Preface

The aim of this book is to prove a complete Gross–Zagier formula on quaternionic Shimura curves over totally real fields. The original formula proved by Benedict Gross and Don Zagier in 1983 relates the Néron–Tate heights of Heegner points on $X_0(N)$ to the central derivatives of some Rankin–Selberg L-functions under the Heegner condition, which is an assumption of mild ramification. Since then, some generalizations were given in various works by Shou-Wu Zhang. The proofs of Gross–Zagier and Shou-Wu Zhang depend on some newform theories. There are essential difficulties to removing all ramification assumptions by these methods. This book is a completion of those generalizations in which all ramification restrictions are removed.

The Gross–Zagier formula in this book is an analogue of the central value formula of Jean-Loup Waldspurger, and has been speculated by Benedict Gross in terms of representation theory in a lecture at MSRI in 2002. In fact, the Waldspurger formula concerns periods of automorphic forms on quaternion algebras over number fields, while the Gross–Zagier formula may be viewed as a formula of periods of "automorphic forms" on *incoherent quaternion algebras*. These incoherent automorphic forms are functions on Shimura curves with values in some abelian varieties.

Besides many ideas of Gross–Zagier and Shou-Wu Zhang, one main new ingredient of this book is to construct the analytic kernel and the geometric kernel systematically using Weil representations and the generating series of Hecke correspondences of Stephen S. Kudla constructed in 1997, though we do not use his program on the *arithmetic Siegel–Weil formula*. The construction is inspired by the Waldspurger formula mentioned above. To simplify many computations in both automorphic forms and arithmetic geometry, we take advantage of representation theory and make use of the concepts of *degenerate Schwartz functions*, *coherence of pseudo-theta series*, and *modularity of generating functions*.

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Loup Waldspurger. We have directly adapted his strategy of proving special value formula to our incoherent situation.

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