Preface

Graduate students in the earth sciences, particularly those in geophysics and atmospheric, oceanographic, planetary, and space physics, as well as astronomy, require a substantial degree of mathematical preparation—for the sake of brevity, we will simply refer to these application areas as being in geophysics. While there is significant overlap between their needs and those of graduate students in physics or in applied mathematics, there are important differences in the preparation needed and, notably, the sequence of presentation required as well as the overall quantity of material that is necessary. Most textbooks that address mathematical methods for physics and engineering begin from the standpoint that the student already knows the underlying equations, generally partial differential equations, but needs to learn how to solve them. Since the background of most entering or second-year graduate students in geophysics is highly variable, I felt it necessary to provide derivations in a number of circumstances for those equations to help students appreciate better where they arise and how their solution must be addressed. Moreover, most mathematical methods textbooks were published before the renaissance in thinking, especially about geophysical problems, that introduced the concepts of chaos and complexity, as well as the significance of probability and statistics and of numerical methods. Significant attention is given to the ordinary and partial differential equations that have played a pivotal role in the evolution of geophysics. In addition, in order to round out our treatment of mathematical methods, a succinct survey of statistical and computational issues is introduced. A brief but comprehensive summary of solution methods is presented, including many exercises. In so doing, it is my hope that this book will address that need during the course of one academic semester or quarter. In essence, we treat some central problem areas in depth, while providing a measure of literacy in others.

Students also can find helpful materials in the following works. The text that is closest to our presentation is that due to

Mathews and Walker (1970) which is out of print. A relatively contemporary text on the topic is that of Arfken and Weber (2005), but its newest edition (Arfken et al., 2013) has become a "comprehensive guide"; a helpful lead-in to the latter, designed more for advanced undergraduates, is Weber and Arfken (2004). The graduate textbook by Stone and Goldbart (2009) is also helpful, although the examples selected are drawn from physics and have a more formal flavor. Finally, the classic two-volume definitive texts on the subject are those by Morse and Feshbach (1999) and by Courant and Hilbert (1962). While the former is now back in print, the latter remains out of print. Regarding specific applications to geophysics and planetary, atmospheric, oceanographic, and space physics, chapters in existing graduate-level textbooks in those specialties contain appropriate derivations. As we encounter each new topic, additional citations to reference materials will be provided. We shall attempt to integrate some of the most important of these into this book.

Given the time available in a single academic quarter or semester, we are fundamentally limited in the quantity of material that can be presented. Basically, we provide an overarching survey of the relevant issues, a brief treatment of how to treat these problems, and an indication for each of these topics where the student can find a more thorough and rigorous treatment. Our objective is to give each student sufficient instruction to solve elementary problems and then advance to more exhaustive treatments of the individual topics, whether they originated in geophysics and its associated disciplines, physics, astronomy, or engineering.

The first chapter reviews many mathematical preliminaries that students should have studied previously, but also serves as a review. Vectors, indicial or "Einstein" notation, vector operators, cylindrical and spherical geometry, and the theorems of Gauss, Green, and Stokes are presented here. Since the focus of this chapter is on geometry, we introduce matrices in the context of the rotation of vectors. Then, we present tensors, which are matrices whose physical properties remain unchanged under a rotation and preserve other physical (e.g., variational) principles, including a very brief description of the eigenvalue problem. Here we introduce the concept of generalized functions through the Dirac δ function, and some of its relatives, inasmuch as they will form the basis later for our treatment of Green's functions. We present a number of assignment problems.

In the second chapter, we review features of ordinary differential equations. The Laplacian operator in partial differential equations permits the use of the method of separation of variables, which yields a set of second-order ordinary differential equations in the different geometric variables. We introduce the concept of Green's functions. Accordingly, we treat the separation of variables issue from the standpoint of ordinary differential equations and we introduce the derivation underlying Bessel functions and spherical harmonics, including the Legendre polynomials. (We complete the discussion of Poisson's, Laplace's, and Helmholtz's equation in chapter 4 because of their utility in solving partial differential equations of elliptic type.) We introduce problems describable by coupled ordinary differential equations, which, ultimately, provide the basis for chaos theory and are largely overlooked in classical mathematical methods of physics textbooks. Geophysical examples provide a wonderful testbed for ordinary differential equation approaches. For example, efforts to model the geodynamo using the interaction of mechanical and electrical components yielded strictly cyclical behavior with no field reversals. Efforts to resolve this problem demonstrated an epiphanic paradigm shift in moving to systems with three equations, such as the Lorenz model for convection and turbulence. This chapter also provides hands-on experience in performing perturbation theory analysis. Since chaotic behavior often yields fractal geometry, as in the Lorenz model trajectory, we provide a brief survey of fractal concepts and applications, as well as mappings as an adjunct to understanding transition to chaos.

In the third chapter, we introduce the evaluation of integrals, including a brief overview of complex analysis and elementary contour integration, saddle point methods, and some special problems in geophysics that yield elliptic integrals. We continue to address integral transforms following a brief introduction to Fourier series and transforms. We prove the sampling theorem and describe the phenomenon of aliasing. While these latter topics are overlooked in most textbooks, they play an important role in geophysics, particularly in the context of data collection and analysis. We introduce the fast Fourier transform and some approximation methods for spectral analysis. We conclude this chapter by briefly touching upon Laplace transforms and the Bromwich integral, and we introduce some integral equations, including the Abel and Radon transforms, as well as the Herglotz–Wiechert problem of seismology.

In chapter 4, we introduce the fundamental partial differential equations of mathematical physics, in general, and geophysics, in particular. We whet the student's appetite by introducing the three fundamental types of partial differential equations that are pervasive in geophysics: the wave equation, the potential equation, and the diffusion equation. This chapter embeds practical examples of real-world problems with the theory. Classic mathematical methods of physics books rarely provide examples, especially those that are appropriate to the earth sciences. Remarkably, some of the most beautiful yet practical examples of these types of equations appear in geophysics. We introduce, for linear problems, integral transform methods, and introduce eigenfunctions, eigenvalues, and Green's functions in those time-dependent contexts. We exploit these methods to solve both the diffusion equation and the wave equation in three dimensions. We employ spherical harmonics, introduced in the second chapter, to solve the gravitational potential equation relating a planet's mass distribution to its potential in three dimensions. Further, we exploit Fourier methods in order to identify dispersion relations for linear problems, including the role of diffusion and dispersion. At this stage, we associate with dispersion relations for partial differential equations the role of instability. Perturbation theory in this context is presented via a simple example, the propagation of sound in a fluid. However, since partial differential equations incorporate an infinite number of modes—associated with spherical harmonics, for example—the chaotic nature of a fundamentally infinite degree of freedom system underscores what is called *complexity*. We consider collective, nonlinear modes of behavior as exemplified by solitary waves and, especially, solitons. As illustrations, we derive the solution for solitary waves exemplified by Burgers's equation and for solitons via the Korteweg–de Vries equation. Scaling arguments underlying the emergence of turbulence are presented, as well as a simple derivation for the Kolmogorov spectrum.

The remaining chapter surveys two topics that are central to modern geophysics yet have been orphaned from essentially all elementary treatments. We briefly survey topics in probability and statistics, including the binomial, Poisson, and Gaussian (normal) distributions as well as the central limit theorem. A sketch is provided for methods of random number generation, central to Monte Carlo simulation. We also identify some of the themes associated with regression and the fitting of experimental data. Finally, we survey some questions emergent from numerical methods. Here, we briefly address the nature of computational and round-off errors. As an example, we survey the determination of the roots of polynomials, which play a fundamental role in the dispersion relations of modern geophysics. We provide a brief overview of numerical methods of solving ordinary and partial differential equations, with a focus on finite difference methods, but mention spectral approaches.

As is evident, this textbook provides a whirlwind survey of many topics and helps bring together many different concepts yet provide a brief practical introduction to problem solving in geophysics. This book was developed in consultation with my colleagues and is the outcome of several offerings at UCLA of this survey course to entering and second-year graduate students in geophysics and planetary and space physics, but was also designed to be helpful to students in allied disciplines, including atmospheric and ocean sciences, and in physics and astronomy. We very much hope that this volume will help stimulate thinking about these problem areas and further investigation and study of the different topics reviewed.

While completing this volume, my editor asked me to provide a cover image for this book and recommended that a photograph be adopted instead of a geometrical design or blank cover as is so often employed in technical books. This presented a special challenge inasmuch as how could a photograph convey what underscores the mathematics implicit to the earth, planetary, and space sciences? What kind of image would capture the outcome of a combination of many different geologic events? Yellowstone National Park is a truly special place, and the Grand Canyon of the Yellowstone is a focal point for much of its varied geologic history. This area was shaped by a caldera eruption 600,000 years ago and a series of lava flows. The area was also faulted by the caldera dome before the eruption. The site of this canyon was possibly established by this faulting, which magnified the rate of erosion. Glaciation also took place, although glacial deposits are largely absent. This photograph features the Lower Falls, 308 feet in height, as viewed from Lookout Point. The rich colors of the rock in this photograph are likely an outcome of the hydrothermal alteration of the rhyolite containing different iron compounds and their subsequent "cooking." Exposure to the elements and oxidation added to this effect, and are not due to sulfur. The falling water provides a quick reminder of the power of the flow. Thinking about all of the various physical and chemical effects present in creating this scene, it is clear how this image captures so many different influences and that challenge of providing a quantitative description of them. I took this photograph on August 24, 2009, with a Sony A350 DSLR at F8 with a 1/320-second exposure time using a 160-mm zoom lens.

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