Chapter One

Introduction

In population genetics one frequently replaces a discrete Markov chain model, which describes the random processes of genetic drift, with or without selection, and mutation with a limiting, continuous time and space, stochastic process. If there are $N+1$ possible types, then the configuration space for the resultant continuous Markov process is typically the N-simplex

$$\mathcal{S}_N = \{(x_1, \ldots, x_N) : \; x_j \geq 0 \text{ and } x_1 + \cdots + x_N \leq 1\}. \tag{1.1}$$

If a different scaling is used to define the limiting process, different domains might also arise. As a geometrical object the simplex is quite complicated. Its boundary is not a smooth manifold, but has a stratified structure with strata of codimensions 1 through N. The codimension 1 strata are

$$\Sigma_{1,l} = \{x_l = 0\} \cap \mathcal{S}_N \text{ for } l = 1, \ldots, N, \tag{1.2}$$

along with

$$\Sigma_{1,0} = \{x_1 + \cdots + x_N = 1\} \cap \mathcal{S}_N. \tag{1.3}$$

Components of the stratum of codimension $1 < l \leq N$ arise by choosing integers $0 \leq i_1 < \cdots < i_l \leq N$ and forming the intersection:

$$\Sigma_{1,i_1} \cap \cdots \cap \Sigma_{1,i_l}. \tag{1.4}$$

The simplex is an example of a *manifold with corners*. The singularity of its boundary significantly complicates the analysis of differential operators acting functions defined in $\mathcal{S}_N$.

In the simplest case, without mutation or selection, the limiting operator of the Wright-Fisher process is the Kimura diffusion operator, with formal generator:

$$L_{\text{Kim}} = \sum_{i,j=1}^{N} x_i(\delta_{ij} - x_j)\partial_{x_i}\partial_{x_j}. \tag{1.5}$$

This is the "backward" Kolmogorov operator for the limiting Markov process. This operator is elliptic in the interior of $\mathcal{S}_N$ but the coefficient of the second order normal derivative tends to zero as one approaches a boundary. We can introduce local coordinates $(x_1, y_1, \ldots, y_{N-1})$ near the interior of a point on one of the faces of $\Sigma_{1,l}$, so that the boundary is given locally by the equation $x_1 = 0$, and the operator then takes the form

$$x_1\partial_{x_1}^2 + \sum_{l=1}^{N-1} x_1 c_{1l}\partial_{x_1}\partial_{y_l} + \sum_{l,m=1}^{N-1} c_{lm}\partial_{y_m}\partial_{y_l}, \tag{1.6}$$

where the matrix c_{lm} is positive definite. To include the effects of mutation, migration and selection, one typically adds a vector field:

$$V = \sum_{i=1}^{N} b_i(x)\partial_{x_i}, \tag{1.7}$$

where V is inward pointing along the boundary of $\mathcal{S}_N$. In the classical models, if only the effect of mutation and migration are included, then the coefficients $\{b_i(x)\}$ can be taken to be linear polynomials, whereas selection requires at least quadratic terms.

The most significant feature is that the coefficient of $\partial_{x_1}^2$ vanishes exactly to order 1. This places L_{Kim} outside the classes of degenerate elliptic operators that have already been analyzed in detail. For applications to Markov processes the difficulty that presents itself is that it is not possible to introduce a square root of the coefficient of the second order terms that is Lipschitz continuous up to the boundary. Indeed the best one can hope for is Hölder-$\frac{1}{2}$. The uniqueness of the solutions to either the forward Kolmogorov equation, or the associated stochastic differential equation, cannot then be concluded using standard methods.

Even in the presence of mutation and migration, the solutions of the heat equation for this operator in 1-dimension was studied by Kimura, using the fact that $L_{\text{Kim}} + V$ preserves polynomials of degree d for each d. In higher dimensions it was done by Karlin and Shimakura by showing the existence of a complete basis of polynomial eigenfunctions for this operator. This in turn leads to a proof of the existence of a polynomial solution to the initial value problem for $[\partial_t - (L_{\text{Kim}} + V)]v = 0$ with polynomial initial data. Using the maximum principle, this suffices to establish the existence of a strongly continuous semi-group acting on $\mathcal{C}^0$, and establish many of its basic properties, see [38]. This general approach has been further developed by Barbour, Etheridge, Ethier, and Griffiths, see [17, 2, 16, 25].

As noted, if selection is also included, then the coefficients of V are at least quadratic polynomials, and can be quite complicated, see [9]. So long as the second order part remains L_{Kim}, then a result of Ethier, using the Trotter product formula, makes it possible to again define a strongly continuous semi-group, see [18]. Various extensions of these results, using a variety of functional analytic frameworks, were made by Athreya, Barlow, Bass, Perkins, Sato, Cerrai, Clément, and others, see [1, 4, 6, 7, 8].

For example Cerrai and Clément constructed a semi-group acting on $\mathcal{C}^0([0,1]^N)$, with the coefficient a_{ij} of $\partial_{x_i}\partial_{x_j}$ assumed to be of the form

$$a_{ij}(x) = m(x)A_{ij}(x_i, x_j). \tag{1.8}$$

Here $m(x)$ is strictly positive. In [1, 4, 3], Bass and Perkins along with several collaborators, study a class of equations, similar to that defined below. Their work has many points of contact with our own, and we discuss it in greater detail at the end of Section 1.5.

We have not yet said anything about boundary conditions, which would seem to be a serious omission for a PDE on a domain with a boundary. Indeed, one would expect that there would be an infinite dimensional space of solutions to the homogeneous equation. It is possible to formulate local boundary conditions that assure uniqueness, but, in some sense, this is not necessary. As a result of the degeneracy of the principal part, uniqueness for these types of equations can also be obtained as a consequence of regularity alone! We illustrate this in the simplest 1-dimensional case, which is the equation, with $b(0) \geq 0,\ b(1) \leq 0$,

$$\partial_t v - [x(1-x)\partial_x^2 + b(x)\partial_x]v = 0 \text{ and } v(x,0) = f(x). \tag{1.9}$$

If we assume that $\partial_x v(x,t)$ extends continuously to $[0,1] \times (0,\infty)$ and

$$\lim_{x\to 0^+} x(1-x)\partial_x^2 v(x,t) = \lim_{x\to 1^-} x(1-x)\partial_x^2 v(x,t) = 0, \tag{1.10}$$

then a simple maximum principle argument shows that the solution is unique. In our approach, such regularity conditions naturally lead to uniqueness, and little effort is expended in the consideration of boundary conditions. In Chapter 3 we prove a generalization of the Hopf boundary point maximum principle that demonstrates, in the general case, how regularity implies uniqueness.

1.1 GENERALIZED KIMURA DIFFUSIONS

In his seminal work, Feller analyzed the most general closed extensions of operators, like those in (1.9), which generate Feller semi-groups in 1-dimension, see [20]. He analyzes the resolvent kernel, using methods largely restricted to ordinary differential equations. In [10], Chen and Stroock use probabilistic methods to prove estimates on the fundamental solution of the parabolic equation, $\partial_t u = x(1-x)\partial_x^2 u$.

Up to now very little is known, in higher dimensions, about the analytic properties of the solution to the initial value problem for the heat equation

$$\partial_t v - (L_{\text{Kim}} + V)v = 0 \text{ in } (0,\infty) \times \mathcal{S}_N \text{ with } v(0,x) = f(x). \tag{1.11}$$

Indeed, if we replace L_{Kim} with a qualitatively similar second order part, which does not take one of the forms described above, then even the existence of a solution is not known. In this monograph we introduce a very flexible analytic framework for studying a large class of equations, which includes all standard models, of this type appearing in population genetics, as well as the SIR model for epidemics, see [19, 38], and models that arise in Mathematical Finance, see [21]. Our approach is to introduce non-isotropic Hölder spaces with respect to which we establish sharp existence and regularity results for the solutions to heat equations of this type, as well as the corresponding elliptic problems. Using the Lumer-Phillips theorem we conclude that the $\mathcal{C}^0$-graph closure of this operator generates a strongly continuous semi-group.

The approach here is an extension of our work on the 1-d case in [15], which allows us to prove existence, uniqueness and regularity results for a class of higher dimensional, degenerate diffusion operators. While our methods also lead to a precise description of the heat kernel in the 1-dimensional case, this has proved considerably more challenging in higher dimensions. It is hoped that a combination of the analytic techniques used here, and the probabilistic techniques from [10] will lead to good descriptions of the heat kernels in higher dimensions.

Our analysis applies to a class of operators that we call *generalized Kimura diffusions*, which act on functions defined on *manifolds with corners.* Such spaces generalize the notion of a regular convex polyhedron in $\mathbb{R}^N$, e.g., the simplex. Working in this more general context allows for a great deal of flexibility, which proves indispensable in the proof of our basic existence result.

Locally a manifold with corners, P, can be described as a subset of $\mathbb{R}^N$ defined by inequalities. Let $\{p_k(x) : k = 1, \dots, K\}$ be smooth functions in the unit ball $B_1(0) \subset \mathbb{R}^N$, vanishing at 0, with $\{dp_k(0) : k = 1, \dots, K\}$ linearly independent; clearly $K \leq N$. Locally P is diffeomorphic to

$$\bigcap_{k=1}^{K} \{x \in B_1(0) : p_k(x) \geq 0\}. \tag{1.12}$$

We let $\Sigma_k = P \cap \{x : p_k(x) = 0\}$; suppose that Σ_k contains a non-empty, open $(N-1)$-dimensional hypersurface and that dp_k is non-vanishing in a neighborhood of Σ_k. The boundary of P is a stratified space, where the strata of codimension n locally consists of points where the boundary is defined by the vanishing of n functions with independent gradients. The components of the codimension 1 part of the bP are called *faces*. As in (1.4), the codimension-n stratum of bP is formed from intersections of n faces.

The *formal* generator is a degenerate elliptic operator of the form

$$L = \sum_{i,j=1}^{N} A_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{N} b_j(x)\partial_{x_j}. \tag{1.13}$$

Here $A_{ij}(x)$ is a smooth, symmetric matrix valued function in P. The second order term is positive definite in the interior of P and degenerates along the hypersurface boundary components in a specific way. For each k

$$\sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_i} p_k(x)\partial_{x_j} p_k(x) \propto p_k(x) \text{ as } x \text{ approaches } \Sigma_k. \tag{1.14}$$

On the other hand,

$$\sum_{i,j=1}^{N} a_{ij}(x)v_i v_j > 0 \text{ for } x \in \text{int}\,\Sigma_k \text{ and } v \neq 0 \in T_x\Sigma_k. \tag{1.15}$$

The first order part of L is an inward pointing vector field

$$Vp_k(x) = \sum_{j=1}^{N} b_j(x)\partial_{x_j} p_k(x) \geq 0 \text{ for } x \in \Sigma_k. \tag{1.16}$$

We call a second order partial differential operator defined on P, which is non-degenerate elliptic in int P, with this local description near any boundary point a *generalized Kimura diffusion* .

If p is a point on the stratum of bP of codimension n, then locally there are coordinates $(x_1, \dots, x_n; y_1, \dots, y_m)$ so that p corresponds to $(\mathbf{0}; \mathbf{0})$, and a neighborhood, U, of p is given by

$$U = \{(\boldsymbol{x}; \boldsymbol{y}) \in [0, 1)^n \times (-1, 1)^m\}. \tag{1.17}$$

In these local coordinates a generalized Kimura diffusion, L, takes the form

$$\begin{aligned} L = \sum_{i=1}^{n} [a_{ii}(\boldsymbol{x}; \boldsymbol{y}) x_i \partial_{x_i}^2 + \widetilde{b}_i(\boldsymbol{x}; \boldsymbol{y})\partial_{x_i}] + \sum_{1 \leq i \neq j \leq n} x_i x_j a_{ij}(\boldsymbol{x}; \boldsymbol{y})\partial_{x_i x_j}^2 + \\ \sum_{i=1}^{n}\sum_{k=1}^{m} x_i b_{ik}(\boldsymbol{x}; \boldsymbol{y})\partial_{x_i y_k}^2 + \sum_{k,l=1}^{m} c_{kl}(\boldsymbol{x}; \boldsymbol{y})\partial_{y_k y_l}^2 + \sum_{k=1}^{m} d_k(\boldsymbol{x}; \boldsymbol{y})\partial_{y_k}; \end{aligned} \tag{1.18}$$

(a_{ij}) and (c_{kl}) are symmetric matrices, the matrices (a_{ii}) and (c_{kl}) are strictly positive definite. The coefficients $\{\widetilde{b}(\boldsymbol{x}; \boldsymbol{y})\}$ are non-negative along $bP \cap U$, so that first order part is inward pointing.

Let P be a compact manifold with corners and L a generalized Kimura diffusion defined on P. Broadly speaking, our goal is to prove the existence, uniqueness and regularity of solutions to the equation

$$\begin{aligned} &(\partial_t - L)u = g \text{ in } P \times (0, \infty) \\ &\text{with } u(p, 0) = f(p), \end{aligned} \tag{1.19}$$

with certain boundary behavior along $bP \times [0, \infty)$, for data g and f satisfying appropriate regularity conditions. These results in turn can be used to prove the existence of a strongly continuous semi-group acting on $\mathscr{C}^0(P)$, with formal generator L. This is the "backward Kolmogorov equation." The solution to the "forward Kolmogorov equation," $(\partial_t - L^*)m = \nu$, is then given by the adjoint semi-group, canonically defined on a (non-dense) domain in $[\mathscr{C}^0(P)]' = \mathscr{M}(P)$, the space of finite Borel measures on P.

1.2 MODEL PROBLEMS

The problem of proving the existence of solutions to a class of PDEs is essentially a matter of finding a good class of model problems, for which existence and regularity can be established, more or less directly, and then finding a functional analytic setting in which to do a perturbative analysis of the equations of interest. The model operators for Kimura diffusions are the differential operators, defined on $\mathbb{R}_+^n \times \mathbb{R}^m$, by

$$L_{\boldsymbol{b},m} = \sum_{j=1}^{n} [x_j \partial_{x_j}^2 + b_j \partial_{x_j}] + \sum_{k=1}^{m} \partial_{y_k}^2. \tag{1.20}$$

Here $\boldsymbol{b} = (b_1, \dots, b_n)$ is a non-negative vector.

We have not been too explicit about the boundary conditions that we impose along $b\mathbb{R}_+^n \times \mathbb{R}^m$. This condition can be defined by a local Robin-type formula involving the value of the solution and its normal derivative along each hypersurface boundary component of bP. For $b > 0$, the 1-dimensional model operator, $L_b = x\partial_x^2 + b\partial_x$, has two indicial roots

$$\beta_0 = 0, \beta_1 = 1 - b, \tag{1.21}$$

that is

$$L_b x^{\beta_0} = L_b x^{\beta_1} = 0. \tag{1.22}$$

If $b \neq 1$, then the boundary condition,

$$\lim_{x \to 0^+} [\partial_x (x^b u(x,t)) - b x^{b-1} u(x,t)] = 0, \tag{1.23}$$

excludes the appearance of terms like x^{1-b} in the asymptotic expansion of solutions along $x = 0$. In fact, this condition insures that u is as smooth as possible along the boundary: if $g = 0$ and f has m derivatives then the solution to (1.19), satisfying (1.23), does as well. This boundary condition can be encoded as a regularity condition, that is $u(\cdot, t) \in \mathscr{C}^1([0,\infty)) \cap \mathscr{C}^2((0,\infty))$, with

$$\lim_{x \to 0^+} x\partial_x^2 u(x,t) = 0 \tag{1.24}$$

for $t > 0$. We call the unique solution to a generalized Kimura diffusion, satisfying this condition, or its higher dimensional analogue, the *regular* solution. The majority of this monograph is devoted to the study of regular solutions.

In applications to probability one often seeks solutions to equations of the form $Lw = g$, where w satisfies a Dirichlet boundary condition: $w \restriction_P = h$. Our uniqueness results often imply that these equations *cannot* have a regular solution, for example, when $g \geq 0$. In the classical case the solutions to these problems can sometimes be written down explicitly, and are seen to involve the non-zero indicial roots. Usually these satisfy the other natural boundary condition, a la [20]. In 1-dimension, when $b \neq 0, 1$ it is:

$$\lim_{x \to 0^+} [\partial_x (x u(x,t)) - (2-b)u] = 0, \tag{1.25}$$

and allows for solutions that are $O(x^{1-b})$ as $x \to 0^+$. These are not smooth up to the boundary, even if the data is. The adjoint of L is naturally defined as an operator on $\mathscr{M}(P)$, the space finite Borel measures on P. It is more common to study this operator using techniques from probability theory, see [40].

For a generalized Kimura diffusion in dimensions greater than 1, the coefficient of the normal first derivative can vary as one moves along the boundary. For example, in 2-dimensions one might consider the operator

$$L = x\partial_x^2 + \partial_y^2 + b(y)\partial_x. \tag{1.26}$$

If $b(y)$ is not constant, then, with the boundary condition

$$\lim_{x\to 0^+} [\partial_x(xu(x, y, t)) - (2 - b(y))u(x, y, t)] = 0, \tag{1.27}$$

one would be faced with the very thorny issue of a varying indicial root on the outgoing face of the heat or resolvent kernel. As it is, we get a varying indicial root on the incoming face, a fact which already places the analysis of this problem beyond what has been achieved using the detailed kernel methods familiar in geometric microlocal analysis. The natural boundary condition for the adjoint operator includes the condition:

$$\lim_{x\to 0^+} [\partial_x(xu(x, y, t)) - b(y)u(x, y, t)] = 0, \tag{1.28}$$

allowing for solutions that behave like $x^{b(y)-1}$, as $x \to 0^+$.

The solution operators for the 1-dimensional model problems are given by simple explicit formulæ. If $b > 0$, then the heat kernel is

$$k_t^b(x, y)dy = \left(\frac{y}{t}\right)^b e^{-\frac{x+y}{t}} \psi_b\left(\frac{xy}{t^2}\right)\frac{dy}{y}, \tag{1.29}$$

where

$$\psi_b(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(j + b)}. \tag{1.30}$$

If $b = 0$ then

$$k_t^0(x, y) = e^{-\frac{x}{t}}\delta_0(y) + \left(\frac{x}{t}\right) e^{-\frac{x+y}{t}} \psi_2\left(\frac{xy}{t^2}\right)\frac{dy}{t}. \tag{1.31}$$

In either case k_t^b is smooth as $x \to 0^+$ and displays a y^{b-1} singularity as $y \to 0^+$. It is notable that the character of the kernel changes dramatically as $b \to 0$, nonetheless the regular solutions to these heat equations satisfy uniform estimates even as $b \to 0^+$. This fact is quite essential for the success of our approach.

The structure of these operators suggests that a natural functional analytic setting in which to do the analysis might be that provided by the anisotropic Hölder spaces defined by the singular, but incomplete metric on $\mathbb{R}_+^n \times \mathbb{R}^m$:

$$ds_{WF}^2 = \sum_{j=1}^{n} \frac{dx_j^2}{x_j} + \sum_{k=1}^{m} dy_m^2. \tag{1.32}$$

Similar spaces have been introduced by other authors for problems with similar degeneracies, see [11, 12, 28, 32, 23, 24]. In [22] Goulaouic and Shimakura proved a priori estimates in a Hölder space of this general sort in a case where the operator has this type of degeneracy, but the boundary is smooth. As was the case in these earlier works, we introduce two families of anisotropic Hölder spaces, which we denote by $\mathscr{C}^{k,\gamma}_{WF}(P)$, and $\mathscr{C}^{k,2+\gamma}_{WF}(P)$, for $k \in \mathbb{N}_0$, and $0 < \gamma < 1$. In this context, heuristically an operator A is "elliptic of second order" if $A^{-1} : \mathscr{C}^{k,\gamma}_{WF}(P) \to \mathscr{C}^{k,2+\gamma}_{WF}(P)$. Note that $\mathscr{C}^{k,2+\gamma}_{WF}(P) \subseteq \mathscr{C}^{k+1,\gamma}_{WF}(P)$, but $\mathscr{C}^{k+2,\gamma}_{WF}(P) \not\subseteq \mathscr{C}^{k,2+\gamma}_{WF}(P)$, which explains the need for two families of spaces.

In this monograph we consider the problem in (1.19) for f and g belonging to these Hölder spaces. The results obtained suffice to prove the existence of a semi-group on the space $\mathscr{C}^0(P)$, but establishing the refined regularity properties of this semi-group and its adjoint require the usage of a priori estimates. These are of a rather different character from the analysis presented here; we will return to this question in a subsequent publication.

In [23, 24], Graham studies model operators of the forms $x(\partial_x^2 + \Delta_{\mathbb{R}^n}) + (1-\lambda)\partial_x$, and $x\partial_x^2 + \Delta_{\mathbb{R}^n} + (1-\lambda)\partial_x$, acting on functions defined on the half-space $[0, \infty) \times \mathbb{R}^n$. Using kernel methods, he proves sharp estimates for the solution of the inhomogeneous Dirichlet problem, in both isotropic and an-isotropic Hölder spaces. Graham works directly with the solution kernels for the elliptic problems, whereas we prove estimates for the parabolic problem and obtain the elliptic case via the Laplace transform.

As manifolds with corners have non-smooth boundaries, and the Kimura diffusions are degenerate elliptic operators, the analysis of (1.19) can be expected to be rather challenging. We have already indicated a variety of problems that arise:

1. The principal part of L degenerates at the boundary.

2. The boundary of P is not smooth.

3. The "indicial roots" vary with the location of the point on bP.

4. The character of the solution operator is quite different at points where the vector field is tangent to bP.

Along the boundary $\{x_j = 0\}$, the first and second order terms in (1.20), $b_j\partial_{x_j}$ and $x_j\partial_{x_j}^2$, respectively, are of comparable "strength." It is a notable and non-trivial fact that estimates for the solutions of these equations can be proved in these Hölder spaces, without regard for the value of $\boldsymbol{b} \geq \boldsymbol{0}$. As there is an explicit formula for the fundamental solution, the analysis of these model operators, while tedious and time consuming, is elementary. Indeed the solution of the homogeneous Cauchy problem,

$$\begin{aligned} &(\partial_t - L_{\boldsymbol{b},m})u = 0 \text{ in } P \times (0, \infty) \\ &\text{with } u(p, 0) = f(p), \end{aligned} \tag{1.33}$$

has an analytic extension to $\operatorname{Re} t > 0$, which satisfies many useful estimates.

To obtain a gain of derivatives where $\operatorname{Re} t > 0$, in a manner that can be extended beyond the model problems, one must address the inhomogeneous problem, which has somewhat simpler analytic properties. By this device, one can also estimate the Laplace transform of the heat semi-group, which is the resolvent operator:

$$(\mu - L_{\boldsymbol{b},m})^{-1} = \int_0^\infty e^{tL_{\boldsymbol{b},m}} e^{-\mu t} dt. \tag{1.34}$$

The estimates for the inhomogeneous problem show that, in an appropriate sense, $(\mu - L_{\boldsymbol{b},m})^{-1}$ gains two derivatives and is analytic in the complement of $(-\infty, 0]$. Finally one can re-synthesize the heat operator from the resolvent, via contour integration:

$$e^{tL_{\boldsymbol{b},m}} = \frac{1}{2\pi i} \int_C (\mu - L_{\boldsymbol{b},m})^{-1} e^{\mu t} d\mu, \tag{1.35}$$

where C is of the form $|\arg \mu| = \frac{\pi}{2} + \alpha$, for an $0 < \alpha < \frac{\pi}{2}$. This shows that, for t with positive real part, $e^{tL_{\boldsymbol{b},m}}$ also gains two derivatives.

Remark. In probability theory it is more common to consider the operators obtained from $L_{\boldsymbol{b},m}$ under the change of variables $x_j = \frac{w_j^2}{2}$. In this coordinate system, the model operators become:

$$L_{\boldsymbol{b},m} = \frac{1}{2} \sum_{j=1}^{n} \left[\partial_{w_j}^2 + \frac{2b_j - 1}{w_j} \partial_{w_j} \right] + \sum_{k=1}^{m} \partial_{y_k}^2. \tag{1.36}$$

The processes these operators generate are called Bessel processes. For a recent paper on this subject see, for example, [5].

1.3 PERTURBATION THEORY

The next step is to use these estimates for the model problems in a perturbative argument to prove existence and regularity for a generalized Kimura diffusion operator L on a manifold with corners, P. The boundary of a manifold with corners is a stratified space, which produces a new set of difficulties. To overcome this we use an induction on the maximal codimension of the strata of bP.

The induction starts with the simplest case where bP is just a manifold, and P is then a manifold with boundary. In this case, we can use the model operators to build a parametrix for the solution operator to the heat equation in a neighborhood of the boundary, $\widehat{Q}_b^t$. It is a classical fact that there is an exact solution operator, $\widehat{Q}_i^t$, defined in the complement of a neighborhood of the boundary, for, in any such subset of P, L is a non-degenerate elliptic operator. Using a partition of unity these operators can be "glued together" to define a parametrix, $\widehat{Q}^t$, for the solution operator. The Laplace transform

$$\widehat{R}(\mu) = \int_0^\infty e^{\mu t} \widehat{Q}^t dt \tag{1.37}$$

is then a right parametrix for $(\mu - L)^{-1}$. Using the estimates and analyticity for the model problems, and the properties of the interior solution operator, we can show that

$$(\mu - L)\widehat{R}(\mu) = \mathrm{Id} + E(\mu), \tag{1.38}$$

where $E(\mu)$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$, and the Neumann series for $(\mathrm{Id} + E(\mu))^{-1}$ converges in the operator norm topology for μ in sectors $|\arg \mu| \leq \pi - \alpha$, for any $\alpha > 0$, if $|\mu|$ is sufficiently large. This allows us to show that

$$(\mu - L)^{-1} = \widehat{R}(\mu)(\mathrm{Id} + E(\mu))^{-1} \tag{1.39}$$

is analytic and satisfies certain estimates in

$$T_{\alpha,R} = \{\mu : |\arg \mu| < \pi - \alpha, \quad |\mu| > R\}, \tag{1.40}$$

for any $0 < \alpha$, and R depending on α.

For t in the right half plane we can now reconstruct the heat semi-group acting on the Hölder spaces:

$$e^{tL} = \frac{1}{2\pi i} \int\limits_{bT_{\alpha,R}} (\mu - L)^{-1} e^{\mu t} d\mu \tag{1.41}$$

for an appropriate choice of α. This allows us to verify that e^{tL} has an analytic continuation to $\mathrm{Re}\, t > 0$, which satisfies the desired estimates with respect to the anisotropic Hölder spaces defined above. The proof for the general case now proceeds by induction on the maximal codimension of the strata of bP. In all cases we use the model operators to construct a boundary parametrix $\widehat{Q}_b^t$ near the maximal codimensional part of bP. The induction hypothesis provides an exact solution operator in the "interior," $\widehat{Q}_i^t$, with certain properties, which we once again glue together to get $\widehat{Q}^t$. A key step in the argument is to verify that the heat operator we finally obtain satisfies the induction hypotheses. The representation of e^{tL} in (1.41) is a critical part of this argument.

1.4 MAIN RESULTS

With these preliminaries we can state our main results. The sharp estimates for operators e^{tL} and $(\mu - L)^{-1}$ are phrased in terms of two families of Hölder spaces. For $k \in \mathbb{N}_0$ and $0 < \gamma < 1$, we define the spaces $\mathscr{C}^{k,\gamma}_{WF}(P)$, $\mathscr{C}^{k,2+\gamma}_{WF}(P)$, and their "heat-space" analogues, $\mathscr{C}^{k,\gamma}_{WF}(P \times [0, T])$, $\mathscr{C}^{k,2+\gamma}_{WF}(P \times [0, T])$, see Chapter 5. For example: in the 1-dimensional case $f \in \mathscr{C}^{0,\gamma}_{WF}([0, \infty))$ if f is continuous and

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|\sqrt{x} - \sqrt{y}|^{\gamma}} < \infty; \tag{1.42}$$

it belongs to $\mathscr{C}^{0,2+\gamma}_{WF}([0, \infty))$ if f, $\partial_x f$, and $x\partial_x^2 f$ all belong to $\mathscr{C}^{0,\gamma}_{WF}([0, \infty))$, with

$$\lim_{x \to 0^+, \infty} x\partial_x^2 f(x) = 0.$$

For $k \in \mathbb{N}$, we say that $f \in \mathscr{C}^{k,\gamma}_{WF}([0,\infty))$, if $f \in \mathscr{C}^k([0,\infty))$, and $\partial_x^k f \in \mathscr{C}^{0,\gamma}_{WF}([0,\infty))$. A function $g \in \mathscr{C}^{0,\gamma}_{WF}([0,\infty)\times[0,\infty))$, if $g \in \mathscr{C}^0([0,\infty)\times[0,\infty))$, and

$$\sup_{(x,t)\neq(y,s)} \frac{|g(x,t)-g(y,s)|}{[|\sqrt{x}-\sqrt{y}|+\sqrt{|t-s|}]^{\gamma}} < \infty, \tag{1.43}$$

etc.

Much of this monograph is concerned with proving detailed estimates for the model problems with respect to these Hölder spaces and then using perturbative arguments to obtain analogous results for a general Kimura diffusion on an arbitrary compact manifold with corners.

To describe the uniqueness properties for solutions to these equations, we need to consider the geometric structure of the boundary of P, and its relationship to L. As noted bP is a stratified space, with hypersurface boundary components $\{\Sigma_{1,j} : j = 1,\dots,N_1\}$. A boundary component of codimension n is a component of an intersection

$$\Sigma_{1,i_1}\cap\cdots\cap\Sigma_{1,i_n}, \tag{1.44}$$

where $1 \le i_1 < \cdots < i_n \le N_1$. A component of bP is *minimal* if it is an isolated point or a positive dimensional manifold without boundary. We denote the set of minimal components by bP_{min}. Fix a generalized Kimura diffusion operator L. Let $\{\rho_j : j = 1,\dots,N_1\}$ be defining functions for the hypersurface boundary components. We say that L is *tangent* to $\Sigma_{1,j}$ if $L\rho_j \restriction_{\Sigma_{1,j}} = 0$, and *transverse* if there is a $c > 0$ so that

$$L\rho_j \restriction_{\Sigma_{1,j}} > c. \tag{1.45}$$

DEFINITION 1.4.1. *The terminal boundary of P relative to L, $bP_{\text{ter}}(L)$, consists of elements of bP_{min} to which L is tangent, along with boundary strata, Σ of P to which L is tangent, and such that $L_\Sigma = L \restriction_\Sigma$ is transverse to all components of $b\Sigma$.*

For the model space we say that

$$f \in \mathscr{D}^2_{WF}(\mathbb{R}^n_+\times\mathbb{R}^m) \subset \dot{\mathscr{C}}^1(\mathbb{R}^n_+\times\mathbb{R}^m)\cap\mathscr{C}^2((0,\infty)^n\times\mathbb{R}^m) \tag{1.46}$$

if the scaled second derivatives

$$x_i\partial^2_{x_i} f(\boldsymbol{x};\boldsymbol{y}),\ x_ix_j\partial^2_{x_ix_j} f(\boldsymbol{x},\boldsymbol{y}),\ x_i\partial^2_{x_iy_l} f(\boldsymbol{x},\boldsymbol{y}),\ \partial^2_{y_ly_k} f(\boldsymbol{x},\boldsymbol{y}) \tag{1.47}$$

extend continuously to $\mathbb{R}^n_+\times\mathbb{R}^m$. We also assume that $x_ix_j\partial^2_{x_ix_j} f(\boldsymbol{x},\boldsymbol{y})$ tends to zero if either x_i or x_j goes to zero, and $x_i\partial^2_{x_i} f(\boldsymbol{x};\boldsymbol{y})$ and $x_i\partial^2_{x_iy_l} f(\boldsymbol{x},\boldsymbol{y})$ go to zero as x_i goes to zero. A function $f \in \mathscr{C}^1(P)\cap\mathscr{C}^2(\text{int}\,P)$ belongs to $\mathscr{D}^2_{WF}(P)$ if it belongs to these local spaces in each local coordinate chart. Using a variant of the Hopf maximum principle, we can prove

THEOREM 1.4.2. *Let P be a compact manifold with corners, and L a generalized Kimura diffusion defined on P. Suppose that L is either tangent or transverse to every hypersurface boundary component of bP, and let $bP_{\text{ter}}(L)$ denote the set of terminal components of the boundary stratification relative to L. The cardinality of the* *set* $bP_{\text{ter}}(L)$ *equals the dimension of the null-space of* L *acting on* $\mathcal{D}^2_{WF}(P)$*, which is also the dimension of* $\ker \overline{L}^*$*. The null-space of* L *is represented by smooth non-negative functions; the null-space of* $\overline{L}^*$ *by non-negative measures supported on the components of* $bP_{\text{ter}}(L)$*.*

The existence and regularity results for the heat equation defined by a general Kimura diffusion, L, on a manifold with corners, P, are summarized in the next two results:

THEOREM 1.4.3. *Let* P *be a compact manifold with corners,* L *a generalized Kimura diffusion on* P*,* $k \in \mathbb{N}_0$ *and* $0 < \gamma < 1$*. If* $f \in \mathcal{C}^{k,\gamma}_{WF}(P)$*, then there is a unique solution*

$$v \in \mathcal{C}^{k,\gamma}_{WF}(P \times [0,\infty)) \cap \mathcal{C}^{\infty}(P \times (0,\infty)),$$

to the initial value problem

$$(\partial_t - L)v = 0 \text{ with } v(p,0) = f(p). \tag{1.48}$$

This solution has an analytic continuation to t *with* $\operatorname{Re} t > 0$*.*

We have a similar result for the inhomogeneous problem:

THEOREM 1.4.4. *Let* P *be a compact manifold with corners,* L *a generalized Kimura diffusion on* P*,* $k \in \mathbb{N}_0$*,* $0 < \gamma < 1$*, and* $T > 0$*. If* $g \in \mathcal{C}^{k,\gamma}_{WF}(P \times [0,T])$*, then there is a unique solution*

$$u \in \mathcal{C}^{k,2+\gamma}_{WF}(P \times [0,T])$$

to

$$(\partial_t - L)u = g \text{ with } u(p,0) = 0, \tag{1.49}$$

which satisfies estimates of the form

$$\|u\|_{WF,k,2+\gamma,T} \leq M_{k,\gamma} \exp(C_{k,\gamma} T)\|g\|_{WF,k,\gamma,T}. \tag{1.50}$$

We also have a result for the resolvent of L acting on the spaces $\mathcal{C}^{k,2+\gamma}_{WF}(P)$, showing that $(\mu - L)^{-1}$ is an elliptic operator with respect to our scales of Banach spaces.

THEOREM 1.4.5. *Let* P *be a compact manifold with corners,* L *a generalized Kimura diffusion on* P*,* $k \in \mathbb{N}_0$*,* $0 < \gamma < 1$*. The spectrum,* E*, of the unbounded, closed operator* L*, with domain*

$$\mathcal{C}^{k,2+\gamma}_{WF}(P) \subset \mathcal{C}^{k,\gamma}_{WF}(P),$$

is independent of k*, and* γ*. It is a discrete set lying in a conic neighborhood of* $(-\infty, 0]$*. The eigenfunctions belong to* $\mathcal{C}^{\infty}(P)$*.*

Remark. Note that $\mathcal{C}^{k,2+\gamma}_{WF}(P)$ is *not* a dense subspace of $\mathcal{C}^{k,\gamma}_{WF}(P)$.

1.5 APPLICATIONS IN PROBABILITY THEORY

The principal sources for operators of the type studied here are infinite population limits of Markov chains in population genetics, and certain classes of "linear" models in mathematical finance. In this context the operator L, acting on a dense domain in $\mathscr{C}^0(P)$ is called the backward Kolmogorov operator. Its formal adjoint, which acts on the dual space, $\mathscr{M}(P)$, of finite signed Borel measures on P, is the forward Kolmogorov operator. The standard way to address the adjoint operator is to study the martingale problem associated with L on $\mathscr{C}^0([0,\infty); P)$. Letting $\omega \in \mathscr{C}^0([0,\infty); P)$, for each $t \in [0,\infty)$, we define

$$x(t) : \mathscr{C}^0([0,\infty); P) \to P,$$

by $x(t)[\omega] = \omega(t)$. We let $\mathscr{F}_t$ denote the σ-field generated by $\{x(s) : 0 \leq s \leq t\}$ and $\mathscr{F}$ the σ-field generated by $\{x(s) : s \geq 0\}$. For each $q \in P$, a probability measure $\mathbb{P}_q$ on $(\mathscr{C}^0([0,\infty); P), \mathscr{F})$ is a solution of the martingale problem associated with L and starting from $q \in P$ at time $t = 0$, if

$$\begin{gathered}\mathbb{P}_q(x(0) = q) = 1 \text{ and} \\ \left\{ f(x(t)) - \int_0^t Lf(x(s))ds \right\}_{t \geq 0}\end{gathered} \tag{1.51}$$

is a $\mathbb{P}_q$-martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$, see [39].

The existence results in Theorems 1.4.3 or 1.4.5 suffice to prove that the associated martingale problem has a unique solution. A standard argument then shows that the paths for associated strong Markov process remain, almost surely, within P. From this we can deduce a wide variety of results about the forward Kolmogorov equation, and the solutions of the associated stochastic differential equation. The precise nature of these results depends on the behavior of the vector field V along bP. As this analysis requires techniques quite distinct from those employed here, we defer these questions to a future, joint publication with Daniel Stroock.

Using the Lumer-Phillips theorem, these results also suffice to prove that the $\mathscr{C}^0(P)$-graph closure, $\overline{L}$, of L acting on $\mathscr{C}^3(P)$ is the generator of a strongly continuous contraction semi-group. At present we have not succeeded in showing that the resolvent of $\overline{L}$ is compact, and will return to this question in a later publication. We have nonetheless been able to characterize the null-space of adjoint operator $\overline{L}^*$, under a natural clean intersection condition for the vector field V. This allows for an analysis of the asymptotic behavior of the solution to $\partial_t \nu - L^*\nu = 0$, as $t \to \infty$, see formula (12.63), and (12.62), for the asymptotics of $e^{tL} f$.

In [1, 4] Bass, Perkins, et al. have employed methods, similar to our own, to study operators of the form

$$L_{BP} = \sum_{i,j=1}^{n} \sqrt{x_i x_j} a_{ij}(\boldsymbol{x}) \partial^2_{x_i x_j} + \sum_{i=1}^{n} b_i(\boldsymbol{x}) \partial_{x_i} \tag{1.52}$$

acting on functions in $\mathscr{C}^2_b(\mathbb{R}^n_+)$. Here $b_i(\boldsymbol{x}) \geq 0$ along $b\mathbb{R}^n_+$. They have also considered other degenerate operators of this general type. Their main goal is to show the uniqueness of the solution to the martingale problem defined by L_{BP}. To that end they introduce *weighted* Hölder spaces, which take the place of our anisotropic spaces. In the 1-dimensional case, the weighted γ-semi-norm is defined by

$$[f]_{BP,\gamma} = \sup_{x\in\mathbb{R}_+;\, h>0} \left[\frac{|f(x) - f(x+h)|}{h^\gamma} x^{\frac{\gamma}{2}}\right]. \tag{1.53}$$

They prove estimates for the heat kernels of model operators, equivalent to $L_{\boldsymbol{b},0}$, with respect to these Hölder spaces. Under a smallness assumption on the off-diagonal elements of the coefficient matrix $(a_{ij}(\boldsymbol{x}))$, they are able to control the error terms introduced by replacing L_{BP} by the model operator

$$L_{BP,0} = \sum_{i=1}^{n} [x_i a_{ii}(\mathbf{0})\partial^2_{x_i} + b_i(\mathbf{0})\partial_{x_i}] \tag{1.54}$$

well enough to construct a resolvent operator $(L_{BP} - \mu)^{-1}$ for $\mu > 0$. This suffices for their applications to the martingale problem defined by L_{BP}. Notice that with this approach, only "pure corner" models are used, and no consideration is given to operators of the form $L_{\boldsymbol{b},m}$ with $m > 0$. For domains much more general than $\mathbb{R}^n_+$ it is difficult to see how to make such an approach viable.

The operators we treat are somewhat more restricted, in that we take the coefficients of the off-diagonal terms to have the form $x_i x_j a_{ij}(\boldsymbol{x}; \boldsymbol{y})$. Our method could equally well be applied to operators of the form considered by Bass and Perkins, i.e., with $x_i x_j$ replaced by $\sqrt{x_i x_j}$, if we were to append smallness hypotheses for the off-diagonal terms, similar to those they employ. We briefly consider this more general class of operators in Section 11.3. After slightly modifying the definitions of the higher order Hölder norms to include certain increasing weights, many of our results could be generalized to include certain non-compact cases.

Our aims were of a more analytic character, and take us far beyond what is needed to show the uniqueness of the solution to the martingale problem. This leads us to consider such things as the higher order regularity of solutions with smoother initial data, the analytic extension of the semi-group in time, and the higher order mapping properties of the resolvent operator. We also show the ellipticity of the resolvent, with a gain of 2 derivatives with respect to the anisotropic Hölder norms. While this does not appear explicitly in [4], a similar result, with respect to the weighted Hölder norms, should follow from what they have proved.

1.6 ALTERNATE APPROACHES

We wish to mention a few other approaches which have been successfully employed to study similar degenerate elliptic and parabolic equations. As outlined above, the approach here is a hybrid one: we use the explicit kernels for the solution operators of the model problems to derive a priori estimates in adapted Hölder spaces, and then, using these a priori estimates, carry out a perturbation argument to pass from the model problem to the actual problem. A guiding principle throughout is to frame definitions and constructions based on the geometry of the underlying WF metrics. It is also viable to study such problems using only a priori estimates, or through a purely parametrix based approach.

We have already mentioned the papers [11], [12] and [28]. Each of these treats very similar classes of degenerate operators on a manifold with boundary, with the defining function for the boundary multiplying either just the second normal derivative or else all second derivative terms, respectively. Motivation for these papers comes out of the study of the porous medium equation. The work of Daskalopoulos and Hamilton is based on a priori estimates derived through the maximum principle. Koch's paper, by contrast, uses some powerful techniques involving singular integral operators to derive similar estimates.

It is not immediately apparent, but the model operators

$$L_1 := x\partial_x^2 + \sum_{j=1}^m \partial_{y_j}^2 + b\partial_x, \quad \text{and}$$
$$L_2 := x(\partial_x^2 + \sum_{j=1}^m \partial_{y_j}^2) + b\partial_x$$

are essentially equivalent. Indeed, replacing L_1 by xL_1 and changing variables by setting $\xi = \sqrt{x}$ gives

$$\frac{1}{4}\xi^2\partial_\xi^2 + \frac{1}{2}(b - \frac{1}{2})\xi\partial_\xi + \xi^2\sum_{j=1}^m \partial_{y_j}^2,$$

which has the same structure as xL_2. In this representation the indicial roots of both xL_1 and xL_2 are $\alpha \in \{0, 1-b\}$. The functions $\xi^{2\alpha}$ (x^α) are solutions of the normal equation

$$\left[\frac{1}{4}\xi^2\partial_\xi^2 + \frac{1}{2}(b - \frac{1}{2})\xi\partial_\xi + \lambda\xi^2\right]\xi^\alpha = O(\xi^{\alpha+1}), \tag{1.55}$$

which evidently do not depend on λ.

The operators xL_1 and xL_2 are *uniformly degenerate operators* on a manifold with boundary. By definition, any such operator is one which can be written as sums of products of the basic vector fields $x\partial_x$ and $x\partial_{y_j}$. Thus, a model prototype for second order "elliptic" differential operators has the form

$$(x\partial_x)^2 + \sum_{j=1}^m (x\partial_{y_j})^2 + bx\partial_x. \tag{1.56}$$

In other words, the linear operators considered in [11], [12], [28] and [23, 24] are subsumed into the more general framework of uniformly degenerate operators.

Another instance of a much-studied operator of this form is the Laplacian on hyperbolic space, $\Delta_{\mathbb{H}^n}$, which has the form (1.56) with $b = 1 - n$. A key difference between the usual theory developed for that operator, and indeed also for the linearizations of the porous medium equations in [11], [23, 24] and [28], and our study of the generalized Kimura diffusions is that the implicit boundary condition (1.23) used here is of Neumann type (at least when $b < 1$). In other words, for the hyperbolic Laplacian one typically picks out the solution determined by a simple growth condition, e.g., $\Delta_{\mathbb{H}^n} u = f$ with $u \in L^2$. Neumann conditions are well-known, even in the nondegenerate setting, to have a more global nature. On the other hand, a very important feature of our class of generalized Kimura operators L is the fact that the indicial root 0 is constant, and in particular is independent of both the first order coefficient b as well as the spectral parameter λ when we study the resolvent of L. This is in marked contrast with $\Delta_{\mathbb{H}^n} - \lambda$ where the indicial roots depend on the spectral parameter. While this property of Kimura diffusions is not inconsistent with the general theory of elliptic uniformly degenerate operators, it indicates a special feature of these operators that makes tractable much of the analysis here.

A comprehensive study of uniformly degenerate elliptic operators on manifolds with boundary, and of the slightly more general class of edge operators, was undertaken in [32]. The emphasis in that paper is on mapping properties on weighted Sobolev and Hölder spaces, and on the construction of parametrices using the methods of geometric microlocal analysis. The focus is on identifying the precise behavior of the Schwartz kernels of the various operators (the resolvent operator, heat kernel, etc.) along the boundaries and near to the intersection of the diagonal with the boundaries. These techniques give very detailed information, but have the disadvantage that they require an elaborate formalism.

The geometric-microlocal methods and those employed in [11, 28] have been developed *only* for operators on manifolds with boundary, but not corners. It has proved challenging to generalize the microlocal approach to situations where the boundary is stratified, but see [33, 34]. It is likely that the amount of effort required to do this would be at least what has been done in the present monograph. Successful completion of such a program would give a precise description of the resolvent kernel itself, which would more than suffice to prove local regularity results. In this monograph we do not give precise descriptions of the resolvent or heat kernels, nor have we established local regularity results.

1.7 OUTLINE OF TEXT

The book is divided in three parts:

I. **Wright-Fisher Geometry and the Maximum Principle: Chapters 2.1-3.** Chapter 2.1 introduces the geometric preliminaries needed to analyze generalized Kimura diffusions. In Chapter 2.2 we show that coordinates

$$(x_1, \ldots, x_M; y_1, \ldots, y_{N-M})$$

can be introduced in the neighborhood of a boundary point of codimension M so that the boundary is locally given by $\{x_1 = \cdots = x_M = 0\}$ and the second order purely normal part of L takes the form

$$\sum_{j=1}^{M} x_i \partial_{x_i}^2. \tag{1.57}$$

This generalizes a 1-dimensional result in [20]. In Chapter 3 we prove maximum principles for the parabolic and elliptic equations,

$$(\partial_t - L)u = g \text{ and } (\mu - L)w = f, \text{ respectively,} \tag{1.58}$$

from which the uniqueness results follow easily. Of particular note is an analogue of the Hopf boundary point maximum principle, which allows very detailed analyses of the ker L and ker $\overline{L}^*$.

II. **Analysis of Model Problems: Chapters 4–9.** In Chapter 4 we introduce the model problems and the solution operator for the associated heat equations. These operators,

$$L_{\boldsymbol{b},m} = \sum_{j=1}^{m} [x_j \partial_{x_j}^2 + b_j \partial_{x_j}] + \sum_{l=1}^{m} \partial_{y_l}^2, \tag{1.59}$$

act on functions defined on $S_{n,m} = \mathbb{R}_+^n \times \mathbb{R}^m$, where $n+m = N$, and give a good approximation for the behavior of the heat kernel $(\partial_t - L)^{-1}$ in neighborhoods of different types of boundary points. We state and prove elementary features of these operators, that generalize results proved in [15], and show that the model heat operators have an analytic continuation to the right half plane:

$$H_+ = \{t : \operatorname{Re} t > 0\}. \tag{1.60}$$

In Chapter 5 we introduce the degenerate Hölder spaces on the spaces $S_{n,m}$, and their heat-space counterparts on $S_{n,m} \times [0, T]$. These are, in essence, Hölder spaces defined by the incomplete metric on $S_{n,m}$ given by

$$ds_{WF}^2 = \sum_{j=1}^{n} \frac{dx_i^2}{x_i} + \sum_{l=1}^{m} dy_l^2. \tag{1.61}$$

We also establish the basic properties of these spaces.

Chapters 6–9 are devoted to analyzing the heat and resolvent operators for the model problems acting on the Hölder spaces defined in Chapter 5. This is a very long and tedious process because many cases need to be considered and, in each case, many estimates are required. Conceptually, however, these results are elementary. The estimates are pointwise estimates done in Hölder spaces, which means one can vary a single variable at a time. As the model heat kernels are products of 1-dimensional heat kernels, this reduces essentially every question one might want to answer to one of proving estimates for the 1-dimensional kernels. We call this the *1-variable-at-a-time* method. In higher dimensions, the resolvent kernel, which is the Laplace transform of the heat kernel, is *not* a product of 1-dimensional kernels. This makes it far more difficult to deduce the mapping properties of the resolvent from its kernel, and explains why we use the representation as a Laplace transform.

The proof of the estimates on the 1-dimensional heat kernels, defined by the operators $x\partial_x^2 + b\partial_x$ are given in Appendix A. Analogous results for the Euclidean heat kernel are stated in Chapter 8. The proofs of these lemmas, which are elementary, are left to the reader. A notable feature of the estimates for the degenerate model problem is the fact that the constants remain uniformly bounded as $b \to 0$. This despite the fact that the character of the heat kernel changes quite dramatically at $b = 0$, see (1.29) and (1.31). This is also in sharp contrast to the analysis of similar problems in [11], where a positive lower bound is assumed for the coefficient of the analogous vector field.

III. **Analysis of Generalized Kimura Diffusions: Chapters 10–12.** This part of the book represents the culmination of all the work done up to this point. We consider a generalized Kimura diffusion operator L defined on a compact manifold with corners P. In Chapter 10 we prove the existence of solutions to the heat equation

$$(\partial_t - L)u = g \text{ in } P \times (0, T] \text{ with } u(p, 0) = f(p), \tag{1.62}$$

with data in $(g, f) \in \mathscr{C}^{k,\gamma}_{WF}(P \times [0, T]) \times \mathscr{C}^{k,2+\gamma}_{WF}(P)$. We show that the solution belongs to $\mathscr{C}^{k,2+\gamma}_{WF}(P \times [0, T])$ (Theorems 10.0.2 and 10.5.2). The case $g = 0$ provides a solution to the Cauchy problem, but it is not optimal as regards either the regularity of the solution, or the domain of the time variable, defects that are corrected in Chapter 11. The proof of these results is an intricate induction argument, where we induct over the maximal codimension of bP. This argument allows us to handle one stratum at a time. The underlying geometric fact is a “doubling theorem,” which shows that any neighborhood, *complementary* to the highest codimension stratum of bP, can be embedded into a manifold with corners $\widetilde{P}$ where the maximum codimension of $b\widetilde{P}$ is one less than that of bP, (Theorem 10.2.1). This explains why we need to consider domains well beyond those that can be easily embedded into Euclidean space.

We first treat the lowest differentiability case ($k = 0$) and then use an extension of the contraction mapping theorem to towers of Banach spaces (Theorem 10.8.1), to obtain the mapping results for $k > 0$. These results (even in the $k = 0$ case) suffice to prove that the graph closure of L acting on $\mathscr{C}^3(P)$ is the generator of strongly continuous semi-group in $\mathscr{C}^0(P)$.

We next consider the operators L_γ, defined as L acting on the domain $\mathscr{C}^{0,2+\gamma}_{WF}(P)$. As a map from $\mathscr{C}^{0,2+\gamma}_{WF}(P)$ to $\mathscr{C}^{0,\gamma}_{WF}(P)$, L_γ is a Fredholm operator of index 0. In

Chapter 11 we use essentially the same parametrix construction as used to prove Theorem 10.0.2 to prove the existence of the resolvent operator

$$(\mu - L_{\gamma})^{-1} : \mathcal{C}^{k,\gamma}_{WF}(P) \longrightarrow \mathcal{C}^{k,2+\gamma}_{WF}(P),$$

for μ in the complement of discrete set lying in a conic neighborhood of $(-\infty, 0]$. These are the expected "elliptic" estimates for operators with this type of degeneracy. In fact the spectrum of L acting on $\mathcal{C}^{k,\gamma}_{WF}(P)$ does not depend on k or γ, as the resolvent is compact and all eigenfunctions belong to $\mathcal{C}^{\infty}(P)$. Using the analyticity properties of the resolvent, we give an alternate construction, using a contour integral, for the semi-group, acting on $\mathcal{C}^{0,\gamma}_{WF}(P)$:

$$e^{tL} = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{t\mu}(\mu - L)^{-1} d\mu, \tag{1.63}$$

where Γ_α bounds a region of the form $|\arg \mu| > \pi - \alpha$, and $|\mu| > R_\alpha$, (Theorem 11.2.1). As we can take α to be any positive number, this shows that the semi-group is holomorphic in the right half plane.

Finally in Chapter 12 we give a good description of the null-space of L_γ, and show that the non-zero spectrum of L_γ lies in a half plane $\operatorname{Re} \mu < \eta < 0$. We also deduce various properties of the semi-group, defined by the graph closure, $\overline{L}$, of L, acting on $\mathcal{C}^0(P)$. The adjoint operator, $\overline{L}^*$, is defined on a domain $\operatorname{Dom}(\overline{L}^*) \subset \mathcal{M}(P)$, which is not dense. We describe the subspace $\mathcal{M}^{\odot}(P)$, defined by Lumer and Phillips, on which $\overline{L}^*$ defines a C_0-semi-group. Although we have not yet proved the compactness of the resolvent of $\overline{L}$, we obtain a rather complete description of the null-space of $\overline{L}^*$. Using this we give the long time asymptotics for $e^{tL} f$, assuming that $f \in \mathcal{C}^{0,\gamma}_{WF}(P)$, for any $0 < \gamma$, as well as those for $e^{t\overline{L}^*}\nu$, for ν in $\mathcal{M}^{\odot}(P)$. Finally we consider the problem of finding "irregular" solutions to the equation $Lw = f$, when the maximum principle precludes the existence of a regular solution.

IV. **Proof of 1-dimensional Estimates: Appendix A.** In the Appendix we give careful proofs of the estimates for the degenerate, 1-dimensional heat kernels used in the perturbation theory. These arguments are complicated by the fact that the heat kernel displays both the additive and multiplicative group structures on $\mathbb{R}_+$:

$$k^b_t(x, y)dy = \left(\frac{y}{t}\right)^b e^{-\frac{x+y}{t}} \psi_b\left(\frac{xy}{t^2}\right) \frac{dy}{y}. \tag{1.64}$$

The arguments involve Taylor's theorem, the asymptotic expansion of the heat kernel where $\frac{|\sqrt{x}-\sqrt{y}|}{\sqrt{t}}$ tends to infinity and Laplace's method. We obtain mapping properties for $b > 0$, with uniform constants as b tends to zero. Using compactness of the embeddings, $\mathcal{C}^{k,\gamma}_{WF} \hookrightarrow \mathcal{C}^{k,\widetilde{\gamma}}_{WF}$, for $\widetilde{\gamma} < \gamma$, e.g., we then extend these results to $b = 0$.

1.8 NOTATIONAL CONVENTIONS

- We use $\mathbb{R}_+ = [0, \infty)$, hence $\mathscr{C}_c^\infty(\mathbb{R}_+)$ consists of smooth functions on $\mathbb{R}_+$, supported in finite intervals $[0, R]$.
- The right half plane is denoted

$$H_+ = \{t \in \mathbb{C} : \operatorname{Re} t > 0\}. \tag{1.65}$$

- We let

$$S_{n,m} = \mathbb{R}_+^n \times \mathbb{R}^m. \tag{1.66}$$

- For $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}_+^n$, the model operator is:

$$L_{\boldsymbol{b},m} = \sum_{j=1}^{m} [x_j \partial_{x_j}^2 + b_j \partial_{x_j}] + \sum_{l=1}^{m} \partial_{y_l}^2. \tag{1.67}$$

- If $a(t)$ is an operator valued function, acting on functions in a space $\mathscr{X}$, and $g : [0, T] \to \mathscr{X}$ is measurable, then the notation $A^t g$ usually means

$$A^t g = \int_0^t a(t-s) g(s) ds. \tag{1.68}$$

- For $\phi \in [0, \frac{\pi}{2})$ we define the sector

$$S_\phi = \{t \in \mathbb{C} : |\arg t| < \frac{\pi}{2} - \phi\}. \tag{1.69}$$

  We also let $S_{\frac{\pi}{2}} = [0, \infty)$.
- For $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}_+^n$ we let

$$\rho_s(\boldsymbol{x}, \boldsymbol{x}') = \sum_{j=1}^{n} |\sqrt{x_j} - \sqrt{x_j'}|; \tag{1.70}$$

  for $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{R}^m$ we let

$$\rho_e(\boldsymbol{y}, \boldsymbol{y}') = \sum_{k=1}^{m} |y_k - y_k'|. \tag{1.71}$$

  We then let

$$\rho((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) = \rho_s(\boldsymbol{x}, \boldsymbol{x}') + \rho_e(\boldsymbol{y}, \boldsymbol{y}'). \tag{1.72}$$

  We also use

$$d_{WF}((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) = \rho((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')). \tag{1.73}$$

  When there is also a time variable,

$$d_{WF}((\boldsymbol{x}, \boldsymbol{y}, t), (\boldsymbol{x}', \boldsymbol{y}', t')) = \rho((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) + \sqrt{|t - t'|}. \tag{1.74}$$

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in $\mathbb{R}^n$, then $\boldsymbol{a} < \boldsymbol{b}$, or $\boldsymbol{a} \leq \boldsymbol{b}$ means that

$$a_j < b_j (\text{ or } a_j \leq b_j) \text{ for } j = 1, \ldots, n. \tag{1.75}$$

  We let $\mathbf{0} = (0, \ldots, 0)$, and $\mathbf{1} = (1, \ldots, 1)$; the dimension will be clear from the context.