Preface

Many years ago, I was sitting in my second-grade classroom when I made what I thought was a remarkable discovery: *there is no largest number*. Whatever number I thought of, I realized I could just add one to it and get a larger number. This remains to this day one of my most vivid childhood memories. What I had "discovered" was the "dot, dot, dot" in the infinite collection of the numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ ,$$

which we know as the *natural numbers*.

As simple as this collection of numbers may appear, humans have been studying these numbers for thousands of years, learning their properties, uncovering their secrets, finding one marvelous thing after another about them, and still we have only barely begun to tap this remarkable and ever-flowing current of ideas. These are the numbers we intend to study.

This book is an introduction to the study of the natural numbers; it evolved from courses I have taught at Colorado College, ranging from a general math course designed for nonmajors to a far more rigorous sophomore-level course required of all math majors. I hope to preserve several fundamental features of these courses in this book:

- Number theory is beautiful. It is fun. That's why people have done it for thousands of years and why people still do it today. Number theory is so naturally appealing that it provides a perfect introduction—either for math majors or for nonmajors—to the idea of doing mathematics for its own sake and for the pleasure we derive from it.
- Although number theory will always remain a part of pure mathematics (as opposed to applied mathematics), it has also in modern times become a spectacular instance of what the physicist Eugene Wigner called the "unreasonable effectiveness of mathematics" in that there are now important real-world applications of number theory. One of the most useful of these applications came along several centuries *after* the original concepts in number theory were developed and will be explored in the chapter on cryptography.
- Number theory is a subject with an extraordinarily long and rich history. Studying number theory with due attention to its history reminds us that this subject has always been an intensely

human activity. Many other mathematical subjects, calculus, for example, would have undoubtedly evolved much as they are today quite independent of the individual people involved in the actual development, but number theory has had a wonderfully quirky evolution that depended heavily upon the particular interests of the people who developed the subject over the years.
- Reading mathematics is very different from, say, reading a novel. It requires enormous patience to read mathematics. You cannot expect to digest new, and often complex, mathematical ideas in a single reading. It is frequently the case that multiple readings are needed. You will discover that individual sentences, paragraphs, and even whole chapters must be read carefully several times before the key ideas all fall into place.
- One of the primary goals of the book is to use the study of number theory as a context within which we learn to prove things. Proof plays a vital role in mathematics and is the way we bridge the gap between what our intuition tells us *might* be true and the certainty about what *is* true. You will encounter several quite different styles of proof as you read (and should feel free to skip any that you find either too difficult or simply not very interesting). In many cases, an informal argument or even a carefully examined example is sufficient to discover truth, but in other cases a far more rigorous and formal argument will be required to achieve certainty.

Another feature of our courses at Colorado College I hope to preserve in this book is the *interactive* nature of our classes. Learning mathematics requires active participation, and this book should be read with paper and pencil in hand, and a good calculator or computer nearby, checking details and working things through as you go. Sometimes, in order to understand an idea, it is best to go through a few examples by hand. Other times it is better to let a computer do the computations, and so an introduction to the computer software Sage has been provided at the back of the book. Sage is an extremely powerful aide to such computations and is a wonderful resource that can be used online or downloaded for free.

The problems at the end of each chapter are an important part of the text and you should try to do as many as you can. In this book problems are not merely exercises for you to do, but they also introduce definitions, explore new ideas, and prove additional results. Much of this material will be used later in the book, and so you should be sure to read *all* of the problems, even ones you make no attempt to solve. Problems that are either particularly important or explicitly referred to later in the book have the symbol $\star$ by them.

Solutions and answers are provided for many of these problems at the back of the book. There is also a separate section containing hints for you to consult to get an idea of how to start on a problem if you are stuck. Problems for which a hint is available have a letter H after them, and problems for which there is a solution or answer have a letter S after them. These solutions and hints can be used in a variety of ways: to check your answers; to compare your solutions with mine (there are often several ways to approach a given problem); to study how to write up a solution once you have figured out how to solve a problem; but, also, just to read as part of the text, since I occasionally make additional, and hopefully useful and interesting, comments about the material in these solutions.

Also at the back of the book are two useful tables. One is simply a short list of prime numbers. The other is a pronunciation guide to help you with the names of foreign mathematicians. Rather than using a phonetic alphabet, these pronunciations are given in a form that should make it easy for any speaker of standard (American) English to get reasonably close to an accurate pronunciation. So, for example, (THA bit) is used for the ninth-century Arab mathematician Thābit ibn Qurra, rather than the phonetically correct (Θa:bıt).

It is probably obvious that covering all of the material in this book in a typical number theory course is not possible. I tend to think of Chapters 1–10 forming the core material and the topics covered in Chapters 11–15 being optional, perhaps to be done by students either individually or in groups as independent study. In the table of contents I have marked individual sections that I consider critical with a ⋆ .

Many people have at various stages helped me write this book. The first and foremost was the long-time chair of our math department, Dave Roeder. It was Dave who put a number theory course at the very core of our math curriculum, and over the years it became my very favorite course to teach. I also owe a deep debt to my colleague Stefan Erickson who, unlike me, is a real number theorist and has used numerous drafts of this book in his own course on number theory. Stefan guided me with enormous patience through draft after draft. He also provided me with extraordinarily detailed student feedback from these courses. One result of this extensive “field-testing” is that there have been many students whose comments greatly improved this book. In particular, two of these students deserve special mention. Gautam Webb’s careful reading of the latest draft uncovered more errors than I would have believed possible. Marina Gresham did the same sort of meticulous reading of several early drafts; more importantly, I relied almost exclusively on Marina’s excellent judgment in deciding which problems needed to be provided with hints and solutions.

Finally, I would like to say that while this book is modeled upon specific courses I have taught at Colorado College, this book is nonetheless intended for a far more general audience; and so there is almost nothing in terms of prerequisites that a readers need to bring along with them except enthusiasm and curiosity. That is one of the fundamental charms of number theory. It really does begin with

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots,$$

and you can't be too young, or too old, to enjoy this amazing story.

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