# Chapter One

## Selected Theorems by Eli Stein

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### INTRODUCTION

The purpose of this chapter is to give the general reader some idea of the scope and originality of Eli Stein's contributions to analysis*. His work deals with representation theory, classical Fourier analysis, and partial differential equations. He was the first to appreciate the interplay among these subjects, and to preceive the fundamental insights in each field arising from that interplay. No one else really understands all three fields; therefore, no one else could have done the work I am about to describe. However, deep understanding of three fields of mathematics is by no means sufficient to lead to Stein's main ideas. Rather, at crucial points, Stein has shown extraordinary originality, without which no amount of work or knowledge could have succeeded. Also, large parts of Stein's work (e.g., the fundamental papers [26, 38, 41, 59] on complex analysis in tube domains) don't fit any simple one-paragraph description such as the one above.

It follows that no single mathematician is competent to present an adequate survey of Stein's work. As I attempt the task, I am keenly aware that many of Stein's papers are incomprehensible to me, while others were of critical importance to my own work. Inevitably, therefore, my survey is biased, as any reader will see. Fortunately, S. Gindikin provided me with a layman's explanation of Stein's contributions to representation theory, thus keeping the bias (I hope) within reason. I am grateful to Gindikin for his help, and also to Y. Sagher for a valuable suggestion.

For purposes of this chapter, representation theory deals with the construction and classification of the irreducible unitary representations of a semisimple Lie group. Classical Fourier analysis starts with the $L^p$-boundedness of two fundamental operators, the maximal function.

$$f^*(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| dy,$$

and the Hilbert transform

$$Hf(x) = \lim_{\epsilon \to 0+} \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{f(x)dy}{x-y}.$$

Finally, we shall be concerned with those problems in partial differential equations that come from several complex variables.

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## COMPLEX INTERPOLATION

Let us begin with Stein's work on interpolation of operators. As background, we state and prove a clasical result, namely the

**M. Riesz Convexity Theorem**. *Suppose X, Y are measure spaces, and suppose T is an operator that carries functions on X to functions on Y. Assume T is bounded from $L^{p_0}(X)$ to $L^{r_0}(Y)$, and from $L^{p_1}(X)$ to $L^{r_1}(Y)$. (Here, $p_0, p_1, r_0, r_1 \in [1, \infty]$.) Then T is bounded from $L^p(X)$ to $L^r(Y)$ for $\frac{1}{p} = \frac{t}{p_1} + \frac{(1-t)}{p_0}$, $\frac{1}{r} = \frac{t}{r_1} + \frac{(1-t)}{r_0}$, $0 \leq t \leq 1$.*

The Riesz Convexity Theorem says that the points $(\frac{1}{p}, \frac{1}{r})$ for which $T$ is bounded from $L^p$ to $L^r$ from a convex region in the plane. A standard application is the Hausdorff-Young inequality: We take T to be the Fourier transform on $\mathbb{R}^n$, and note that $T$ is obviously bounded from $L^1$ to $L^\infty$, and from $L^2$ to $L^2$. Therefore, $T$ is bounded from $L^p$ to the dual class $L^{p'}$ for $1 \leq p \leq 2$.

The idea of the proof of the Riesz Convexity Theorem is to estimate $\int_Y (Tf) \cdot g$ for $f \in L^p$ and $g \in L^{r'}$. Say $f = Fe^{i\phi}$ and $g = Ge^{i\psi}$ with $F, G \geq 0$ and $\phi, \psi$ real. Then we can define analytic families of functions $f_z, g_z$ by setting $f_z = F^{az+b}e^{i\phi}$, $g_z = G^{cz+d}e^{i\psi}$, for real $a, b, c, d$ to be picked in a moment.

Define

$$\Phi(z) = \int_Y (Tf_z)g_z. \tag{1}$$

Evidently, $\Phi$ is an analytic function of $z$.

For the correct choice of $a, b, c, d$ we have

$$|f_z|^{p_0} = |f|^p \quad \text{and} \quad |g_z|^{r_0'} = |g|^{r'} \quad \text{when} \quad \text{Re } z = 0; \tag{2}$$

$$|f_z|^{p_1} = |f|^p \quad \text{and} \quad |g_z|^{r_1'} = |g|^{r'} \quad \text{when} \quad \text{Re } z = 1; \tag{3}$$

$$f_z = f \quad \text{and} \quad g_z = g \quad \text{when} \quad z = t. \tag{4}$$

From (2) we see that $\|f_z\|_{L^{p_0}}, \|g_z\|_{L^{r'}_0} \leq C$ for Re $z = 0$. So the definition (1) and the assumption $T : L^{p_0} \to L^{r_0}$ show that

$$|\Phi(z)| \leq C' \quad \text{for} \quad \text{Re } z = 0. \tag{5}$$

Similarly, (3) and the assumption $T : L^{p_1} \to L^{r_1}$ imply

$$|\Phi(z)| \leq C' \quad \text{for} \quad \text{Re } z = 1. \tag{6}$$

Since $\Phi$ is analytic, (5) and (6) imply $|\Phi(z)| \leq C'$ for $0 \leq \text{Re } z \leq 1$ by the maximum principle for a strip. In particular, $|\Phi(t)| \leq C'$. In view of (4), this means that $|\int_Y (Tf)g| \leq C'$, with $C'$ determined by $\|f\|_{L^p}$ and $\|g\|_{L^{r'}}$.

Thus, $T$ is bounded from $L^p$ to $L^r$, and the proof of the Riesz Convexity Theorem is complete.

This proof had been well-known for over a decade, when Stein discovered an amazingly simple way to extend its usefulness by an order of magnitude. He realized that an ingenious argument by Hirschman [H] on certain multiplier operators on $L^p(\mathbb{R}^n)$ could be viewed as a Riesz Convexity Theorem for analytic families of operators. Here is the result.

**Stein Interpolation Theorem**. *Assume $T_z$ is an operator depending analytically on $z$ in the strip $0 \leq \text{Re } z \leq 1$. Suppose $T_z$ is bounded from $L^{p_0}$ to $L^{r_0}$ when* Re $z = 0$, *and from $L^{p_1}$ to $L^{r_1}$ when* Re $z = 1$. *Then $T_t$ is bounded from $L^p$ to $L^r$, where $\frac{1}{p} = \frac{t}{p_1} + \frac{(1-t)}{p_o}$, $\frac{1}{r} = \frac{t}{r_1} + \frac{(1-t)}{r_0}$ and $0 \leq t \leq 1$.*

Remarkably, the proof of the theorem comes from that of the Riesz Convexity Theorem by adding a single letter of the alphabet. Instead of taking $\Phi(z) = \int_y (Tf_z)g_z$ as in (1), we set $\Phi(z) = \int_Y (T_z f_z)g_z$. The proof of the Riesz Convexity Theorem then applies with no further changes.

Stein's Interpolation Theorem is an essential tool that permeates modern Fourier analysis. Let me just give a single application here, to illustrate what it can do. The example concerns Cesaro summability of multiple Fourier integrals.

We define an operator $T_{\alpha R}$ on functions on $\mathbb{R}^n$ by setting

$$\widehat{T_{\alpha R} f}(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\alpha \hat{f}(\xi).$$

Then

$$\|T_{\alpha R} f\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha p} \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{if} \quad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\alpha}{n-1}. \tag{7}$$

This follows immediately from the Stein Interpolation Theorem. We let $\alpha$ play the role of the complex parameter $z$, and we interpolate between the elementary cases $p = 1$ and $p = 2$. Inequality (7), due to Stein, was the first non-trivial progress on spherical summation of multiple Fourier series.

## REPRESENTATION THEORY 1

Our next topic is the Kunze-Stein phenomenon, which links the Stein Interpolation Theorem to representations of Lie groups. For simplicity we restrict attention to $G = SL(2, \mathbb{R})$, and begin by reviewing elementary Fourier analysis on $G$. The irreducible unitary representations of $G$ are as follows:

The *principal series*, parametrized by a sign $\sigma = \pm 1$ and a real parameter $t$;
The *discrete series*, parametrized by a sign $\sigma = \pm 1$ and an integer $k \geq 0$; and
The *complementary series*, parametrized by a real number $t \in (0, 1)$.

We don't need the full description of these representations here.

The irreducible representations of $G$ give rise to a Fourier transform. If $f$ is a function on $G$, and $U$ is an irreducible unitary representation of $G$, then we define

$$\hat{f}(U) = \int_G f(g) U_g dg,$$

where $dg$ denotes Haar measure on the group. Thus, $\hat{f}$ is an operator-valued function defined on the set of irreducible unitary representations of $G$. As in the Euclidean case, we can analyze convolutions in terms of the Fourier transform. In fact,

$$\widehat{f * g} = \hat{f} \cdot \hat{g} \tag{8}$$

as operators. Moreover, there is a Plancherel formula for $G$, which asserts that

$$\|f\|^2_{L^2(G)} = \int \|\hat{f}(U)\|^2_{\text{Hilbert-Schmidt}} d\mu(U)$$

for a measure $\mu$ (the Plancherel measure). The Plancherel measure for $G$ is known, but we don't need it here. However, we note that the complementary series has measure zero for the Plancherel measure.

These are, of course, the analogues of familiar results in the elementary Fourier analysis of $\mathbb{R}^n$. Kunze and Stein discovered a fundamental new phenomenon in Fourier analysis on $G$ that has no analogue on $\mathbb{R}^n$. Their result is as follows.

**Theorem (Kunze-Stein Phenomenon).** *There exists a uniformly bounded representation $U_{\sigma,\tau}$ of $G$, parametrized by a sign $\sigma = \pm 1$ and a complex number $\tau$ in a strip $\Omega$, with the following properties.*

(A) *The $U_{\sigma,\tau}$ all act on the same Hilbert space $H$.*
(B) *For fixed $\sigma = \pm 1$, $g \in G$, and $\xi, \eta \in H$, the matrix element $\langle (U_{\sigma,\tau})_g \xi, \eta\rangle$ is an analytic function of $\tau \in \Omega$.*
(C) *The $U_{\sigma,\tau}$ for $\mathrm{Re}\,\tau = \frac{1}{2}$ are equivalent to the representations of the principal series.*
(D) *The $U_{+1,\tau}$ for suitable $\tau$ are equivalent to the representations of the complementary series.*

(See [14] for the precise statement and proof, as well as Ehrenpreis-Mautner [EM] for related results.)

The Kunze-Stein Theorem suggests that analysis on $G$ resembles a fictional version of classical Fourier analysis in which the basic exponential $\xi \longmapsto \exp(i\xi \cdot x)$ is a bounded analytic function on strip $|\mathrm{Im}\,\xi| \le C$, uniformly for all $x$.

As an immediate consequence of the Kunze-Stein Theorem, we can give an analytic continuation of the Fourier transform for $G$. In fact, we set $\hat{f}(\sigma,\tau) = \int_G f(g)(U_{\sigma,\tau})_g dg$ for $\sigma = \pm 1$, $\tau \in \Omega$.

Thus, $f \in L^1(G)$ implies $\hat{f}(\sigma, \cdot)$ analytic and bounded on $\Omega$. So we have continued analytically the restriction of $\hat{f}$ to the principal series. It is as if the Fourier transform of an $L^1$ function on $(-\infty, \infty)$ were automatically analytic in a strip. If $f \in L^2(G)$, then $\hat{f}(\sigma,\tau)$ is still defined on the line $\{\mathrm{Re}\,\tau = \frac{1}{2}\}$, by virtue of the Plancherel formula and part (C) of the Kunze-Stein Theorem. Interpolating between $L^1(G)$ and $L^2(G)$ using the Stein Interpolation Theorem, we see that $f \in L^p(G)(1 \le p < 2)$ implies $\hat{f}(\sigma, \cdot)$ analytic and satisfying an $L^{p'}$-inequality on a strip $\Omega_p$. As $p$ increases from 1 to 2, the strip $\Omega_p$ shrinks from $\Omega$ to the line $\{\mathrm{Re}\,\tau = \frac{1}{2}\}$. Thus we obtain the following results.

**Corollary 1.** *If $f \in L^p(G)(1 \le p < 2)$, then $\hat{f}$ is bounded almost everywhere with respect to the Plancherel measure.*

**Corollary 2.** *For $1 \le p < 2$ we have the convolution inequality $\|f * g\|_{L^2(G)} \le C_p \|f\|_{L^p(G)} \|g\|_{L^2(G)}$.*

To check Corollary 1, we look separately at the principal series, the discrete series, and the complementary series. For the principal series, we use the $L^{p'}$-inequality established above for the analytic function $\tau \longmapsto \hat{f}(\sigma, \tau)$ on the strip $\Omega_p$. Since an $L^{p'}$-function analytic on a strip $\Omega_p$ is clearly bounded on an interior line $\{\text{Re}\ \tau = \frac{1}{2}\}$, it follows at once that $\hat{f}$ is bounded on the principal series. Regarding the discrete series $U_{\sigma,k}$ we note that

$$\left( \sum_{\sigma,k} \mu_{\sigma,k} \|\hat{f}(U_{\sigma,k})\|^{p'} \right)^{1/p'} \le \|f\|_{L^p(G)} \tag{9}$$

for suitable weights $\mu_{\sigma,k}$ and for $1 \le p \le 2$. The weights $\mu_{\sigma,k}$ amount to the Plancherel measure on the discrete series, and (9) is proved by a trivial interpolation, just like the standard Hausdorff-Young inequality. The boundedness of the $\|\hat{f}(U_{\sigma,k})\|$ is immediate from (9). Thus the Fourier transform $\hat{f}$ is bounded on both the principal series and the discrete series, for $f \in L^p(G)$ $(1 \le p < 2)$. The complementary series has measure zero with respect to the Plancherel measure, so the proof of Corollary 1 is complete. Corollary 2 follows trivially from Corollary 1, the Plancherel formula, and the elementary formula (8).

This proof of Corollary 2 poses a significant challenge. Presumably, the corollary holds because the geometry of $G$ at infinity is so different from that of Euclidean space. For example, the volume of the ball of radius $R$ in $G$ grows exponentially as $R \to \infty$. This must have a profound impact on the way mass piles up when we take convolutions on $G$. On the other hand, the statement of Corollary 2 clearly has nothing to do with cancellation; proving the corollary for two arbitrary functions $f$, $g$ is the same as proving it for $|f|$ and $|g|$. When we go back over the proof of Corollary 2, we see cancellation used crucially, e.g., in the Plancherel formula for $G$; but there is no explicit mention of the geometry of $G$ at infinity. Clearly there is still much that we do not understand regarding convolutions on $G$.

The Kunze-Stein phenomenon carries over to other semisimple groups, with profound consequences for representation theory. We will continue this discussion later in the chapter. Now, however, we turn our attention to classical Fourier analysis.

## CURVATURE AND THE FOURIER TRANSFORM

One of the most fascinating themes in Fourier analysis in the last two decades has been the connection between the Fourier transform and curvature. Stein has been the most important contributor to this set of ideas. To illustrate, I will pick out two of his results. The first is a "restriction theorem," i.e., a result on the restriction $\hat{f}|_\Gamma$ of the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$ to a set $\Gamma$ of measure zero. If $p > 1$, then the standard inequality $\hat{f} \in L^{p'}(\mathbb{R}^n)$ suggests that $\hat{f}$ should not even be well-defined on $\Gamma$, since $\Gamma$ has measure zero. Indeed, if $\Gamma$ is (say) the $x$-axis in the plane $\mathbb{R}^2$, then we can easily find functions $f(x_1, x_2) = \varphi(x_1)\psi(x_2) \in L^p(\mathbb{R}^2)$ for which $\hat{f}|_\Gamma$ is infinite everywhere. Fourier transforms of $f \in L^p(\mathbb{R}^2)$ clearly cannot be restricted to straight lines. Stein proved that the situation changes drastically when $\Gamma$ is curved. His result is as follows.

**Stein's Restriction Theorem**. *Suppose $\Gamma$ is the unit circle, $1 \le p < \frac{8}{7}$, and $f \in C_0^\infty(\mathbb{R}^2)$. Then we have the a priori inequality $\|\hat{f}|_\Gamma\|_{L^2} \le C_p \|f\|_{L^p(\mathbb{R}^2)}$, with $C_p$ depending only on p.*

Using this a priori inequality, we can trivially pass from the dense subspace $C_0^\infty$ to define the operator $f \longmapsto \hat{f}|_\Gamma$ for all $f \in L^p(\mathbb{R}^2)$. Thus, the Fourier transform of $f \in L^p (p < \frac{8}{7})$ may be restricted to the unit circle.

Improvements and generalizations were soon proven by other analysts, but it was Stein who first demonstrated the phenomenon of restriction of Fourier transforms.

Stein's proof of his restriction theorem is amazingly simple. If $\mu$ denotes uniform measure on the circle $\Gamma \subset \mathbb{R}^2$, then for $f \in C_0^\infty(\mathbb{R}^2)$ we have

$$\int_\Gamma |\hat{f}|^2 = \int_{\mathbb{R}^2} (\hat{f}\mu)\overline{(\hat{f})} = \langle f * \hat{\mu}, f\rangle \le \|f\|_{L^p} \|f * \hat{\mu}\|_{L^{p'}}. \tag{10}$$

The Fourier transform $\hat{\mu}(\xi)$ is a Bessel function. It decays like $|\xi|^{-1/2}$ at infinity, a fact intimately connected with the curvature of the circle. In particular, $\hat{\mu} \in L^q$ for $4 < q \le \infty$, and therefore $\|f * \hat{\mu}\|_{L^{p'}} \le C_p \|f\|_{L^p}$ for $1 \le p < \frac{8}{7}$, by the usual elementary estimates for convolutions. Putting this estimate back into (10), we see that $\int_\Gamma |\hat{f}|^2 \le C_p \|f\|^2_{L^p}$, which proves Stein's Restriction Theorem. The Stein Restriction Theorem means a lot to me personally, and has strongly influenced my own work in Fourier analysis.

The second result of Stein's relating the Fourier transform to curvature concerns the differentiation of integrals on $\mathbb{R}^n$.

**Theorem**. *Suppose $f \in L^p(\mathbb{R}^n)$ with $n \ge 3$ and $p > \frac{n}{n-1}$. For $x \in \mathbb{R}^n$ and $r > 0$, let F(x, r) denote the average of f on the sphere of radius r centered at x. Then* $\lim_{r\to 0} F(x, r) = f(x)$ *almost everywhere.*

The point is that unlike the standard Lebesgue Theorem, we are averaging $f$ over a small sphere instead of a small ball. As in the restriction theorem, we are seemingly in trouble because the sphere has measure zero in $\mathbb{R}^n$, but the curvature of the sphere saves the day. This theorem is obviously closely connected to the smoothness of solutions of the wave equation.

The proof of the above differentiation theorem relies on an

**Elementary Tauberian Theorem**. *Suppose that* $\lim_{R\to 0} \frac{1}{R}\int_0^R F(r)dr$ *exists and* $\int_0^\infty r|\frac{dF}{dr}|^2 dr < \infty$. *Then* $\lim_{R\to 0} F(R)$ *exists, and equals* $\lim_{R\to 0} \frac{1}{R}\int_0^R F(r)dr$.

This result had long been used, e.g., to pass from Cesaro averages of Fourier series to partial sums. (See Zygmund [Z].) On more than one occasion, Stein has shown the surprising power hidden in the elementary Tauberian Theorem. Here we apply it to $F(x, r)$ for a fixed $x$. In fact, we have $F(x, r) = \int f(x + ry)d\mu(y)$, with $\mu$ equal to normalized surface measure on the unit sphere, so that the Fourier transforms of $F$ and $f$ are related by $\hat{F}(\xi, r) = \hat{f}(\xi)\hat{\mu}(r\xi)$ for each fixed $r$.

Therefore, assuming $f \in L^2$ for simplicity, we obtain

$$\int_{\mathbb{R}^n} \left( \int_0^\infty r \left| \frac{\partial}{\partial r} F(x, r) \right|^2 dr \right) dx = \int_0^\infty r \left[ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial r} F(x, r) \right|^2 dx \right] dr$$

$$= \int_0^\infty r \left[ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial r} \hat{F}(\xi, r) \right|^2 d\xi \right] dr = \int_{\mathbb{R}^n} \int_0^\infty r \left| \frac{\partial}{\partial r} \hat{\mu}(r\xi) \right|^2 |\hat{f}(\xi)|^2 dr d\xi$$

$$= \int_{\mathbb{R}^n} \left\{ \int_0^\infty r \left| \frac{\partial}{\partial r} \hat{\mu}(r\xi) \right|^2 dr \right\} |\hat{f}(\xi)|^2 d\xi = (\text{const.}) \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi < \infty.$$

(Here we make crucial use of curvature, which causes $\hat{\mu}$ to decay at infinity, so that the integral in curly brackets converges.) It follows that $\int_0^\infty r|\frac{\partial}{\partial r} F(x, r)|^2 dr < \infty$ for almost every $x \in \mathbb{R}^n$. On the other hand, $\frac{1}{R} \int_0^R F(x, r)dr$ is easily seen to be the convolution of $f$ with a standard approximate identity. Hence the usual Lebesgue differentiation theorem shows that $\lim_{R \to 0} \frac{1}{R} \int_0^R F(x, r)dr = f(x)$ for almost every $x$.

So for almost all $x \in \mathbb{R}^n$, the function $F(x, r)$ satisfies the hypotheses of the elementary Tauberian theorem. Consequently,

$$\lim_{r \to 0} F(x, r) = \lim_{R \to 0} \frac{1}{R} \int_0^R F(x, r)dr = f(x)$$

almost everywhere, proving Stein's differentiation theorem for $f \in L^2(\mathbb{R}^n)$.

To prove the full result for $f \in L^p(\mathbb{R}^n)$, $p > \frac{n}{n-1}$, we repeat the above argument with surface measure $\mu$ replaced by an even more singular distribution on $\mathbb{R}^n$. Thus we obtain a stronger conclusion than asserted, when $f \in L^2$. On the other hand, for $f \in L^{1+\varepsilon}$ we have a weaker result than that of Stein, namely Lebesgue's differentiation theorem. Interpolating between $L^2$ and $L^{1+\varepsilon}$, one obtains the Stein differentiation theorem.

The two results we picked out here are only a sample of the work of Stein and others on curvature and the Fourier transform. For instance, J. Bourgain has dramatic results on both the restriction problem and spherical averages.

We refer the reader to Stein's address at the Berkeley congress [128] for a survey of the field.

## $H^p$-SPACES

Another essential part of Fourier analysis is the theory of $H^p$-spaces. Stein transformed the subject twice, once in a joint paper with Guido Weiss, and again in a joint paper with me. Let us start by recalling how the subject looked before Stein's work. The classical theory deals with analytic functions $F(z)$ on the unit disc. Recall that $F$ belongs to $H^p$ $(0 < p < \infty)$ if the norm $\|F\|_{H^p} \equiv \lim_{r\to 1-} (\int_0^{2\pi} |F(re^{i\theta})|^p d\theta)^{1/p}$ is finite.

The classical $H^p$-spaces serve two main purposes. First, they provide growth conditions under which an analytic function tends to boundary values on the unit circle. Secondly, $H^p$ serves as a substitute for $L^p$ to allow basic theorems on Fourier series to extend from $1 < p < \infty$ to all $p > 0$. To prove theorems about $F \in H^p$, the main tool is the Blaschke product

$$B(z) = \Pi_v e^{i\theta v} \frac{z_v - z}{1 - \bar{z}_v z}, \tag{11}$$

where $\{z_v\}$ are the zeroes of the analytic function $F$ in the disc, and $\theta_v$ are suitable phases. The point is that $B(z)$ has the same zeroes as $F$, yet it has absolute value 1 on the unit circle. We illustrate the role of the Blaschke product by sketching the proof of the Hardy-Littlewood maximal theorem for $H^p$. The maximal theorem says that $\|F^*\|_{L^p} \leq C_p \|F\|_{H^p}$ for $0 < p < \infty$, where $F^*(\theta) = \sup_{z\in\Gamma(\theta)} |F(z)|$, and $\Gamma(\theta)$ is the convex hull of $e^{i\theta}$ and the circle of radius $\frac{1}{2}$ about the origin.

This basic result is closely connected to the pointwise convergence of $F(z)$ as $z \in \Gamma(\theta)$ tends to $e^{i\theta}$. To prove the maximal theorem, we argue as follows.

First suppose $p > 1$. Then we don't need analyticity of $F$. We can merely assume that $F$ is harmonic, and deduce the maximal theorem from real variables. In fact, it is easy to show that $F$ arises as the Poisson integral of an $L^p$ function $f$ on the unit circle. The maximal theorem for $f$, a standard theorem of real variables, says that $\|Mf\|_{L^p} \leq C_p \|f\|_{L^p}$, where $Mf(\theta) = \sup_{h>0}(\frac{1}{2h}\int_{\theta-h}^{\theta+h} |f(t)|dt)$. It is quite simple to show that $F^*(\theta) < CMf(\theta)$. Therefore $\|F^*\|_{L^p} \leq C\|Mf\|_{L^p} < C'\|F\|_{H^p}$, and the maximal theorem is proven for $H^p \quad (p > 1)$.

If $p \leq 1$, then the problem is more subtle, and we need to use analyticity of $F(z)$. Assume for a moment that $F$ has no zeroes in the unit disc. Then for $0 < q < p$, we can define a single-valued branch of $(F(z))^q$, which will belong to $H^{p/q}$ since $F \in H^p$. Since $\tilde{p} \equiv \frac{p}{q} > 1$, the maximal theorem for $H^{\tilde{p}}$ is already known. Hence, $\max_{z\in\Gamma(\theta)} |(F(z))^q| \in L^{p/q}$, with norm

$$\int_{-\pi}^{\pi} \left( \max_{z\in\Gamma(\theta)} |(F(z))^q| \right)^{p/q} d\theta \leq C_{p,q} \|F^q\|_{H^{p/q}}^{p/q} = C_{p,q} \|F\|_{H^p}^{p}.$$

That is, $\|F^*\|_{L^p} \leq C_{p,q}\|F\|_{H^p}$, proving the maximal theorem for functions without zeroes.

To finish the proof, we must deal with the zeroes of an $F \in H^p$ $(p \leq 1)$. We bring in the Blaschke product $B(z)$, as in (11). Since $B(z)$ and $F(z)$ have the same zeroes and since $|B(z)| = 1$ on the unit circle, we can write $F(z) = G(z)B(z)$ with $G$ analytic, and $|G(z)| = |F(z)|$ on the unit circle. Thus, $\|G\|_{H^p} = \|F\|_{H^p}$. Inside the circle, $G$ has no zeroes and $|B(z)| \leq 1$. Hence $|F| \leq |G|$, so

$$\| \max_{z\in\Gamma(\theta)} |F(z)| \|_{L^p} \leq \| \max_{z\in\Gamma(\theta)} |G(z)| \|_{L^p} \leq C_p \|G\|_{H^p} = C_p \|F\|_{H^p},$$

by the maximal theorem for functions withour zeroes. The proof of the maximal theorem is complete. (We have glossed over difficulties that should not enter an expository paper.)

Classically, $H^p$ theory works only in one complex variable, so it is useful only for Fourier analysis in one real variable. Attempts to generalize $H^p$ to several complex variables ran into a lot of trouble, because the zeroes of an analytic function $F(z_1, \dots z_n) \in H^p$ form a variety $V$ with growth conditions. Certainly $V$ is much more complicated than the discrete set of zeroes $\{z_v\}$ in the disc. There is no satisfactory substitute for the Blaschke product. For a long time, this blocked all attempts to extend the deeper properties of $H^p$ to several variables.

Stein and Weiss [13] realized that several complex variables was the wrong generalization of $H^p$ for purposes of Fourier analysis. They kept clearly in mind what $H^p$-spaces are supposed to do, and they kept an unprejudiced view of how to achieve it. They found a version of $H^p$ theory that works in several variables.

The idea of Stein and Weiss was very simple. They viewed the real and imaginary parts of an analytic function on the disc as the gradient of a harmonic function. In several variables, the gradient of a harmonic function is a system $\vec{u} = (u_1, u_2, \dots, u_n)$ of functions on $\mathbb{R}^n$ that satisfies the Stein-Weiss Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad \sum_k \frac{\partial u_k}{\partial x_k} = 0. \tag{12}$$

In place of the Blaschke product, Stein and Weiss used the following simple observation. If $\vec{u} = (u_1, \dots, u_n)$ satisfies (12), then $|\vec{u}|^p = (u_1^2 + u_2^2 + \cdots + u_n^2)^{p/2}$ is subharmonic for $p > \frac{n-2}{n-1}$. We sketch the simple proof of this fact, then explain how an $H^p$ theory can be founded on it.

To see that $|\vec{u}|^p$ is subharmonic, we first suppose $|\vec{u}| \neq 0$ and calculate $\Delta(|\vec{u}|^p)$ in coordinates that diagonalize the symmetric matrix $(\frac{\partial u_j}{\partial x_k})$ at a given point. The result is

$$\Delta(|\vec{u}|^p) = p|\vec{u}|^{p-2}\{|\vec{w}|^2|\vec{u}|^2 - (2-p)|\vec{v}|^2\}, \tag{13}$$

with $w_k = \frac{\partial u_k}{\partial x_k}$ and $v_k = u_k w_k$.

Since $\Sigma_{k=1}^n w_k = 0$ by the Cauchy-Riemann equations, we have

$$|w_k|^2 = \left|\sum_{j \neq k} wj\right|^2 \leq (n-1)\sum_{j \neq k}|w_j|^2 = (n-1)|\vec{w}|^2 - (n-1)|w_k|^2,$$

i.e., $|w_k|^2 \leq \frac{n-1}{n}|\vec{w}|^2$. Hence $|\vec{v}|^2 \leq (\max_k |w_k|^2)|\vec{u}|^2 \leq (\frac{n-1}{n})|\vec{w}|^2|\vec{u}|^2$, so the expression in curly brackets in (13) is non-negative for $p \geq \frac{n-2}{n-1}$, and $|\vec{u}|^p$ is subharmonic.

So far, we know that $|\vec{u}|^p$ is subharmonic where it isn't equal to zero. Hence for $0 < r < r(x)$ we have

$$|\vec{u}(x)|^p \leq Av_{|y-x|=r}|\vec{u}(y)|^p, \tag{14}$$

provided $|\vec{u}(x)| \neq 0$. However, (14) is obvious when $|\vec{u}(x)| = 0$, so it holds for any $x$. That is, $|\vec{u}|^p$ is a subharmonic function for $p \geq \frac{n-2}{n-1}$, as asserted.

Now let us see how to build an $H^p$ theory for Cauchy-Riemann systems, based on subharmonicity of $|\vec{u}|^p$. To study functions on $\mathbb{R}^{n-1} (n \geq 2)$, we regard $\mathbb{R}^{n-1}$ as the boundary of $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) | x_n > 0\}$, and we define $H^p(\mathbb{R}^n_+)$ as the space of all Cauchy-Riemann systems $(u_1, u_2, \ldots, u_n)$ for which the norm

$$\|\vec{u}\|^p_{H^p} = \sup_{t>0} \int_{\mathbb{R}^{n-1}} |\vec{u}(x_1, \ldots, x_{n-1}, t)|^p dx_1 \cdots dx_{n-1}$$

is finite. For $n = 2$ this definition agrees with the usual $H^p$-spaces for the upper half-plane.

Next we show how the Hardy-Littlewood maximal theorem extends from the disc to $\mathbb{R}^n_+$.

Define the maximal function $M(\vec{u})(x) = \sup_{|y-x|<t} |\vec{u}(y, t)|$ for $x \in \mathbb{R}^{n-1}$.

Then for $\vec{u} \in H^p(\mathbb{R}^n_+)$, $\frac{n-2}{n-1} < p$, we have $M(\vec{u}) \in L^p(\mathbb{R}^{n-1})$ with norm $\int_{\mathbb{R}^{n-1}} (M(\vec{u}))^p dx \leq C\|\vec{u}\|^p_{H^p}$.

As in the classical case, the proof proceeds by reducing the problem to the maximal theorem for $L^p (p > 1)$. For small $h > 0$, the function $F_h(x, t) = |\vec{u}(x, t + h)|^{\frac{n-2}{n-1}} (x \in \mathbb{R}^{n-1}, t \geq 0)$ is subharmonic on $\mathbb{R}^n_+$ and continuous up to the boundary. Therefore,

$$F_h(x, t) \leq \text{P.I.}(f_h), \tag{15}$$

where P.I. is the Poisson integral and $f_h(x) = F_h(x, 0) = |\vec{u}(x, h)|^{\frac{n-2}{n-1}}$. By definition of the $H^p$-norm, we have

$$\int_{\mathbb{R}^{n-1}} |f_h(x)|^{\tilde{p}} dx \leq \|\vec{u}\|^p_{H^p}, \quad \text{with} \quad \tilde{p} = \left(\frac{n-1}{n-2}\right) p > 1. \tag{16}$$

On the other hand, since the Poisson integral arises by convolving with an approximate identity, one shows easily that

$$\sup_{|y-x|<t} \text{P.I.}(f_h)(y, t) \leq C f^*_h(x) \tag{17}$$

with

$$f^*_h(x) = \sup_{r>0} r^{-(n-1)} \int_{|x-y|<r} |f_h(y)| dy \quad (x \in \mathbb{R}^{n-1}).$$

The standard maximal theorem of real variables gives

$$\int_{\mathbb{R}^{n-1}} (f^*_h)^{\tilde{p}} \leq C_p \int_{\mathbb{R}^{n-1}} |f_h|^{\tilde{p}},$$

since $\tilde{p} > 1$. Hence (15), (16), and (17) show that

$$\int_{x\in\mathbb{R}^{n-1}} \left( \sup_{|y-x|<t} F_h(y, t) \right)^{\tilde{p}} dy \leq C_p \|\vec{u}\|^p_{H^p}, \text{ i.e.,}$$

$$\int_{x\in\mathbb{R}^{n-1}} \left( \sup_{|y-x|+h<t} |\vec{u}(y, t)| \right)^p dy \leq C_p \|\vec{u}\|^p_{H^p}. \tag{18}$$

The constant $C_p$ is independent of $h$, so we can take the limit of (18) as $h \to 0$ to obtain the maximal theorem for $H^p$. The point is that subharmonicity of $|\vec{u}|^{\frac{n-2}{n-1}}$ substitutes for the Blaschke product in this argument.

Stein and Weiss go on in [13] to obtain n-dimensional analogues of the classical theorems on existence of boundary values of $H^p$ functions. They also extend to $\mathbb{R}^n_+$ the classical F. and M. Riesz theorem on absolute continuity of $H^1$ boundary values. They begin the program of using $H^p(\mathbb{R}^n_+)$ in place of $L^p(\mathbb{R}^{n-1})$, to extend the basic results of Fourier analysis to $p = 1$ and below. We have seen how they deal with the maximal function. They prove also an $H^p$-version of the Sobolev theorem.

It is natural to try to get below $p = \frac{n-2}{n-1}$, and this can be done by studying higher gradients of harmonic functions in place of (12). See Calderón-Zygmund [CZ].

A joint paper [60] by Stein and me completed the task of developing basic Fourier analysis in the setting of the $H^p$-spaces. In particular, we showed in [60] that singular integral operators are bounded on $H^p(\mathbb{R}^n_+)$ for $0 < p < \infty$. We proved this by finding a good viewpoint, and we found our viewpoint by repeatedly changing the definition of $H^p$. With each new definition, the function space $H^p$ remained the same, but it became clearer to us what was going on. Finally we arrived at a definition of $H^p$ with the following excellent properties. First of all, it was easy to prove that the new definition of $H^p$ was equivalent to the Stein-Weiss definition and its extensions below $p = \frac{n-2}{n-1}$. Secondly, the basic theorems of Fourier analysis, which seemed very hard to prove from the original definition of $H^p(\mathbb{R}^n_+)$, became nearly obvious in terms of the new definition. Let me retrace the steps in [60].

Burkholder-Gundy-Silverstein [BGS] had shown that an analytic function $F = u + iv$ on the disc belongs to $H^p$ $(0 < p < \infty)$ if and only if the maximal function $u^*(\theta) = \sup_{z\in\Gamma(\theta)} |u(z)|$ belongs to $L^p$ (Unit Circle). Thus, $H^p$ can be defined purely in terms of harmonic functions $u$, without recourse to the harmonic conjugate $v$. Stein and I showed in [60] that the same thing happens in $n$ dimensions. That is, a Cauchy-Riemann system $(u_1, u_2, \ldots, u_n)$ on $\mathbb{R}^n_+$ belongs to the Stein-Weiss $H^p$-space $(p > \frac{n-2}{n-1})$ if and only if the maximal function $u^*(x) = \sup_{|y-x|<t} |u_n(y,t)|$ belongs to $L^p(\mathbb{R}^{n-1})$. (Here, the $n$th function $u_n$ plays a special role because $\mathbb{R}^n_+$ is defined by $\{x_n > 0\}$.) Hence, $H^p$ may be viewed as a space of harmonic functions $u(x,t)$ on $\mathbb{R}^n_+$. The result extends below $p = \frac{n-2}{n-1}$ if we pass to higher gradients of harmonic functions.

The next step is to view $H^p$ as a space of distributions $f$ on the boundary $\mathbb{R}^{n-1}$. Any reasonable harmonic function $u(x,t)$ arises as the Poisson integral of a distribution $f$, so that $u(x,t) = \varphi_t * f(x)$, $\varphi_t$ = Poisson kernel. Thus, it is natural to say that $f \in H^p(\mathbb{R}^{n-1})$ if the maximal function

$$f^*(x) \equiv \sup_{|y-x|<t} |\varphi_t * f(y)| \tag{19}$$

belongs to $L^p$. Stein and I found in [60] that this definition is independent of the choice of the approximate identity $\varphi_t$, and that the "grand maximal function"

$$\mathcal{M}f(x) = \sup_{\{\varphi_t\}\in\mathcal{A}} \sup_{|y-x|<t} |\varphi_t * f(y)| \tag{20}$$

belongs to $L^p$, provided $f \in H^p$. Here $\mathcal{A}$ is a neighborhood of the origin in a suitable space of approximate identities. Thus, $f \in H^p$ if and only if $f^* \in L^p$ for some reasonable approximate identity. Equivalently, $f \in H^p$ if and only if the grand maximal function $\mathcal{M}f$ belongs to $L^p$. The proofs of these various equivalencies are not hard at all.

We have arrived at the good definition of $H^p$ mentioned above.

To transplant basic Fourier analysis from $L^p(1 < p < \infty)$ to $H^p(0 < p < \infty)$, there is a simple algorithm. Take Calderón-Zygmund theory, and replace every application of the standard maximal theorem by an appeal to the grand maximal function. Only small changes are needed, and we omit the details here. Our paper [60] also contains the duality of $H^1$ and BMO. Before leaving [60], let me mention an application of $H^p$ theory to $L^p$-estimates. If $\sigma$ denotes uniform surface measure on the unit sphere in $\mathbb{R}^n$, then $f \longmapsto (\frac{\partial}{\partial x})^\alpha \sigma * f$ is bounded on $L^p(\mathbb{R}^n)$, provided $n \geq 3$ and $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{2} - \frac{\alpha}{n-1}$. Clearly, this result gives information on solutions to the wave equation.

The proof uses complex interpolation involving the analytic family of operators

$$T_\alpha : f \longmapsto (-\Delta)^{\alpha/2} \sigma * f \quad (\alpha \text{ complex}),$$

as is clear to anyone familiar with the Stein interpolation theorem. The trouble here is that $(-\Delta)^{\alpha/2}$ fails to be bounded on $L^1$ when $\alpha$ is imaginary. This makes it impossible to prove the sharp result $(|\frac{1}{p} - \frac{1}{2}| = \frac{1}{2} - \frac{\alpha}{n-1})$ using $L^p$ alone. To overcome the difficulty, we use $H^1$ in place of $L^1$ in the interpolation argument. Imaginary powers of the Laplacian are singular integrals, which we know to be bounded on $H^1$. To show that complex interpolation works on $H^1$, we combined the duality of $H^1$ and BMO with the auxiliary function $f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - (\text{mean}_Q f)| dy$. We refer the reader to [60] for an explanation of how to use $f^\#$, and for other applications.

Since [60], Stein has done a lot more on $H^p$, both in “higher rank” settings, and in contexts related to partial differential equations.

## REPRESENTATION THEORY II

Next we return to representation theory. We explain briefly how the Kunze-Stein construction extends from $SL(2, \mathbb{R})$ to more general semisimple Lie groups, with profound consequences for representation theory. The results we discuss are contained in the series of papers by Kunze-Stein [20, 22, 33, 63], Stein [35, 48, 70], and Knapp-Stein [43, 46, 50, 53, 58, 66, 73, 93, 97]. Let $G$ be a semisimple Lie group, and let $U^\pi$ be the unitary principal series representations of $G$, or one of its degenerate variants. The $U^\pi$ all act on a common Hilbert space, whose inner product we denote by $\langle \xi, \eta \rangle$. We needn't write down $U^\pi$ here, nor even specify the parameters on which it depends. A finite group $W$, the Weyl group, acts on the parameters $\pi$ in such a way that the representations $U^\pi$ and $U^{w\pi}$ are unitarily equivalent for $w \in W$. Thus there is an *intertwining operator* $A(w, \pi)$ so that

$$A(w, \pi) U_g^{w\pi} = U_g^\pi A(w, \pi) \quad \text{for} \quad g \in G, \ w \in W, \text{and for all } \pi. \qquad (21)$$

If $U^\pi$ is irreducible (which happens for most $\pi$), then $A(w, \pi)$ is uniquely determined by (21) up to multiplication by an arbitrary scalar $a(w, \pi)$. The crucial idea is as follows. If the $A(w, \pi)$ are correctly normalized (by the correct choice of $a(w, \pi)$), then $A(w, \pi)$ continues analytically to complex parameter values $\pi$. Moreover, for certain complex $(w, \pi)$, the quadratic form

$$((\xi, \eta))_{w,\pi} = \langle (\text{TRIVIAL FACTOR}) A(w, \pi)\, \xi, \eta \rangle \tag{22}$$

is positive definite.

In addition, the representation $U^\pi$ (defined for complex $\pi$ by a trivial analytic continuation) is unitary with respect to the inner product (22). Thus, starting with the principal series, we have constructed a new series of unitary representations of $G$. These new representations generalize the complementary series for $SL(2, \mathbb{R})$. Applications of this basic construction are as follows.

(1) Starting with the unitary principal series, one obtains understanding of the previously discovered complementary series, and construction of new ones, e.g., on $Sp(4, \mathbb{C})$. Thus, Stein exposed a gap in a supposedly complete list of complementary series representations of $Sp(4, \mathbb{C})$ [GN]. See [33].

(2) Starting with a degenerate unitary principal series, Stein constructed new irreducible unitary representations of $SL(n, \mathbb{C})$, in startling contradiction to the standard, supposedly complete list [GN] of irreducible unitary representations of that group. Much later, when the complete list of representations of $SL(n, \mathbb{C})$ was given correctly, the representations constructed by Stein played an important role.

(3) The analysis of intertwining operators required to carry out analytic continuation also determines which exceptional values of $\pi$ lead to reducible principal series representations. For example, such reducible principal series representations exist already for $SL(n, \mathbb{R})$, again contradicting what was "known." See Knapp-Stein [53].

A very recent result of Sahi and Stein [139] also fits into the same philosophy. In fact, Speh's representation can also be constructed by a more complicated variant of the analytic continuation defining the complementary series. Speh's representation plays an important role in the classification of the irreducible unitary representations of $SL(2n, \mathbb{R})$.

The main point of Stein's work in representation theory is thus to analyze the interwining operators $A(w, \pi)$. In the simplest non-trivial case, $A(w, \pi)$ is a singular integral operator on a nilpotent group $N$. That is, $A(w, \pi)$ has the form

$$Tf(x) = \int_N K(xy^{-1}) f(y) dy, \tag{23}$$

with $K(y)$ smooth away from the identity and homogeneous of the critical degree with respect to "dilations"

$$(\delta_t)_{t>0} : N \to N.$$

In (23), $dy$ denotes Haar measure on $N$. We know from the classical case $N = \mathbb{R}^1$ that (23) is a bounded operator only when the convolution kernel $K(y)$ satisfies a cancellation condition. Hence we assume $\int_{B_1 \setminus B_0} K(y)dy = 0$, where the $B_i$ are dilates ($B_i = \delta_{t_i}(B)$) of a fixed neighborhood of the identity in $N$.

It is crucial to show that such singular integrals are bounded on $L^2(N)$, generalizing the elementary $L^2$-boundedness of the Hilbert transform.

## COTLAR-STEIN LEMMA

In principle, $L^2$-boundedness of the translation-invariant operator (23) should be read off from the representation theory of $N$. In practice, representation theory provides a necessary and sufficient condition for $L^2$-boundedness that no one knows how to check. This fundamental analytic difficulty might have proved fatal to the study of intertwining operators. Fortunately, Stein was working simultaneously on a seemingly unrelated question, and made a discovery that saved the day. Originally motivated by desire to get a simple proof of Calderón's theorem on commutator integrals [Ca], Stein proved a simple, powerful lemma in functional analysis. His contribution was to generalize to the critically important non-commutative case the remarkable lemma of Cotlar [Co]. The Cotlar-Stein lemma turned out to be the perfect tool to prove $L^2$-boundedness of singular integrals on nilpotent groups. In fact, it quickly became a basic, standard tool in analysis. We will now explain the Cotlar-Stein lemma, and give its amazingly simple proof. Then we will return to its application to singular integrals on nilpotent groups.

The Cotlar-Stein lemma deals with a sum $T = \Sigma_v T_v$ of operators on a Hilbert space. The idea is that if the $T_v$ are almost orthogonal, like projections onto the various coordinate axes, then the sum $T$ will have norm no larger than $\max_v \|T_v\|$. The precise statement is as follows.

**COTLAR-STEIN LEMMA**. *Suppose* $T = \Sigma_{k=1}^M T_k$ *is a sum of operators on Hilbert space. Assume* $\|T_j^* T_k\| \leq a(j-k)$ *and* $\|T_j T_k^*\| \leq a(j-k)$. *Then* $\|T\| \leq \Sigma_{-M}^M \sqrt{a(j)}$.

*Proof.* $\|T\| = (\|TT^*\|)^{1/2}$, so

$$\|T\|^{2s} \leq \|(TT^*)^s\| = \left\| \sum_{j_1 \cdots j_{2s=1}}^M T_{j_1} T_{j_2}^* \cdots T_{j_{2s-1}} T_{j_{2s}}^* \right\|$$
$$\leq \sum_{j_1 \cdots j_{2s=1}}^M \|T_{j_1} T_{j_2}^* \cdots T_{j_{2s-1}} T_{j_{2s}}^*\|. \qquad (24)$$

We can estimate the summand in two different ways.

Writing $T_{j_1} T_{j_2}^* \cdots T_{j_{2s-1}} T_{j_{2s}}^* = (T_{j_1} T_{j_2}^*)(T_{j_3} T_{j_4}^*) \cdots (T_{j_{2s-1}} T_{j_{2s}}^*)$, we get

$$\|T_{j_1} T_{j_2}^* \cdots T_{j_{2s-1}} T_{j_{2s}}^*\| \leq a(j_1 - j_2) a(j_3 - j_4) \cdots a(j_{2s-1} - j_{2s}). \qquad (25)$$

On the other hand, writing

$$T_{j_1} T_{j_2}^* \cdots T_{j_{2s-1}} T_{j_{2s}}^* = T_{j_1} (T_{j_2}^* T_{j_3})(T_{j_4}^* T_{j_5}) \cdots (T_{j_{2s-2}}^* T_{j_{2s-1}}) T_{j_{2s}}^*,$$

we see that

$$\|T_{j_1}T^*_{j_2}\cdots T_{j_{2s-1}}T^*_{j_{2s}}\| \le (\max_j \|T_j\|)^2 a(j_2-j_3)a(j_4-j_5)\cdots a(j_{2s-2}-j_{2s-1}). \tag{26}$$

Taking the geometric mean of (25), (26) and putting the result into (24), we conclude that

$$\begin{aligned}\|T\|^{2s} &\le \sum_{j_1\cdots j_{2s}=1}^{M}\left(\max_j \|T_j\|\right)\sqrt{a(j_1-j_2)}\sqrt{a(j_2-j_3)}\cdots\sqrt{a(j_{2s-1}-j_{2s})}\\ &\le \left(\max_j \|T_j\|\cdot M\right)\left(\sum_\ell \sqrt{a(\ell)}\right)^{2s-1}.\end{aligned}$$

Thus, $\|T\| \le (\max_j \|T_j\|\cdot M)^{1/2s}\cdot(\Sigma_\ell\sqrt{a(\ell)})^{\frac{2s-1}{2s}}$. Letting $s\to\infty$, we obtain the conclusion of the Cotlar-Stein Lemma. □

To apply the Cotlar-Stein lemma to singular integral operators, take a partition of unity $1 = \Sigma_v\varphi_v(x)$ on $N$, so that each $\varphi_v$ is a dilate of a fixed $C_0^\infty$ function that vanishes in a neighborhood of the origin. Then $T : f \to K * f$ may be decomposed into a sum $T = \Sigma_v T_v$, with $T_v : f \to (\varphi_v K) * f$. The hypotheses of the Cotlar-Stein lemma are verified trivially, and the boundedness of singular integral operators follows. The $L^2$-boundedness of singular integrals on nilpotent groups is the Knapp-Stein Theorem.

Almost immediately after this work, the Cotlar-Stein lemma became the standard method to prove $L^2$-boundedness of operators. Today one knows more, e.g., the $T(1)$ theorem of David and Journé. Still it is fair to say that the Cotlar-Stein lemma remains the most important tool for $L^2$-boundedness.

Singular integrals on nilpotent groups were soon applied by Stein in a context seemingly far from representation theory.

## $\bar\partial$-PROBLEMS

We prepare to discuss Stein's work on the $\bar\partial$-problems of several complex variables and related questions. Let us begin with the state of the subject before Stein's contributions. Suppose we are given a domain $D\subset\mathbb{C}^n$ with smooth boundary. If we try to construct analytic functions on $D$ with given singularities at the boundary, then we are led naturally to the following problems.

I. Given a (0, 1) form $\alpha = \Sigma_{k=1}^n f_k\overline{dz_k}$ on $D$, find a function $u$ on $D$ that solves $\bar\partial u = \alpha$, where $\bar\partial u = \Sigma_{k=1}^n \frac{\bar\partial u}{\partial\bar z_k}\overline{dz_k}$. Naturally, this is possible only if $\alpha$ satisfies the consistency condition $\bar\partial\alpha = 0$, i.e., $\frac{\partial}{\partial\bar z_k}f_j = \frac{\partial}{\partial\bar z_j}f_k$. Moreover, $u$ is determined only modulo addition of an arbitrary analytic function on $D$. To make $u$ unique, we demand that $u$ be orthogonal to analytic functions in $L^2(D)$.

II. There is a simple analogue of the $\bar{\partial}$-operator for functions defined only on the boundary $\partial D$. In local coordinates, we can easily find $(n-1)$ linearly independent complex vector fields $L_1 \cdots L_{n-1}$ of type (0, 1) (i.e., $L_j = a_{j1}\frac{\partial}{\partial \bar{z}_1} + a_{j2}\frac{\partial}{\partial \bar{z}_2} + \cdots + a_{jn}\frac{\partial}{\partial \bar{z}_n}$ for smooth, complex-valued $a_{jk}$) whose real and imaginary parts are all tangent to $\partial D$. The restriction $u$ of an analytic function to $\partial D$ clearly satisfies $\bar{\partial}_b u = 0$, where in local coordinates $\bar{\partial}_b u = (L_1 u, L_2 u, \ldots, L_{n-1} u)$. The boundary analogue of the $\bar{\partial}$-problem (I) is the inhomogeneous $\bar{\partial}_b$-equation $\bar{\partial}_b u = \alpha$. Again, this is possible only if $\alpha$ satisfies a consistency condition $\bar{\partial}_b \alpha = 0$, and we impose the side condition that $u$ be orthogonal to analytic functions in $L^2(\partial D)$.

Just as analytic functions of one variable are related to harmonic functions, so the first-order systems (I) and (II) are related to second-order equations $\Box$ and $\Box_b$, the $\bar{\partial}$-Neumann and Kohn Laplacians. Both fall outside the scope of standard elliptic theory. Even for the simplest domains $D$, they posed a fundamental challenge to workers in partial differential equations. More specifically, $\Box$ is simply the Laplacian in the interior of $D$, but it is subject to non-elliptic boundary conditions. On the othe hand, $\Box_b$ is a non-elliptic system of partial differential operators on $\partial D$, with no boundary conditions (since $\partial D$ has no boundary). Modulo lower-order terms (which, however, are important), $\Box_b$ is the scalar operator $\mathcal{L} = \Sigma_{k=1}^{n-1}(X_k^2 + Y_k^2)$, where $X_k$ and $Y_k$ are the real and imaginary parts of the basic complex vector fields $L_k$. At a given point in $\partial D$, the $X_k$ and $Y_k$ are linearly independent, but they don't span the tangent space of $\partial D$. This poses the danger that $\mathcal{L}$ will behave like a partial Laplacian such as $\Delta' = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ acting on function $u(x, y, z)$. The equation $\Delta' u = f$ is very bad. For instance, we can take $u(x, y, z)$ to depend on $z$ alone, so that $\Delta' u = 0$ with $u$ arbitrarily rough. Fortunately, $\mathcal{L}$ is more like the full Laplacian than like $\Delta'$, because the $X_k$ and $Y_k$ together with their commutators $[X_k, Y_k]$ span the tangent space of $\partial D$ for suitable $D$. Thus, $\mathcal{L}$ is a well-behaved operator, thanks to the intervention of commutators of vector fields.

It was Kohn in the 1960s who proved the basic $C^\infty$ regularity theorems for $\Box$, $\Box_b$, $\bar{\partial}$ and $\bar{\partial}_b$ on strongly pseudoconvex domains (the simplest case). His proofs were based on subelliptic estimates such as $\langle \Box_b w, w \rangle \geq c\|w\|^2_{(\varepsilon)} - C\|w\|^2$, and brought to light the importance of commutators. Hörmander proved a celebrated theorem on $C^\infty$ regularity of operators,

$$L = \sum_{j=1}^{N} X_j^2 + X_0,$$

where $X_0, X_1, \ldots, X_N$ are smooth, real vector fields which, together with their repeated commutators, span the tangent space at every point.

If we allow $X_0$ to be a complex vector field, then we get a very hard problem that is not adequately understood to this day, except in very special cases.

Stein made a fundamental change in the study of the $\bar{\partial}$-problems by bringing in constructive methods. Today, thanks to the work of Stein with several collaborators, we know how to write down explicit solutions to the $\bar{\partial}$-problems modulo negligible errors on strongly pseudoconvex domains. Starting from these explicit solutions, it is then possible to prove sharp regularity theorems. Thus, the $\bar{\partial}$ equations on strongly pseudoconvex domains are understood completely. It is a major open problem to achieve comparable understanding of weakly pseudoconvex domains.

Now let us see how Stein and his co-workers were able to crack the strongly pseudoconvex case. We begin with the work of Folland and Stein [67]. The simplest example of a strongly pseudoconvex domain is the unit ball. Just as the disc is equivalent to the half-plane, the ball is equivalent to the Siegel domain $D_{\text{Siegel}} = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid \text{Im } w > |z|^2\}$. Its boundary $H = \partial D_{\text{Siegel}}$ has an important symmetry group, including the following.

(a) *Translations* $(z, w) \longmapsto (z, w) \cdot (z', w') \equiv (z + z', w + w' + 2iz \cdot \bar{z}')$ for $(z', w') \in H$;
(b) *Dilations* $\delta_t : (z, w) \longmapsto (tz, t^2 w)$ for $t > 0$;
(c) *Rotations* $(z, w) \longmapsto (Uz, w)$ for unitary $(n-1) \times (n-1)$ matrices $U$.

The multiplication law in (a) makes $H$ into a nilpotent Lie group, the Heisenberg group. Translation-invariance of the Seigel domain allows us to pick the basic complex vector-fields $L_1 \cdots L_{n-1}$ to be translation-invariant on $H$. After we make a suitable choice of metric, the operators $\mathcal{L}$ and $\Box_b$ become translation- and rotation-invariant, and homogeneous with respect to the dilations $\delta_t$. Therefore, the solution[1] of $\Box_b w = \alpha$ should have the form of a convolution $w = K * \alpha$ on the Heisenberg group. The convolution kernel $K$ is homogeneous with respect to the dilations $\delta_t$ and invariant under rotations. Also, since $K$ is a fundamental solution, it satisfies $\Box_b K = 0$ away from the origin. This reduces to an elementary ODE after we take the dilation- and rotation-invariance into account. Hence one can easily find $K$ explicitly and thus solve the $\Box_b$-equation for the Siegel domain. To derive sharp regularity theorems for $\Box_b$, we combine the explicit fundamental solution with the Knapp-Stein theorem on singular integrals on the Heisenberg group. For instance, if $\Box_b w \in L^2$, then $L_j L_k w$, $\bar{L}_j L_k w$, $L_j \bar{L}_k w$, and $\bar{L}_j \bar{L}_k w$ all belong to $L^2$. To see this, we write

$$\Box_b w = \alpha, \quad w = K * \alpha, \quad L_j L_k w = (L_j L_k K) * \alpha,$$

and note that $L_j L_k K$ has the critical homogeneity and integral 0. Thus $L_j L_k K$ is a singular integral kernel in the sense of Knapp and Stein, and it follows that $\|L_j L_k w\| \leq C\|\alpha\|$. For the first time, nilpotent Lie groups have entered into the study of $\bar{\partial}$-problems.

Folland and Stein viewed their results on the Heisenberg group not as ends in themselves, but rather as a tool to understand general strongly pseudoconvex CR-manifolds. A CR-manifold $M$ is a generalization of the boundary of a smooth domain $D \subset \mathbb{C}^n$. For simplicity we will take $M = \partial D$ here. The key idea is that near any point $w$ in a strongly pseudoconvex $M$, the CR-structure for $M$ is very nearly equivalent to that of the Heisenberg group $H$ via a change of coordinates $\Theta_w : M \to H$. More precisely, $\Theta_w$ carries $w$ to the origin, and it carries the CR-structure on $M$ to a CR-structure on $H$ that agrees with the usual one at the origin.

[1] Kohn's work showed that $\Box_b w = \alpha$ has a solution if we are in complex dimension $> 2$. In two complex dimensions, $\Box_b w = \alpha$ has no solution for most $\alpha$. We assume dimension $> 2$ here.

Therefore, if $w = K * \alpha$ is our known solution of $\Box_b w = \alpha$ on the Heisenberg group, then it is natural to try

$$w(z) = \int_M K(\Theta_w(z))\alpha(w)dw \tag{27}$$

as an approximate solution of $\Box_b w = \alpha$ on $M$. (Since $w$ and $\alpha$ are sections of bundles, one has to explain carefully what (27) really means.) If we apply $\Box_b$ to the $w$ defined by (27), then we find that

$$\Box_b w = \alpha - \mathcal{E}\alpha, \tag{28}$$

where $\mathcal{E}$ is a sort of Heisenberg version of $(-\Delta)^{-1/2}$. In particular, $\mathcal{E}$ gains smoothness, so that $(I - \mathcal{E})^{-1}$ can be constructed modulo infinitely smoothing operators by means of a Neumann series. Therefore (27) and (28) show that the full solution of $\Box_b w = \alpha$ is given (modulo infinitely smoothing errors) by

$$w(z) = \sum_k \int_M K(\Theta_w(z))(\mathcal{E}^k\alpha)(w)dw, \tag{29}$$

from which one can deduce sharp estimates to understand completely $\Box_b^{-1}$ on $M$.

The process is analogous to the standard method of "freezing coefficients" to solve variable-coefficient elliptic differential equations. Let us see how the sharp results are stated. As on the Heisenberg group, there are smooth, complex vector fields $L_k$ that span the tangent vectors of type (0, 1) locally. Let $X_j$ be the real and imaginary parts of the $L_k$. In terms of the $X_j$ we define "non-Euclidean" versions of standard geometric and analytic concepts. Thus, the non-Euclidean ball $\mathbb{B}(z, \rho)$ may be defined as an ellipsoid with principal axes of length $\sim \rho$ in the codimension 1 hyperplane spanned by the $X_j$, and length $\sim \rho^2$ perpendicular to that hyperplane. In terms of $\mathbb{B}(z, \rho)$, the non-Euclidean Lipschitz spaces $\Gamma_\alpha(M)$ are defined as the set of functions $u$ for which $|u(z) - u(w)| < C\rho^\alpha$ for $w \in \mathbb{B}(z, \rho)$. (Here, $0 < \alpha < 1$. There is a natural extension to all $\alpha > 0$.) The non-Euclidean Sobolev spaces $S_{m,p}(M)$ consist of all distributions $u$ for which all $X_{j_1} X_{j_2} \cdots X_{j_s} u \in L^p(M)$ for $0 \le s \le m$.

Then the sharp results on $\Box_b$ are as follows. If $\Box_b w = \alpha$ and $\alpha \in S_{m,p}(M)$, then $w \in S_{m+2,p}(M)$ for $m \ge 0$, $1 < p < \infty$. If $\Box_b w = \alpha$ and $\alpha \in \Gamma_\alpha(M)$, then $X_j X_k w \in \Gamma_\alpha(M)$ for $0 < \alpha < 1$ (say). For additional sharp estimates, and for comparisons between the non-Euclidean and standard function spaces, we refer the reader to [67].

To prove their sharp results, Folland and Stein developed the theory of singular integral operators in a non-Euclidean context. The Cotlar-Stein lemma proves the crucial results on $L^2$-boundedness of singular integrals. Additional difficulties arise from the non-commutativity of the Heisenberg group. In particular, standard singular integrals or pseudodifferential operators commute modulo lower-order errors, but non-Euclidean operators are far from commuting. This makes more difficult the passage from $L^p$ estimates to the Sobolev spaces $S_{m,p}(M)$.

Before we continue with Stein's work on $\bar{\partial}$, let me explain the remarkable paper of Rothschild-Stein [72]. It extends the Folland-Stein results and viewpoint to general Hörmander operators $\mathcal{L} = \Sigma_{j=1}^N X_j^2 + X_0$. Actually, [72] deals with systems whose second-order part is $\Sigma_j X_j^2$, but for simplicity we restrict attention here to $\mathcal{L}$. In explaining the proofs, we simplify even further by supposing $X_0 = 0$. The goal of the Rothschild-Stein paper is to use nilpotent groups to write down an explicit parametrix for $\mathcal{L}$ and prove sharp estimates for solutions of $\mathcal{L}u = f$. This ambitious hope is seemingly dashed at once by elementary examples. For instance, take $\mathcal{L} = X_1^2 + X_2^2$ with

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y} \quad \text{on} \quad \mathbb{R}^2. \tag{30}$$

Then $X_1$ and $[X_1, X_2]$ span the tangent space, yet $\mathcal{L}$ clearly cannot be approximated by translation-invariant operators on a nilpotent Lie group in the sense of Folland-Stein. The trouble is that $\mathcal{L}$ changes character completely from one point to another. Away from the $y$-axis $\{x = 0\}$, $\mathcal{L}$ is elliptic, so the only natural nilpotent group we can reasonably use is $\mathbb{R}^2$. On the $y$-axis, $\mathcal{L}$ degenerates, and evidently cannot be approximated by a translation-invariant operator on $\mathbb{R}^2$. The problem is so obviously fatal, and its solution by Rothschild and Stein so simple and natural, that [72] must be regarded as a gem. Here is the idea:

Suppose we add an extra variable $t$ and "lift" $X_1$ and $X_2$ in (30) to vector fields

$$\tilde{X}_1 = \frac{\partial}{\partial x}, \quad \tilde{X}_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial t} \quad \text{on} \quad \mathbb{R}^3. \tag{31}$$

Then the Hörmander operator $\tilde{\mathcal{L}} = \tilde{X}_1^2 + \tilde{X}_2^2$ looks the same at every point of $\mathbb{R}^3$, and may be readily understood in terms of nilpotent groups as in Folland-Stein [67]. In particular, one can essentially write down a fundamental solution and prove sharp estimates for $\tilde{\mathcal{L}}^{-1}$. On the other hand, $\tilde{\mathcal{L}}$ reduces to $\mathcal{L}$ when acting on functions $u(x, y, t)$ that do not depend on $t$. Hence, sharp results on $\tilde{\mathcal{L}}u = f$ imply sharp results on $\mathcal{L}u = f$.

Thus we have the Rothschild-Stein program: First, add new variables and lift the given vector fields $X_1, \cdots, X_N$ to new vector fields $\tilde{X}_1 \cdots \tilde{X}_N$ whose underlying structure does not vary from point to point. Next, approximate $\tilde{\mathcal{L}} = \Sigma_1^N \tilde{X}^2$ by a translation-invariant operator $\hat{\mathcal{L}} = \Sigma_1^N Y_j^2$ on a nilpotent Lie group $\mathcal{N}$. Then analyze the fundamental solution of $\hat{\mathcal{L}}$, and use it to write down an approximate fundamental solution for $\tilde{\mathcal{L}}$. From the approximate solution, derive sharp estimates for solutions of $\tilde{\mathcal{L}}u = f$. Finally, descend to the original equation $\mathcal{L}u = f$ by restricting attention to functions $u, f$ that do not depend on the extra variables.

To carry out the first part of their program, Rothschild and Stein prove the following result.

**Theorem A.** *Let $X_1 \cdots X_N$ be smooth vector fields on a neighborhood of the origin in $\mathbb{R}^n$. Assume that the $X_j$ and their commutators $[[[X_{j_1}, X_{j_2}], X_{j_3}] \cdots X_{j_s}]$ of order up to r span the tangent space at the origin. Then we can find smooth vector fields $\tilde{X}_1 \cdots \tilde{X}_N$ on a neighborhood $\tilde{U}$ of the origin in $\mathbb{R}^{n+m}$ with the following properties.*

(a) *The $\tilde{X}_j$ and their commutators up to oder r are linearly independent at each point of $\tilde{U}$, except for the linear relations that follow formally from the antisymmetry of the bracket and the Jacobi identity.*

(b) *The* $\tilde{X}_j$ *and their commutators up to order r span the tangent space of* $\tilde{U}$.
(c) *Acting on functions on* $\mathbb{R}^{n+m}$ *that do not depend on the last m coordinates, the* $\tilde{X}_j$ *reduce to the given* $X_j$.

Next we need a nilpotent Lie group $\mathcal{N}$ appropriate to the vector fields $\tilde{X}_1 \cdots \tilde{X}_N$. The natural one is the free nilpotent group $\mathcal{N}_{Nr}$ of step $r$ on $N$ generators. Its Lie algebra is generated by $Y_1 \cdots Y_N$ whose Lie brackets of order higher than $r$ vanish, but whose brackets of order $\le r$ are linearly independent, except for relations forced by antisymmetry of brackets and the Jacobi identity. We regard the $Y_j$ as translation-invariant vector fields on $\mathcal{N}_{Nr}$. It is convenient to pick a basis $\{y_\alpha\}_{\alpha \in A}$ for the Lie algebra of $\mathcal{N}_{Nr}$, consisting of $Y_1 \cdots Y_N$ and some of their commutators.

On $\mathcal{N}_{Nr}$ we form the Hörmander operator $\hat{\mathcal{L}} = \Sigma_1^N Y_j^2$. Then $\hat{\mathcal{L}}$ is translation-invariant and homogeneous under the natural dilations on $\mathcal{N}_{Nr}$. Hence $\hat{\mathcal{L}}^{-1}$ is given by convolution on $\mathcal{N}_{Nr}$ with a homogeneous kernel $K(\cdot)$ having a weak singularity at the origin. Hypoellipticity of $\hat{\mathcal{L}}$ shows that $K$ is smooth away from the origin. Thus we understand the equation $\hat{\mathcal{L}}u = f$ very well.

We want to use $\hat{\mathcal{L}}$ to approximate $\tilde{\mathcal{L}}$ at each point $y \in \tilde{U}$. To do so, we have to identify a neighborhood of $y$ in $\tilde{U}$ with a neighborhood of the origin in $\mathcal{N}_{Nr}$. This has to be done just right, or else $\hat{\mathcal{L}}$ will fail to approximate $\tilde{\mathcal{L}}$. The idea is to use exponential coordinates on both $\tilde{U}$ and $\mathcal{N}_{Nr}$. Thus, if $x = \exp(\Sigma_{\alpha \in A} t_\alpha Y_\alpha)(\text{identity}) \in \mathcal{N}_{Nr}$, then we use $(t_\alpha)_{\alpha \in A}$ as coordinates for $x$. Similarly, let $(\tilde{X}_\alpha)_{\alpha \in A}$ be the commutators of $\tilde{X}_1 \cdots \tilde{X}_N$ analogous to the $Y_\alpha$, and let $y \in \tilde{U}$ be given. Then given a nearby point $x = \exp(\Sigma_{\alpha \in A} t_\alpha \tilde{X}_\alpha) y \in \tilde{U}$, we use $(t_\alpha)_{\alpha \in A}$ as coordinates for $x$.

Now we can identify $\tilde{U}$ with a neighborhood of the identity in $\mathcal{N}_{Nr}$, simply by identifying points with the same coordinates. Denote the identification by $\Theta_y : \tilde{U} \to \mathcal{N}_{Nr}$, and note that $\Theta_y(y) = \text{identity}$.

In view of the identification $\Theta_y$, the operators $\hat{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ live on the same space. The next step is to see that they are approximately equal. To formulate this, we need some bookkeeping on the nilpotent group $\mathcal{N}_{Nr}$. Let $\{\delta_t\}_{t>0}$ be the natural dilations on $\mathcal{N}_{Nr}$. If $\varphi \in C_0^\infty(\mathcal{N}_{Nr})$, then write $\varphi_t$ for the function $x \longmapsto \varphi(\delta_t x)$. When $\varphi$ is fixed and $t$ is large, then $\varphi_t$ is supported in a tiny neighborhood of the identity. Let $\mathcal{D}$ be a differential operator acting on functions on $\mathcal{N}_{Nr}$. We say that $\mathcal{D}$ has "degree" at most $k$ if for each $\varphi \in C_0^\infty(\mathcal{N}_{Nr})$ we have $|\mathcal{D}(\varphi_t)| = O(t^k)$ for large, positive $t$. According to this definitions, $Y_1, \ldots Y_N$ have degree 1 while $[Y_j, Y_k]$ has degree 2, and the degree of $a(x)[Y_j, Y_k]$ depends on the behavior of $a(x)$ near the identify. Now we can say in what sense $\tilde{\mathcal{L}}$ and $\hat{\mathcal{L}}$ are approximately equal. The crucial result is as follows.

**THEOREM B**. *Under the map* $\Theta_y^{-1}$, *the vector field* $\tilde{X}_j$ *pulls back to* $Y_j + Z_{y,j}$, *where* $Z_{y,j}$ *is a vector field on* $\mathcal{N}_{Nr}$ *of "degree"* $\le 0$.

Using Theorem B and the map $\Theta_y$, we can produce a parametrix for $\tilde{\mathcal{L}}$ and prove that it works. In fact, we take

$$\tilde{K}(x, y) = K(\Theta_y x), \tag{32}$$

where $K$ is the fundamental solution of $\hat{\mathcal{L}}$. For fixed $y$, we want to know that

$$\tilde{\mathcal{L}}\tilde{K}(x, y) = \delta_y(x) + \mathcal{E}(x, y), \tag{33}$$

where $\delta_y(\cdot)$ is the Dirac delta-function and $\mathcal{E}(x, y)$ has only a weak singularity at $x = y$. To prove this, we use $\Theta_y$ to pull back to $\mathcal{N}_{Nr}$. Recall that $\tilde{\mathcal{L}} = \Sigma_1^N \tilde{X}_j^2$ while $\hat{\mathcal{L}} = \Sigma_1^N Y_j^2$. Hence by Theorem B, $\tilde{\mathcal{L}}$ pulls back to an operator of the form $\hat{L} + \mathcal{D}_y$, with $\mathcal{D}_y$ having "degree" at most 1. Therefore (33) reduces to proving that

$$(\hat{\mathcal{L}} + \mathcal{D}_y)K(x) = \delta_{id.}(x) + \hat{\mathcal{E}}(x), \tag{34}$$

where $\hat{\mathcal{E}}$ has only a weak singularity at the identity. Since $\hat{\mathcal{L}}K(x) = \delta_{\text{id.}}(x)$, (34) means simply that $\mathcal{D}_y K(x)$ has only a weak singularity at the identity. However, this is obvious from the smoothness and homogeneity if $K(x)$, and from the fact that $\mathcal{D}_y$ has degree $\leq 1$. Thus, $\tilde{K}(x, y)$ is an approximate fundamental solution for $\tilde{\mathcal{L}}$.

From the explicit fundamental solution for the lifted operator $\tilde{\mathcal{L}}$, one can "descend" to deal with the original Hörmander operator $\mathcal{L}$ in two different ways.

a. Prove sharp estimates for the lifted problems, then specialize to the case of functions that don't depend on the extra variables.

b. Integrate out the extra variables from the fundamental solution for $\tilde{\mathcal{L}}$, to obtain a fundamental solution for $\mathcal{L}$.

Rothschild and Stein used the first approach. They succeeded in proving the estimate

$$\|X_0 u\|_{L^p(U)} + \|X_j X_k u\|_{L^p(U)} \leq C_p \left\| \left( \sum_{j=1}^N X_j^2 + X_0 \right) u \right\|_{L^p(V)}$$
$$+ C_p \|u\|_{L^p(V)} \quad \text{for} \quad 1 < p < \infty \text{ and } U \subset\subset V. \tag{35}$$

This is the most natural and the sharpest estimate for Hörmander operators. It was new even for $p = 2$. Rothschild and Stein also proved sharp estimates in spaces analogous to the $\Gamma_\alpha$ and $S_{m,p}$ of Folland-Stein [67], as well as in standard Lipschitz and Sobolev spaces. We omit the details, but we point out that commuting derivatives past a general Hörmander operator here requires additional ideas.

Later, Nagel, Stein, and Wainger [119] returned to the second approach ("b" above) and were able to estimate the fundamental solution of a general Hörmander operator. This work overcomes substantial problems.

In fact, once we descend from the lifted problem to the original equation, we again face the difficulty that Hörmander operators cannot be modelled directly on nilpotent Lie groups. So it isn't even clear how to state a theorem on the fundamental solution of a Hörmander operator. Nagel, Stein, and Wainger [119], realized that a family of non-Euclidean "balls" $B_{\mathcal{L}}(x, \rho)$ associated to the Hörmander operator $\mathcal{L}$ plays the basic role. They defined the $B_{\mathcal{L}}(x, \rho)$ and proved their essential properties. In particular, they saw that the family of balls survives the projection from the lifted problem back to the original equation, even though the nilpotent Lie group structure is destroyed. Non-Euclidean balls had already played an important part in Folland-Stein [67]. However, it was simple in [67] to guess the correct family of balls. For general Hörmander operators $\mathcal{L}$ the problem of defining and controlling non-Euclidean balls is much more subtle. Closely related results appear also in [FKP], [FS].

Let us look first at a nilpotent group such as $\mathcal{N}_{Nr}$, with its family of dilations $\{\delta_t\}_{t>0}$. Then the correct family of non-Euclidean balls $B_{\mathcal{N}_{Nr}}(x, \rho)$ is essentially dictated by translation and dilation-invariance, starting with a more or less arbitrary harmless "unit ball" $B_{\mathcal{N}_{Nr}}$ (identity, 1). Recall that the fundamental solution for $\hat{\mathcal{L}} = \sum_1^N Y_j^2$ on $\mathcal{N}_{Nr}$ is given by a kernel $K(x)$ homogeneous with respect to the $\delta_t$. Estimates that capture the size and smoothness of $K(x)$ may be phrased entirely in terms of the non-Euclidean balls $B_{\mathcal{N}_{Nr}}(x, \rho)$. In fact, the basic estimate is as follows.

$$|Y_{j_1} Y_{j_2} \dots Y_{j_m} K(x)| \leq \frac{C_m \rho^{2-m}}{\text{vol } B_{\mathcal{N}_{Nr}}(0, \rho)}$$

$$\text{for } x \in B_{\mathcal{N}_{Nr}}(0, \rho) \backslash B_{\mathcal{N}_{Nr}}\left(0, \frac{\rho}{2}\right) \quad \text{and } m \geq 0. \tag{36}$$

Next we associate non-Euclidean balls to a general Hörmander operator. For simplicity, take $\mathcal{L} = \sum_1^N X_j^2$ as in our discussion of Rothschild-Stein [72]. One definition of the balls $B_{\mathcal{L}}(x, \rho)$ involves a moving particle that starts at $x$ and travels along the integral curve of $X_{j_1}$ for time $t_1$. From its new position $x'$ the particle then travels along the integral curve of $X_{j_2}$ for time $t_2$. Repeating the process finitely many times, we can move the particle from its initial position $x$ to a final position $y$ in a total time $t = t_1 + \cdots + t_m$. The ball $B_{\mathcal{L}}(x, \rho)$ consists of all $y$ that can be reached in this way in time $t < \rho$. For instance, if $\mathcal{L}$ is elliptic, then $B_{\mathcal{L}}(x, \rho)$ is essentially the ordinary (Euclidean) ball about $x$ of radius $\rho$. If we take $\hat{\mathcal{L}} = \sum_1^N Y_j^2$ on $\mathcal{N}_{N,r}$, then the balls $B_{\hat{\mathcal{L}}}(x, \rho)$ behave naturally under translations and dilations; hence they are essentially the same as the $B_{\mathcal{N}_{N,r}}(x, \rho)$ appearing in (36). Nagel-Wainger-Stein analyzed the relations between $B_{\hat{\mathcal{L}}}(x, \rho)$, $B_{\tilde{\mathcal{L}}}(x, \rho)$ and $B_{\mathcal{L}}(x, \rho)$ for an arbitrary Hörmander operator $\mathcal{L}$. (Here $\tilde{\mathcal{L}}$ and $\hat{\mathcal{L}}$ are as in our previous discussion of Rothschild-Stein.) This allowed them to integrate out the extra variables in the fundamental solution of $\tilde{\mathcal{L}}$, to derive the following sharp estimates from (36).

**Theorem**. *Suppose $X_1 \cdots X_N$ and their repeated commutators span the tangent space. Also, suppose we are in dimension greater than 2. Then the solution of $(\sum_1^N X_j^2)u = f$ is given by $u(x) = \int K(x, y) f(y) dy$ with*

$$|X_{j_1} \cdots X_{j_m} K(x, y)| \leq \frac{C_m \rho^{2-m}}{(\text{vol} B_{\mathcal{L}}(y, \rho))} \quad \textit{for} \quad x \in B_{\mathcal{L}}(y, \rho) \backslash B_{\mathcal{L}}\left(y, \frac{\rho}{2}\right)$$

and $m \geq 0$.

*Here the $X_{j_i}$ act either in the $x$- or the $y$- variable.*

Let us return from Hörmander operators to the $\bar{\partial}$-problems on strongly pseudoconvex domains $D \subset \mathbb{C}^n$. Greiner and Stein derived sharp estimates for the Neumann Laplacian $\Box w = \alpha$ in their book [78]. This problem is hard, because two different families of balls play an important role. On the one hand, the standard (Euclidean) balls arise here, because $\Box$ is simply the Laplacian in the interior of $D$. On the other hand, non-Euclidean balls (as in Folland-Stein [67]) arise on $\partial D$, because they are adapted to the non-elliptic boundary Conditions for $\Box$. Thus, any understanding of $\Box$ requires notions that are natural with respect to either family of balls. A key notion is that of an allowable vector field on $\bar{D}$. We say that a smooth vector field $X$ is allowable if its restriction to the boundry $\partial D$ lies in the span of the complex vector fields $L_1, \ldots, L_{n-1}, \bar{L}_1 \ldots \bar{L}_{n-1}$. Here we have retained the notation of our earlier discussion of $\bar{\partial}$-problems. At an interior point, an allowable vector field may point in any direction, but at a boundary point it must be in the natural codimension-one subspace of the tangent space of $\partial D$. Allowable vector fields are well-suited both to the Euclidean and the Heisenberg balls that control $\Box$. The sharp estimates of Greiner-Stein are as follows.

**Theorem.** *Suppose $\Box w = \alpha$ on a strictly pseudoconvex domain $D \subset \mathbb{C}^n$. If $\alpha$ belongs to the Sobolev space $L^p_k$, then $w$ belongs to $L^p_{k-1}(1 < p < \infty)$. Moreover, if $X$ and $Y$ are allowable vector fields, then $XYw$ belongs to $L^p_k$. Also, $\bar{L}w$ belongs to $L^p_{k+1}$ if $\bar{L}$ is a smooth complex vector field of type* $(0, 1)$. *Similarly, if $\alpha$ belongs to the Lipschitz space* $\mathrm{Lip}(\beta)$ *($0 < \beta < 1$), then the gradient of $w$ belongs to* $\mathrm{Lip}(\beta)$ *as well. Also the gradient of $\bar{L}w$ belongs to* $\mathrm{Lip}(\beta)$ *if $L$ is a smooth complex vector field of type* $(0, 1)$*; and $XYw$ belongs to* $\mathrm{Lip}(\beta)$ *for $X$ and $Y$ allowable vector fields.*

These results for allowable vector fields were new even for $L^2$. We sketch the proof.

Suppose $\Box w = \alpha$. Ignoring the boundary conditions for a moment, we have $\Delta w = \alpha$ in $D$, so

$$w = G\alpha + \mathrm{P.I.}(\tilde{w}) \tag{37}$$

where $\tilde{w}$ is defined on $\partial D$, and $G$, P.I. denote the standard Green's operator and Poisson integral, respectively. The trouble with (37) is that we know nothing about $\tilde{w}$ so far. The next step is to bring in the boundary condition for $\Box w = \alpha$. According to Calderón's work on general boundary-value problems, (37) satisfies the $\bar{\partial}$-Neumann boundary conditions if and only if

$$A\tilde{w} = \{B(G\alpha)\}|_{\partial D} \tag{38}$$

for a certain differential operater $B$ on $D$, and a certain pseudodifferential operator $A$ on $\partial D$. Both $A$ and $B$ can be determined explicitly from routine computation.

Greiner and Stein [78] derive sharp regularity theorems for the pseudodifferential equation $A\tilde{w} = g$, and then apply those results to (38) in order to understand $\tilde{w}$ in terms of $\alpha$. Once they know sharp regularity theorems for $\tilde{w}$, formula (37) gives the behavior of $w$.

Let us sketch how Greiner-Stein analyzed $A\tilde{w} = g$. This is really a system of $n$ pseudodifferential equations for $n$ unknown functions ($n = \dim \mathbb{C}^n$). In a suitable frame, one component of the system decouples from the rest of the problem (modulo negligible errors) and leads to a trivial (elliptic) pseudodifferential equation.

The non-trivial part of the problem is a first-order system of $(n-1)$ pseudodifferential operators for $(n-1)$ unknowns, which we write as

$$\Box_+ w^\# = \alpha^\#. \tag{39}$$

Here $\alpha^\#$ consists of the non-trivial components of $\{B(G\alpha)\}|_{\partial D}$, $w^\#$ is the unknown, and $\Box_+$ may be computed explicitly.

Greiner and Stein reduce (39) to the study of the Kohn-Laplacian $\Box_b$. In fact, they produce a matrix $\Box_-$ of first-order pseudodifferential operators similar to $\Box_+$, and then show that $\Box_-\Box_+ = \Box_b$ modulo negligible errors.[2] Applying $\Box_-$ to (39) yields

$$\Box_b w^\# = \Box_- \alpha^\# + \text{negligible}. \tag{40}$$

From Folland-Stein [67] one knows an explicit integral operator $K$ that inverts $\Box_b$ modulo negligible errors. Therefore,

$$w^\# = K\Box_- \alpha^\# + \text{negligible}. \tag{41}$$

Equations (37) and (41) express $w$ in terms of $\alpha$ as a composition of various explicit operators, including: the Poisson integral; restriction to the boundary; $\Box_-$; $K$; $G$. Because the basic notion of allowable vector fields is well-behaved with respect to both the natural families of balls for $\Box w = \alpha$, one can follow the effect of each of these very different operators on the relevant function spaces without losing information. To carry this out is a big job. We refer the reader to [78] for the rest of the story.

There have been important recent developments in the Stein program for several complex variables. In particular, we refer the reader to Phong's paper in *Essays in Fourier Analysis in Honor of Elias M. Stein* (Princeton University Press, 1995) for a discussion of singular Radon transforms; and to Nagel-Rosay-Stein-Wainger [131], D.-C. Chang-Nagel-Stein [132], and [McN], [Chr], [FK] for the solution of the $\bar\partial$-problems on weakly pseudoconvex domains of finite type in $\mathbb{C}^2$.

Particularly in several complex variables we are able to see in retrospect the fundamental interconnections among classical analysis, representation theory, and partial differential equations, which Stein was the first to perceive.

I hope this chapter has conveyed to the reader the order of magnitude of Stein's work. However, let me stress that it is only a selection, picking out results which I could understand and easily explain. Stein has made deep contributions to many other topics, e.g.,

Limits of sequences of operators
Extension of Littlewood-Paley Theory from the disc to $\mathbb{R}^n$
Differentiability of functions on sets of positive measure
Fourier analysis on $\mathbb{R}^N$ when $N \to \infty$
Function theory on tube domains
Analysis of diffusion semigroups
Pseudodifferential calculus for subelliptic problems.

The list continues to grow.

[2] This procedure requires significant changes in two complex variables, since then $\Box_b$ isn't invertible.

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