Preface

This monograph concerns a new theory of an algebra of singular integral operators which we refer to as "multi-parameter singular integral operators." These are operators which act on functions on $\mathbb{R}^n$, and have an underlying geometry which is given by a family of "balls" on $\mathbb{R}^n$ with many "radius" parameters: $B(x, \delta_1, \ldots, \delta_\nu)$. The classical theory of singular integrals corresponds to the case when $\nu = 1$. For higher ν, there are several different theories which have been developed by many authors. Some examples are the product theory of singular integrals, convolution with "flag kernels" on graded groups, and compositions of two singular integrals corresponding to different geometries. The goal of this monograph is to develop a general algebra of singular integrals which generalizes and unifies each of these (and other) examples.

Because our goal is to define a new kind of singular integral, it is best to start with the basic question "what is a singular integral?" For this, we start with the well-understood "single-parameter" case. The most classical example of a singular integral operator is the Hilbert transform. To understand the Hilbert transform, we wish to make sense of $\int f(t)/t\, dt$, for $f \in C_0^\infty(\mathbb{R})$. This integral does not make sense classically, because t^{-1} is not in L^1. However, t^{-1} has "cancellation": t^{-1} is an *odd* function. Since t^{-1} is odd, we make the convention that $\int_{-1}^{1} t^{-1}\, dt = 0$. With this convention we have

$$\int f(t)/t\, dt = \int_{-1}^{1} (f(t) - f(0))/t\, dt + \int_{|t|\geq 1} f(t)/t\, dt. \tag{1}$$

This allows us to define $\int f(t)\frac{1}{t}\, dt$ for $f \in C_0^\infty(\mathbb{R})$ by (1) and sees t^{-1} as a distribution on $\mathbb{R}$. The Hilbert transform is defined as the operator $Hf = f * t^{-1}$, for $f \in C_0^\infty(\mathbb{R})$; i.e., $Hf(x) = \int f(x-t)/t\, dt$. A classical theorem states that H extends to a bounded operator $H : L^p \to L^p$, $1 < p < \infty$. We call H a "singular integral operator" because it is defined by an integral which does not converge in the classical sense,[1] but can be made sense of using some sort of "cancellation." Some notion of cancellation is central to any theory of singular integrals.

In this single-parameter case, the above ideas have been greatly generalized. The most well-known such generalization is the theory of spaces of homogeneous type as developed by Coifman and Weiss [CW71] (see, also, Stein's book [Ste93]). In this theory, one is given a (quasi-)metric ρ on $\mathbb{R}^n$ and a Borel measure[2] Vol on $\mathbb{R}^n$. Define the balls by $B(x,\delta) := \left\{y \mid \rho(x,y) < \delta\right\}$. Provided these balls satisfy certain axioms,[3]

[1] But the integral nearly converges in the sense that $|t|^{-\delta} \in L^1([-1,1])$ for $\delta < 1$.

[2] We are mostly interested in the case when Vol is given by Lebesgue measure.

[3] A key such axiom is the "doubling condition": $\mathrm{Vol}(B(x, 2\delta)) \leq C\mathrm{Vol}(B(x,\delta))$.

a theory of "singular integral operators" can be developed. These operators are given, formally, by $f \mapsto Tf$ where

$$Tf(x) = \int K(x, y) f(y) \, d\mathrm{Vol}(y),$$

where $K(x, y)$ is not necessarily integrable, and instead satisfies estimates like

$$|K(x, y)| \lesssim \frac{1}{\mathrm{Vol}(B(x, \rho(x, y)))}, \tag{2}$$

along with a cancellation condition so that we may make sense of the integral. When the right conditions are imposed, these operators are bounded on L^p $(1 < p < \infty)$; again, an important point is to make appropriate sense of the "cancellation condition." Furthermore, in many situations the operators that arise form an algebra: the composition of two singular integral operators is again a singular integral operator.

In Chapters 1 and 2 we discuss two important cases of these single-parameter singular integral operators. Much of the theory in these chapters is well-known, but we present these results as a way to motivate the more general multi-parameter theory discussed later; moreover, some of the results we prove are key tools in studying the multi-parameter situation. In Chapter 1, we discuss the case when ρ is the usual distance on $\mathbb{R}^n$. There, we obtain the most classical theory of singular integrals, which we see is useful for studying elliptic partial differential operators.

In Chapter 1 we are introduced to a running theme of the monograph: singular integral operators can be defined in three equivalent ways. Each way is useful for different purposes. The three ways, roughly speaking, are as follows:

- $Tf(x) = \int K(x, y) f(y) \, dy$, where K satisfies certain estimates like (2); we refer to these estimates as "growth conditions." In addition we need to assume a "cancellation condition." This condition takes the form of bounds for $T\phi$ (and $T^*\phi$) where ϕ ranges over certain test functions. In this "single-parameter" case, the cancellation condition is closely related to the conditions of the $T(1)$ theorem of David and Journé [DJ84]. This type of definition is the most classical type we consider.

- The second equivalent definition introduces a type of "elementary operator." The condition states, roughly, that if E is an elementary operator, then so is TE. This condition is useful for showing that the operators in question form an algebra. In this simplest case, an example of an elementary operator is a Littlewood-Paley cut-off.

- The third equivalent definition sees T as an appropriate sum of elementary operators. This definition is useful for proving the L^p $(1 < p < \infty)$ boundedness of singular integral operators. In this Euclidean case, such decompositions of singular integral operators are often called "Littlewood-Paley decompositions."

Already in Chapter 1, we see this trichotomy three times, in increasing generality: Theorems 1.1.23, 1.1.26, and 1.2.10. Results like the ones in Chapter 1 can be found in many sources (e.g., [Ste93]). One way in which the thrust of our presentation in Chapter 1 differs is the emphasis of the above trichotomy. Indeed, we develop it even when the operators are not translation invariant (many authors discuss such ideas only for translation invariant, or nearly translation invariant operators). We also present these ideas in a slightly different way than is usual, which helps to motivate later results and definitions.

In Chapter 2 we remain in the single-parameter case, and turn to the case when the metric is a Carnot-Carathéodory (or sub-Riemannian) metric.[4] We define a class of singular integral operators adapted to this metric. The setting here is an instance of a space of homogeneous type, but we have more structure to work with. Indeed, there is a natural way to discuss "smoothness" with respect to a Carnot-Carathéodory structure. This makes these ideas useful for studying regularity properties of certain partial differential operators.

Chapter 2 has two major themes. The first is a more general reprise of the trichotomy described above (Theorem 2.0.29); this accounts for much of the work in Chapter 2 and paves the way to proving many properties of these operators (e.g., that they form an algebra). The second theme is a generalization of the fact (from Chapter 1) that Euclidean singular integral operators are closely related to elliptic partial differential equations. In fact, there is a far-reaching generalization of ellipticity, known as maximal hypoellipticity, and the singular integrals defined in Chapter 2 are an essential tool in studying this concept. The concept of maximal hypoellipticity was developed by several authors [Hör67, RS76, Koh78, HN85]. The connection between maximal hypoellipticity and singular integrals has also been used by several authors [RS76, NRSW89, CNS92, Koe02, Str09], but this seems to be the first time that the connection was made explicit in full generality.

A major tool we introduce in Chapter 2 is a quantitative version of the classical Frobenius theorem from differential geometry. This "quantitative Frobenius theorem" can be thought of as yielding "scaling maps" which are well adapted to the Carnot-Carathéodory geometry, and is of central use throughout the rest of the monograph. The statement of the result is quite technical so we devote time to carefully explaining it and its uses. We briefly indicate the proof, but refer the reader to the original source [Str11] for the full details.

Chapters 1 and 2 should be thought of as the background and motivation for the main goal of this monograph: to develop a general theory of "multi-parameter" singular integral operators. To understand this concept, consider the fact that one can reconstruct the metric ρ from the metric balls:

$$\rho(x, y) = \inf\left\{\delta > 0 \,\middle|\, y \in B(x, \delta)\right\}.$$

Thus, when defining a class of singular integrals, the most basic ingredient is the corresponding family of balls. The remainder of this monograph is concerned with the following questions: what if, instead of balls of the form $B(x, \delta)$, we are given balls with many "radius" parameters $B(x, \delta_1, \ldots, \delta_\nu)$? What should we assume on these balls to develop a notion of a singular integral operator? What is the right definition of

[4] We also work on a compact manifold, instead of $\mathbb{R}^n$.

a singular integral operator? A major difficulty is that (2) involves the metric ρ in an essential way, and there is no one natural metric associated to the balls $B(x, \delta_1, \ldots, \delta_\nu)$. It is therefore not obvious what a natural generalization of bounds like (2) might be. Chapters 3, 4, and 5 are devoted to these questions.

We again restrict attention to Carnot-Carathéodory type balls, and we offer answers to the above questions in that situation. We refer to the corresponding singular integral operators as "multi-parameter singular integral operators."

The most basic and well understood example of a multi-parameter singular integral comes from the so-called product theory of singular integrals. In that case, the ambient space $\mathbb{R}^n$ is decomposed into factors $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_\nu}$. On each factor, $\mathbb{R}^{n_\mu}$, one is given a metric ρ_μ. We denote the corresponding metric balls by $B_\mu(x_\mu, \delta_\mu) \subseteq \mathbb{R}^{n_\mu}$. We obtain multi-parameter balls by

$$B((x_1, \ldots, x_\nu), (\delta_1, \ldots, \delta_\nu)) = B_1(x_1, \delta_1) \times \cdots \times B_\nu(x_\nu, \delta_\nu).$$

Generalizing (2) to this situation is easy; we use estimates like

$$|K((x_1, \ldots, x_\nu), (y_1, \ldots y_\nu))| \lesssim \prod_{\mu=1}^{\nu} \mathrm{Vol}_\mu (B_\mu (x_\mu, \rho_\mu (x_\mu, y_\mu)))^{-1}.$$

In this situation, it is well known how to develop a theory of singular integrals. See, for example, Section 4.1. These ideas were developed by many authors, beginning with foundational work of Fefferman and Stein [FS82] and Journé [Jou85] (see Section 4.1.2 for more references). The theory we develop in this monograph incorporates this product theory, but the main point is to develop a theory of multi-parameter singular integral operators when the balls are not necessarily of product type.

In Chapter 3 we develop the theory of multi-parameter Carnot-Carathéodory geometry which we need to study these singular integral operators. In the case when the balls are of product type, all of the results from Chapter 3 are simple variants of results in the single-parameter theory. When the balls are not of product type, these ideas become more difficult. What saves the day is the quantitative Frobenius theorem given in Chapter 2. Using this we can estimate certain integrals, and also develop an appropriate maximal function and an appropriate Littlewood-Paley square function, all of which are essential to our study of singular integral operators.

There are a few special cases where such a theory of multi-parameter singular integral operators has already been developed, and we discuss these in Chapter 4. These include the product theory of singular integrals, convolution with flag kernels on graded groups, convolution with both the left and right invariant Calderón-Zygmund singular integral operators on stratified Lie groups, and composition of standard pseudodifferential operators with certain singular integrals corresponding to non-Euclidean geometries. We outline these examples and their applications and relate them to the trichotomy discussed above.

Finally, in Chapter 5, we turn to a general theory which generalizes and unifies all of the above examples. As mentioned above, a main issue is that the first definition from our trichotomy does not generalize to the multi-parameter situation (there is no useful analog of the "growth conditions" in general). To deal with this, we introduce strengthened cancellation conditions. We do this in two different ways, leaving us with four total definitions for singular integral operators (the first two use the strengthened cancellation conditions, while the later two are generalizations of the later two parts of the above trichotomy). Thus we obtain four classes of singular integral operators, which we denote by $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$. The main theorem of Chapter 5 is $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4$; i.e., all four of these definitions are equivalent. This leads to many nice properties of these singular integral operators. For instance, it proves they are an algebra, and proves their boundedness on appropriate non-isotropic Sobolev spaces. We also include several special cases of this algebra, and relate it to the examples from Chapter 4; in particular, we see that this algebra arises naturally in some questions from partial differential equations.

There are three appendices. The first gives the background theory on functional analysis which we need throughout the monograph. This includes the basic aspects of locally convex topological vector spaces, and also categorical limits and tensor products of locally convex topological vector spaces. The second appendix records three results from calculus which are useful to us: the smoothness of exponentials of vector fields, a version of the inverse function theorem which is "uniform on compact sets," and a technical change of variables which is used several times in the monograph. The third appendix is meant to be a quick reference for some notation which is used throughout the monograph which may be somewhat nonstandard.

This work uses the ideas of many authors. We have included several sections titled "Further reading and references" at the end of Chapters 1, 2, and 3, and at the end of each section in Chapter 4. These sections include references to the literature for the results we have used, along with some comments on surrounding ideas and theories of other authors. Especially in Chapters 1, 2, and 4, few of the ideas are new, and we have attempted to give appropriate credit and surrounding history in these final sections.

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