > 0$. For $b > 0$, if $x_s, 0 \leq s \leq t$ is a smooth curve with values in X with fixed $(x_0, \dot{x}_0), (x_t, \dot{x}_t)$, set

$$H_{b,t}(x.) = \frac{1}{2} \int_0^t \left|\dot{x}\right|^2 + b^4 \left|\ddot{x}\right|^2 \, ds. \tag{0.1}$$

This action functional was introduced in [B05] for smooth curves in X, and the corresponding variational problem was studied by Lebeau [L05]. Still problems remained because of the possible nonsmoothness of the solution of the associated Hamilton-Jacobi equation. However, it turns out that the estimate of the Malliavin covariance matrix represents a tangent variational problem, which can be controlled by the solution of a related variational problem, where X is replaced by the Euclidean vector space $\mathfrak{p}$. This problem had precisely been studied by Lebeau in [L05] as a warm-up to a full understanding of the variational problem on X.

This is why, prior to chapter 12, we devote chapter 10 to a detailed study of the above variational problem on an Euclidean vector space. The results of chapter 10 are used in chapter 12 to obtain a control of the integration by parts formula. Besides, when properly interpreted, chapter 10 can be viewed as an explicit verification of the soundness of our method of proof, when G is an Euclidean vector space. The fact that in this case, the intermediate steps can be made completely explicit is of special interest.

In chapter 12, we also obtain the limit as $b \to 0$ of the scalar hypoelliptic heat kernel. As explained before, when the base manifold is compact, a functional analytic version of this problem was solved in Bismut-Lebeau [BL08, chapter 17]. Here, this result is reobtained by probabilistic methods for the noncompact manifold X.

In chapter 13, we obtain a Gaussian decay of the scalar heat kernel similar to (0.1) using the rough bounds in chapter 12, and also by exploiting the semigroup property. Probabilistic methods are still used, but they are more elementary than in chapter 12. One difficulty is to show that as $b \to 0$, in spite of the fact that the energy of the underlying diffusion in X tends to $+\infty$ as $b \to 0$, this diffusion does not escape to infinity in X, the energy being absorbed by random fluctuations. Ultimately, the estimates follow easily from the rough bounds, and from Mehler's formula for the heat kernel of the harmonic oscillator.

In chapter 14, the transfer of the estimates for the scalar heat kernel to estimates on $q^X_{b,t}$ is obtained using a matrix version of the Feynman-Kac formula. In the case where F is nontrivial, the symmetric space associated with the complexification $K_{\mathbf{C}}$ of K plays an important role.

Let us point out that many of our estimates are still valid in the case of variable curvature. In particular, the techniques developed in the present book can be used to give a different proof of some of results established in [BL08], with explicit estimates on the heat kernels.

Finally, let us explain in more detail how we use our estimates. The uniform bounds on the heat kernels are needed to prove that the orbital integrals are well defined, and also to show that dominated convergence can be applied to the integrand defining the orbital integral when $b \to 0$ and $b \to +\infty$. The bounds on the higher derivatives are needed when computing the limit of the orbital integrals. This is done by establishing uniform bounds over compact subsets on the kernels and their derivatives, and by proving that the heat kernels converge in a weak sense. Ultimately, we get pointwise convergence, which combined with the uniform bounds on the heat kernels, gives the convergence of the orbital integrals.

0.10 The heat kernel for large b, Toponogov, and local index

In chapter 15, the hypoelliptic heat kernel is studied as $b \to +\infty$. Note that by (0.3), for large $b > 0$, after rescaling $Y \in TX \oplus N$,

$$\mathcal{L}_b^X \simeq \frac{1}{2} b^4 \left|\left[Y^{TX}, Y^N\right]\right|^2 + \frac{1}{2} \left|Y\right|^2 + \nabla_{Y^{TX}}^{C^\infty\left(TX \oplus N, \pi^*\left(\Lambda^{\cdot}\left(T^*X \oplus N^*\right)\widehat{\otimes} F\right)\right)} \\ + c\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta \mathrm{ad}\left(Y^N\right)\right) - i\rho^E\left(Y^N\right) + \mathcal{O}\left(1/b^2\right). \tag{0.1}$$

Equation (0.1) indicates that the diffusion associated with the scalar part of $\mathcal{L}_b^X$ tends to propagate along the geodesic flow. Still, because we have to control the corresponding heat kernel, we ultimately need to obtain a quantitative estimate on how much this diffusion differs from the geodesic flow.

This question is dealt with in two steps.

1. Let $\varphi_t|_{t \in \mathbf{R}}$ be the geodesic flow in $\mathcal{X}$. Then $\gamma \varphi_1^{-1}$ is a symplectic transformation of $\mathcal{X}$, whose fixed point set $\mathcal{F}_\gamma$ is simply related to $X(\gamma)$, the critical set of d_γ. A purely geometric question, which is dealt with in the end of chapter 3, is to find how much the return map $\varphi_1^{-1}\gamma$ differs from the identity away from $\mathcal{F}_\gamma$. The corresponding quantitative estimates are obtained by using Toponogov's theorem repeatedly.

2. In chapter 15, these estimates are combined with the rough bounds on the heat kernel $q_{b,t}^X$ to obtain the proper uniform estimates for b large. Local index theoretic methods are used to control local cancellations in the supertrace of the heat kernel near $\mathcal{F}_\gamma$ as $b \to +\infty$.

0.11 The hypoelliptic Laplacian and the wave equation

A crucial observation is made in sections 12.3 and 14.2, which relates the heat equation for the scalar version of $\mathcal{L}_b^X$ to the classical wave equation on X. It is shown that after averaging in the fibre variables, the heat equation on $\mathcal{X}$ or $\mathcal{X}$ descends to a nonlinear version of the wave equation on X. This observation is at the heart of some of the key probabilistic arguments used in chapters 12 and 14 to establish the uniform Gaussian decay in (0.1). More fundamentally, it is connected with the fact that the Hamiltonian differential equation of order 1 for the Hamiltonian flow on $\mathcal{X}$ descends to a differential equation of order 2 on X for the geodesics. In some sense, this descent argument propagates to the heat equation for the hypoelliptic Laplacian.

0.12 The organization of the book

The book is divided into two parts. A first part, which includes chapters 1–9, contains the construction of the objects which are considered in the book, the geometric results which are needed and their proof, the statement of the main results and their proofs. The analytic results which are needed in the proofs are themselves stated without proof.

The detailed proof of the analytic results is deferred to a second part, which includes chapters 10–15.

This book is organized as follows. In chapter 1, we recall general results on Clifford and Heisenberg algebras.

In chapter 2, we construct the hypoelliptic Laplacian $\mathcal{L}_b^X$ over $\mathcal{X}$.

In chapter 3, we establish various geometric results. If $\gamma \in G$ is semisimple, we introduce the displacement function d_γ, its critical set $X(\gamma)$, and the coordinate system on X, which one obtains from the normals to $X(\gamma)$. Also we study the return map $\varphi_1^{-1}\gamma$ using Toponogov's theorem.

In chapter 4, we define the elliptic and hypoelliptic orbital integrals, and we establish identity (0.1).

In chapter 5, if $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, we evaluate the heat kernel for a hypoelliptic operator $\mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ acting on $\mathfrak{p} \oplus \mathfrak{g}$, and we compute the supertrace $J_\gamma\left(Y_0^{\mathfrak{k}}\right)$ of this heat kernel. In chapter 9, the operator $\mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ will appear as a rescaled limit of $\mathcal{L}_b^X$ when $b \to +\infty$. This chapter can be read independently.

In chapter 6, we state without proof our main result, which expresses the elliptic orbital integrals associated with the heat kernel as a Gaussian integral on $\mathfrak{k}(\gamma)$. Also we show how to derive from this formula a corresponding formula for other kernels, which include the wave kernel.

In chapter 7, we show the compatibility of our formula for the orbital integrals to the Atiyah-Singer index theorem [AS68a, AS68b] and to the Lefschetz fixed point formulas of Atiyah-Bott [ABo67, ABo68]. Also we recover results of Moscovici-Stanton [MoSt91] that are related to the evaluation of the Ray-Singer analytic torsion of locally symmetric spaces.

In chapter 8, we evaluate explicitly the integrals over $\mathfrak{k}(\gamma)$ when γ verifies a simple commutation relation, and we recover Selberg's trace formula when $G = \mathrm{SL}_2(\mathbf{R})$.

In chapter 9, we prove the formula that was stated in chapter 6. The proof relies on estimates established in the second part of the book.

In chapter 10, we establish detailed results on the variational problem associated with the action $H_{b,t}$ in (0.1) when X is an Euclidean vector space E, and we state various versions of Mehler's formula. Also we establish key estimates on integrals involving the heat kernel of the harmonic oscillator. This chapter can be read independently.

In chapter 11, given $b > 0$, we adapt the functional analytic methods of [BL08] to construct the resolvent and the heat kernel for the hypoelliptic Laplacian.

In chapter 12, we obtain rough estimates for the heat kernel that is associated with a scalar hypoelliptic operator on $\mathcal{X}$ and on $\mathcal{X}$. Such estimates are given for b bounded, and for b large. Also we study the limit of the heat kernel as $b \to 0$.

In chapter 13, we obtain the analogue of the uniform estimates in (0.1) for the scalar heat kernels on $\mathcal{X}$ and on $\mathcal{X}$.

In chapter 14, we establish estimates for the heat kernel $q^X_{b,t}$ for bounded b, which include (0.1).

Finally, in chapter 15, we establish the required estimates on $q^X_{b,t}$ for large b.

As explained before, many ideas and techniques used in the present book have already been tested in our previous work with Lebeau [BL08]. Still the present book is largely self-contained, except for the analytic results of [BL08, chapter 15], whose use can in part be avoided. Some familiarity with [B08b] could be useful. Also we have tried to make the reading easier, by separating the results from their proofs, and geometric arguments from analytic arguments. Also we tried to justify the role of probabilistic arguments as much as possible. The length of the book can be partly explained by the fact that we rederive many of the technical results of [BL08] by different methods.

If $\mathcal{A}$ is a $\mathbf{Z}_2$-graded algebra, if $a, b \in \mathcal{A}$, $[a, b]$ will be our notation for the supercommutator of a, b, so that

$$[a, b] = ab - (-1)^{\deg(a)\deg(\mathrm{b})}\, ba. \tag{0.1}$$

Also, in most of the book, we will use Einstein summation conventions.

Moreover, in the whole text, constants C will always be positive. If a constant depends on a parameter ϵ, it will often be written as C_ϵ. Also positive constants will often be denoted using the same notation, although they may well be different.

The results contained in this book have been announced in [B09b].