Preface

Trust me, it's not that hard

> In 1951, I had the good fortune of listening to Professor Racah's lecture on Lie groups at Princeton. After attending these lectures, I thought, "This is really too hard. I cannot learn all this . . . too damned hard and unphysical."
>
> —A. Salam, 1979 Nobel laureate in physics[1]

Trust me, it's not that hard. And as Salam's own Nobel-winning work helped show, group theory is certainly relevant to physics. We now know that the interweaving gauge bosons underlying our world dance to the music[2] of Lie groups and Lie algebras.

This book is about the use of group theory in theoretical physics. If you are looking for a mathematics book on group theory complete with rigorous proofs, the abstract[3] modern definition[4] of tensors and the like, please go elsewhere. I will certainly prove every important statement, but only at a level of rigor acceptable to most physicists. The emphasis will be on the intuitive, the concrete, and the practical.

I would like to convince a present day version of Salam that group theory is in fact very physical. With due respect to Racah, I will try to do better than him pedagogically. My goal is to show that group theoretic concepts are natural and easy to understand.

Elegant mathematics and profound physics: Honor your inheritance

> In great mathematics, there is a very high degree of unexpectedness, combined with inevitability and economy.
>
> —G. H. Hardy[5]

Group theory is a particularly striking example of what Hardy had in mind. For me, one of the attractions of group theory is the sequence of uniqueness theorems, culminating in Cartan's classification of all Lie algebras (discussed in part VI). Starting from a few innocuous sounding axioms defining what a group is, an elegant mathematical structure emerges, with many unexpected theorems.

My colleague Greg Huber pointed out to me that group theory is an anagram for rough poetry. Rough? I've always thought that it's close to pure poetry.

Although group theory is certainly relevant for nineteenth-century physics, it really started to play an important role with the work of Lorentz and Poincaré, and became essential with quantum mechanics. Heisenberg opened up an entire new world with his vision of an internal symmetry, the exploration of which continues to this very day in one form or another. Beginning in the 1950s, group theory has come to play a central role in several areas of physics, perhaps none more so than in what I call fundamental physics, as we will see in parts V, VII, VIII, and IX of this book. There are of course some areas[6] of physics that, at least thus far, seem not[7] to require much of group theory.

I understand that group theory has also played a crucial role in many areas of mathematics, for example, algebraic topology, but that is way outside the scope of this book. As a one-time math major who saw the light, while I do not know what mathematicians know about groups, I know enough to know that what I cover here is a tiny fraction of what they know.

This is a book about a branch of mathematics written by a physicist for physicists. One immediate difficulty is the title: the disclaimer "for physicists" has to be there; also the phrase "in a nutshell" because of my contractual obligations to Princeton University Press. The title "Group Theory for Physicists in a Nutshell" would amount to a rather lame joke, so the actual title is almost uniquely determined.

This is my third Nutshell book. As for my motivation for writing yet another textbook, Einstein said it better than I could: "Bear in mind that the wonderful things that you learn in your schools are the work of many generations. All this is put into your hands as your inheritance in order that you may receive it, honor it, add to it, and one day faithfully hand it on to your children."[8]

Advice to the reader

Some advice to the reader, particularly if you are self-studying group theory in physics. The number one advice is, of course, "Exercise!" I strongly recommend doing exercises as you read along, rather than waiting until the end of the chapter, especially in the early chapters. To the best of my knowledge, nobody made it into the NBA by watching the sports channels. Instead of passively letting group theory seep into your head, you should frequently do the mental equivalent of shooting a few baskets. When given a theorem, you should, in the spirit of doubting Thomas, try to come up with counterexamples.

I am particularly worried about the readers who are shaky about linear algebra. Since this is not a textbook on linear algebra, I did not provide lots of exercises in my coverage (see below) of linear algebra. So, those readers should make up their own (even straightforward) exercises, multiplying and inverting a few numerical matrices, if only to get a sense of how matrices work.

For whom is this book written

This brings me to prerequisites. If you know linear algebra, you can read this book. For the reader's convenience, I had planned to provide a brief appendix reviewing linear algebra. It grew and grew, until it became essentially self-contained and clamored to move up front. I ended up covering, quite completely, at least those aspects of linear algebra needed for group theory. So, yes, even if you don't know linear algebra, you could still tackle this book.

Several blurbers and reviewers of my Quantum Field Theory in a Nutshell[9] have said things along the line of "This is the book I wish I had when I was a student."[10] So that's roughly the standard I set for myself here: I have written the book on group theory I wished I had when I was a student.[11]

My pedagogical strategy is to get you to see some actual groups, both finite and continuous, in action as quickly as possible. You will, for example, be introduced to Lie algebra by the third chapter. In this strategy, one tactic is to beat the rotation group to death early on. It got to the point that I started hearing the phrase "beat rotation to death" as a rallying cry.

Group theory and quantum mechanics

I was not entirely truthful when I said "If you know linear algebra, you can read this book." You have to know some quantum mechanics as well. For reasons to be explained in chapter III.1, group theory has played much more of a role in quantum mechanics than in classical mechanics. So for many of the applications, I necessarily have to invoke quantum mechanics. But fear not! What is needed to read this book is not so much quantum mechanics itself as the language of quantum mechanics. I expect the reader to have heard of states, probability amplitudes, and the like. You are not expected to solve the Schrödinger equation blindfolded, and certainly those murky philosophical issues regarding quantum measurements will not come in at all.

For some chapters in parts V, VII, and IX, some rudimentary knowledge of quantum field theory will also be needed. Some readers may wish to simply skip these chapters. For braver souls, I try to provide a gentle guide, easing into the subject with a review of the Lagrangian and Hamiltonian in chapters III.3 and IV.9. The emphasis will be on the group theoretic aspects of quantum field theory, rather than on the dynamical aspects, of course. I believe that readers willing to work through these chapters will be rewarded with a deeper understanding of the universe. In case you run into difficulties, my advice is to muddle through and provisionally accept some statements as given. Of course, I also hope that some readers will go on and master quantum field theory (smile).

Applications of group theory

My philosophy here is not to provide a compendium of applications, but to endow the reader with enough of an understanding of group theory to be able to approach individual problems. The list of applications clearly reflects my own interests, for instance, the Lorentz group and its many implications, such as the Weyl, Dirac, and Majorana equations. I think that this is good. What is the sense of my transporting calculations from some crystallography and materials science textbooks (as some colleagues have urged me to do "to broaden the market"), when I do not feel the subject in my bones, so to speak? In the same way, I do not expect existing books on group theory in solid state physics to cover the Majorana fermion. To be sure, I cover some standard material, such as the nonexistence of crystals with 5-fold symmetry. But, judging from recent advances at the frontier of condensed matter theory, some researchers may need to get better acquainted with Weyl and Majorana rather than to master band structure calculations.

I try to give the reader some flavor of a smattering of various subjects, such as Euler's function and Wilson's theorem from number theory. In my choice of topics, I tend to favor those not covered in most standard books, such as the group theory behind the expanding universe. My choices reflect my own likes or dislikes.[12] Since field theory, particle physics, and relativity are all arenas in which group theory shines, it is natural and inevitable that this book overlaps my two previous textbooks.

The genesis of this book

This book has sat quietly in the back of my mind for many years. I had always wanted to write textbooks, but I am grateful to Steve Weinberg for suggesting that I should write popular physics books first. He did both, and I think that one is good training for the other. My first popular physics book is Fearful Symmetry,[13] and I am pleased to say that, as it reaches its thirtieth anniversary, it is still doing well, with new editions and translations coming out. As the prospective reader of this book would know, I could hardly talk about symmetry in physics without getting into group theory, but my editor at the time[14] insisted that I cut out my attempt to explain group theory to the intelligent public. What I wrote was watered down again and again, and what remained was relegated to an appendix. So, in some sense, this book is a follow-up on Fearful, for those readers who are qualified to leap beyond popular books.

Physics students here at the University of California, Santa Barbara, have long asked for more group theory. In an interesting pedagogical year,[15] I taught a "physics for poets" course for nonphysics majors, discussing Fearful over an entire quarter, and during the following quarter, a special topics course on group theory addressed to advanced undergraduates and graduate students who claimed to know linear algebra.

Teaching from this book and self-studying

As just mentioned, I taught this material (more than once) at the University of California, Santa Barbara, in a single quarter course lasting ten weeks with two and a half hours' worth of lectures per week. This is too short to cover all the material in the book, but with some skipping around, I managed to get through a major fraction. Here is the actual syllabus.

Week 1: Definition and examples of groups, Lagrange's theorem, constructing multiplication tables, direct product, homomorphism, isomorphism

Week 2: Finite group, permutation group, equivalence classes, cycle structure of permutations, dihedral group, quarternionic group, invariant subgroup, simple group

Week 3: Cosets, quotient group, derived subgroup, rotation and Lie's idea, Lie algebra

Week 4: Representation theory, unitary representation theorem, orthogonality theorem, character orthogonality

Week 5: Regular representation; character table is square; constructing character table

Week 6: Tray method; real, pseudoreal, and complex; crystals; Fermat's little theorem (statement only);[16] group theory and quantum mechanics

Week 7: $SO(N)$: why $SO(3)$ is special, Lie algebra of $SO(3)$, ladder operators, Casimir invariants, spherical harmonics, $SU(N)$

Week 8: $SU(2)$ double covers $SO(3)$, $SO(4)$, integration over group manifolds

Week 9: $SU(3)$, roots and weights, spinor representations of $SO(N)$

Week 10: Cartan classification, Dynkin diagrams

Thus, the single quarter course ends with part VI.

Students were expected to do some reading and to fill in some gaps on their own. Of course, instructors may want to deviate considerably from this course plan, emphasizing one topic at the expense of another. It would be ideal if they could complement this book with material from their own areas of expertise, such as materials science. They might also wish to challenge the better students by assigning the appendices and some later chapters. A semester would be ideal.

Some notational alerts

Some books denote Lie groups by capital letters, for example, $SU(2)$, and the corresponding algebras by lower case letters, for example, $su(2)$. While I certainly understand the need to distinguish group and algebra, I find the constant shifting between upper and lower case letters rather fussy looking. Most physicists trust the context to make clear whether the group or the algebra is being discussed. Thus, I will follow the standard physics usage and write $SU(2)$ for both group and algebra. An informal survey of physics students indicates that most agree with me. Of course, I am careful to say that, for example, $SU(2)$ and $SO(3)$ are only locally isomorphic and that one covers the other (as explained in detail in part IV).

In general, I vote for clarity over fussiness; I try not to burden the reader with excessive notation.

Parting comments: Regarding divines and dispensable erudition

> A foolish consistency is the hobgoblin of little minds, adored by little statesmen and philosophers and divines.
>
> —Ralph Waldo Emerson

I made a tremendous effort to be consistent in my convention, but still I have to invoke Emerson and hope that the reader is neither a little statesman nor a divine. At a trivial level, I capriciously use "1, 2, 3" and "x, y, z" to denote the same three Cartesian axes. Indeed, I often intentionally switch between writing superscript and subscript (sometimes driven by notational convenience) to emphasize that it doesn't matter. But eventually we come to a point when it does matter. I will then explain at length why it matters.

In Zvi Bern's Physics Today review of QFT Nut, he wrote this lovely sentence: "The purpose of Zee's book is not to turn students into experts—it is to make them fall in love with the subject."[17] I follow the same pedagogical philosophy in all three of my textbooks. This echoes a sage[18] who opined "One who knows does not compare with one who likes, one who likes does not compare with one who enjoys."

As I have already said, this is not a math book, but a book about math addressed to physicists. To me, math is about beauty, not rigor, unexpected curves rather than rock hard muscles.

Already in the nineteenth century, some mathematicians were concerned about the rising tide of rigor. Charles Hermite, who figures prominently in this book, tried to show his students the simple beauty of mathematics, while avoiding what Einstein would later refer to as "more or less dispensable erudition."[19] In this sense, I am Hermitean, and also Einsteinian.

Indeed, Einstein's aphorism, that "physics should be made as simple as possible, but not any simpler," echoes throughout my textbooks. I have tried to make group theory as simple as possible.[20]

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Notes

1. Quoted in J-Q. Chen, Group Representation Theory for Physicists, p. 1.
2. I will get up to a "faint echo" in the closing parts of this book.
3. George Zweig, who independently discovered the notion of quarks, had this to say about the abstract approach: "Mathematics in the US was taught in a very formal manner. I learned algebra from a wonderful algebraist, Jack McLaughlin, but the textbook we used was Jacobson's two-volume set, 'Lectures in Abstract Algebra,' and abstract it was! It seemed like there were as many definitions as results, and it was impossible to see how Mr. Jacobson actually thought. The process was hidden, only polished proofs were presented." Hear, hear!
4. In contrast to the concrete "archaic" definition that physicists use.
5. G. H. Hardy, A Mathematician's Apology, Cambridge University Press, 1941.
6. I am often surprised by applications in areas where I might not expect group theory to be of much use. See, for example, "An Induced Representation Method for Studying the Stability of Saturn's Ring," by S. B. Chakraborty and S. Sen, arXiv:1410.5865. Readers who saw the film Interstellar might be particularly interested in this paper.
7. To paraphrase Yogi Berra, if some theoretical physicists do not want to learn group theory, nobody is going to make them.
8. Albert Einstein, speaking to a group of school children, 1934.
9. Henceforth, QFT Nut.
10. See, for example, F. Wilczek on the back cover of the first edition of QFT Nut or the lead page of the second edition.
11. Indeed, it would have been marvelous if I had had something like this book after I had learned linear algebra in high school.
12. Or even other people's dislikes. For instance, my thesis advisor told me to stay the heck away from Young tableaux, and so I have ever since.
13. Henceforth, Fearful.
14. He had heard Hawking's dictum that every equation in a popular physics book halves its sale.
15. At the urging of the distinguished high energy experimentalist Harry Nelson.
16. That is, the full proof is given here, but was not entered into when I taught the course.

17. Niels Bohr: "A expert is someone . . . who goes on to know more and more about less and less, and ends up knowing everything about nothing."
18. This represents one of the few instances in which I agree with Confucius. Alas, I am often surrounded by people who know but do not enjoy.
19. In fact, Einstein was apparently among those who decried "die Gruppenpest." See chapter I.1. I don't know of any actual documentary evidence, though.
20. While completing this book, I came across an attractive quote by the mathematician H. Khudaverdyan: "I remember simple things, I remember how I could not understand simple things, this makes me a teacher." See A. Borovik, Mathematics under the Microscope, p. 61.