Preface

In the 1950s and 1960s the celebrated Kolmogorov-Arnol′d-Moser theorem(s) (commonly abbreviated with KAM from the initials of the three main pioneers) finally settled the old question concerning the existence of *quasi-periodic* motions for *nearly-integrable* Hamiltonian systems, i.e., Hamiltonian systems that are slight perturbations of an integrable one. In the integrable case, in fact, the whole phase space is foliated by invariant Lagrangian submanifolds that are diffeomorphic to tori and on which the dynamics is conjugate to a rigid rotation. On the other hand, it is natural to ask what happens to such a foliation and to these *stable motions* once the system is perturbed.

In 1954 Kolmogorov [48]—and later Arnol′d [3] and Moser [71, 70] in different contexts—proved that, in spite of the generic disappearence of the invariant tori filled by periodic orbits (already pointed out in the works of Henri Poincaré), for small perturbations of an integrable system it is still possible to find invariant Lagrangian tori corresponding to "strongly non-resonant" rotation vectors. This result, commonly referred to as *KAM theorem*, besides opening the way to a new understanding of the nature of Hamiltonian systems and their stable motions, contributed to raising new, interesting questions. For instance: *what is the destiny of the stable motions (orbits on KAM tori) that are destroyed by the effect of the perturbation? Is it still possible to identify something reminiscent of their former presence? What can be said about a system that is not close to an integrable one?*

While all these questions are concerned with the investigation of stable motions of the perturbed system, another interesting issue soon took the stage: *Does the breakdown of this stable picture open the way to orbits with unstable or chaotic behaviours?*

An answer to this latter question did not take long to arrive. In 1964 V. I. Arnol′d [4] constructed an example of a perturbed integrable system in which *unstable orbits*—resulting from the breaking of unperturbed KAM tori—coexist with the stable picture drawn by the KAM theorem. This striking, and somewhat unexpected, phenomenon, yet not completely understood, is nowadays called *Arnol′d diffusion*.

This new insight led to a change in perspective, and in order to make sense of the complex balance between stable and unstable motions that was becoming more evident, new approaches needed to be exploited. Among these, variational methods turned out to be particularly suitable and successful. Mostly inspired by the so-called *principle of least action*,[1] a sort of widely accepted "thriftiness" of nature in all its actions, they seemed to provide the natural setting to overcome the local view given by the analytic methods and make progress towards a global understanding of the dynamics.

Aubry-Mather theory represented undoubtedly a great leap forward in this direction. Developed independently by Serge Aubry [6, 7] and John Mather [57] in the 1980s, this novel approach to the study of the dynamics of *twist diffeomorphisms of the annulus* (which correspond to Poincaré maps of one-dimensional Hamiltonian systems [71]) pointed out the existence of many *action-minimizing sets*, which in some sense generalize invariant rotational curves and which always exist, even after rotational curves are destroyed. Besides providing a detailed structure theory for these new sets, this powerful approach yielded to a better understanding of the destiny of invariant rotational curves and led to the construction of interesting chaotic orbits as a result of their destruction [59, 43, 63].

Motivated by these achievements, John Mather [62, 64]—and later Ricardo Mañé [53, 26, 27] and Albert Fathi [33, 34, 35, 36, 37] in different ways—developed a generalization of this theory to higher dimensional systems. Positive definite superlinear Lagrangians on compact manifolds, also called *Tonelli Lagrangians* (see Definition 1.1.1), were the appropriate setting to work in. Under these conditions, in fact, it is possible to prove the existence of interesting invariant (action-minimizing) sets, known as *Mather*, *Aubry*, and *Mañé* sets, which are obtained as minimizing solutions to variational problems. As a result, these objects present a much richer structure and rigidity than one might generally expect and, quite surprisingly, play a leading role in determining the global dynamics of the system. In particular, they can be considered as a sort of generalization of the KAM tori, but they continue to exist even after KAM tori's disappearance or when it does not make sense to speak of them (for example when the system is "far" from any integrable one).

These tools—while dealing with stable motions—were also quite promising in the construction of *chaotic orbits*, for instance, *connecting orbits* among the above-mentioned invariant sets [64, 10, 30]. Therefore they raised hopes for the possibility of proving the generic existence of Arnol′d diffusion in nearly integrable Hamiltonian systems [65, 47, 19, 25, 24]. However, in contrast with the case of twist diffeomorphisms, the situation turns out to be more complicated, due to a general lack of information on the topological structure of these action-minimizing sets. These sets, in fact, play a twofold role. Whereas on the one hand they may provide an obstruction to the existence of "diffusing or-

[1] "*Nature is thrifty in all its actions*," according to Pierre-Louis Moreau de Maupertuis (1744). A better-known special case of this principle is what is generally called *Maupertius' principle.* Actually, as I learned from Leo Butler, König published a note arguing for priority to Leibniz in the Berlin Academy correspondences overseen by Maupertuis. A priority dispute brought in Euler and Voltaire, and ultimately a committee was convened by the King of Prussia. In 1913, the Berlin Academy reversed its previous decision and found that Leibniz had priority.

bits," on the other hand their topological structure plays a fundamental role in the variational methods that have been developed for the construction of orbits with prescribed behaviors. We will not enter further into the discussion of this problem, but we refer the interested readers to [10, 14, 30, 65, 68, 47, 19, 25, 24].

In addition to its fundamental impact on the modern study of classical dynamics, Mather's theory has also contributed to pointing out interesting and sometimes unexpected links to other fields of research (both pure and applied), fostering a multidisciplinary interest in its ideas and in the techniques involved.

These lecture notes aim at satisfying a twofold demand. On the one hand, they are addressed to researchers and students in the field of dynamical systems and intend to provide a comprehensive and pedagogic introduction to Mather's theory and its subsequent developments. On the other hand, they are tailored to be used as an "interdisciplinary bridge," which should help researchers and students coming from different fields and backgrounds, to get acquainted with the main ideas and techniques, and help orient them to the huge body of research literature available on the topic.

Starting from the mathematical background on which Mather's theory was born and has developed, these notes firstly focus on the main questions this theory aims to answer (the destiny of broken invariant KAM tori and the onset of chaos) and try to highlight the sense in which this theory can be viewed as a generalization of KAM theory (or a natural counterpart to it). This will be achieved by their guiding the reader through a detailed illustrative example, which, at the same time, will provide the pretext for introducing the main ideas and concepts of the general theory. Once the crucial ideas are clarified, the whole theory and its subsequent developments will be described in their full generality. We will consider only the autonomous case (i.e., no dependence on time in the Lagrangian and Hamiltonian). This choice has been made only to make the discussion easier and to avoid some technical issues that would be otherwise involved. However, all the theory that we are going to describe can be generalized, with some small modifications, to the non-autonomous time-periodic case. During our discussion, in order to draw the most complete picture of the theory, we will point out and discuss such differences and the needed modifications.

These lecture notes are organized as follows. They consist of five chapters:

1. Tonelli Lagrangian and Hamiltonians on compact manifolds. In this chapter we introduce the basic setting: Tonelli Lagrangians and Hamiltonians on a compact manifold. Besides discussing their main properties and some examples, this chapter provides the opportunity to recall some basic facts on Lagrangian and Hamiltonian dynamics (and on their mutual relation), which will be of fundamental importance in the discussion thereafter.

2. From KAM theory to Aubry-Mather theory. Before entering into the description of the general theory, in this chapter we discuss an illustra-

tive example, namely the properties of invariant probability measures and orbits on KAM tori (or more generally, on invariant Lagrangian graphs). This will prepare the ground for understanding the main ideas and techniques that will be developed in the following chapters, without several technicalities that might be confusing to a neophyte.

3. ACTION-MINIMIZING INVARIANT MEASURES FOR TONELLI LAGRANGIANS. In this chapter we discuss the notion of action-minimizing measures, recalling the needed measure-theoretical material. In particular, this will allow us to define a first family of invariant sets, the so-called *Mather sets*, and discuss their main dynamical and symplectic properties. As a by-product, we introduce the *minimal average actions*, sometimes called Mather's α- and β-functions. A thorough discussion of their properties (differentiability, strict convexity or lack thereof) will be provided and related to the dynamical and structural properties of the Mather sets. We also describe these concepts in a concrete physical example: the simple pendulum.

4. GLOBAL ACTION-MINIMIZING CURVES FOR TONELLI LAGRANGIANS. In this chapter we discuss the notion of action-minimizing orbits. In particular, this will allow us to define other two families of invariant sets, the so-called *Aubry* and *Mañé sets*. We will discuss their main dynamical and symplectic properties, comparing them with the results obtained in the preceding chapter for the Mather sets. The relation between these new invariant sets and the Mather sets will be carefully described. As a by-product, we will introduce the Mañé's potential, Peierls' barrier, and Mañé's critical value. We will throughly discuss their properties; in particular, we will highlight how this critical value is related to the minimal average action and describe these new concepts in the case of the simple pendulum, completing the picture started in the preceding chapter.

5. HAMILTON-JACOBI EQUATION AND WEAK KAM THEORY. Another interesting approach to the study of these invariant sets is provided by the so-called *weak KAM theory*, developed by Albert Fathi. In this chapter we briefly describe this approach, which could be considered as the functional analytic counterpart of the variational methods discussed in the previous chapters. The starting point is the relation between KAM tori (or more generally, invariant Lagrangian graphs) and classical solutions and subsolutions of the Hamilton-Jacobi equation. We will introduce the notion of weak (non-classical) solutions of the Hamilton-Jacobi equation and a special class of subsolutions (critical subsolutions). In particular, we will point out their relation to Aubry-Mather theory.

These notes also include two appendices with some extra (complementary) material:

A. ON THE EXISTENCE OF INVARIANT LAGRANGIAN GRAPHS. In this appendix we recall some basic notions of symplectic geometry and illustrate what kind of information Aubry-Mather theory conveys about the study of the integrability of Hamiltonian systems, and, more generally, how this information relates to the existence or nonexistence of invariant Lagrangian graphs.

B. Schwartzman asymptotic cycle and dynamics. In this appendix we introduce and discuss the notion of the Schwartzman asymptotic cycle of a flow, introduced by Sol Schwartzman in [74], as a first attempt to develop an algebraic topological approach to the study of dynamics. We will discuss its relation to the notion of rotation vector (or homology class) introduced in the previous chapters of these notes, and investigate its main properties.

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