Chapter One

Introduction to Kähler Manifolds

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INTRODUCTION

This chapter is intended to provide an introduction to the basic results on the topology of compact Kähler manifolds that underlie and motivate Hodge theory. Although we have tried to define carefully the main objects of study, we often refer to the literature for proofs of the main results. We are fortunate in that there are several excellent books on this subject and we have freely drawn from them in the preparation of these notes, which make no claim of originality. The classical references remain the pioneering books by Weil [34], Chern [6], Morrow and Kodaira [17, 19], Wells [35], Kobayashi [15], Demailly [8], and Griffiths and Harris [10]. In these notes we refer most often to two superb recent additions to the literature: Voisin's two-volume work [30, 31] and Huybrechts' book [13].

We assume from the outset that the reader is familiar with the basic theory of smooth manifolds at the level of [1], [18], or [28]. The book by Bott and Tu [2] is an excellent introduction to the algebraic topology of smooth manifolds.

This chapter consists of five sections which correspond, roughly, to the five lectures in the course given during the Summer School at ICTP. There are also two appendices. The first collects some results on the linear algebra of complex vector spaces, Hodge structures, nilpotent linear transformations, and representations of $\mathfrak{sl}(2,\mathbb{C})$ and serves as an introduction to many other chapters in this volume. The second is due to Phillip Griffiths and contains a new proof of the Kähler identities by reduction to the symplectic case.

There are many exercises interspersed throughout the text, many of which ask the reader to prove or complete the proof of some result in the notes.

I am grateful to Loring Tu for his careful reading of this chapter.

1.1 COMPLEX MANIFOLDS

1.1.1 Definition and Examples

Let $U \subset \mathbb{C}^n$ be an open subset and $f\colon U \to \mathbb{C}$. We say that f is *holomorphic* if and only if it is holomorphic as a function of each variable separately; i.e., if we fix $z_\ell = a_\ell$, $\ell \neq j$, then the function $f(a_1, \ldots, z_j, \ldots, a_n)$ is a holomorphic function of z_j. A map $F = (f_1, \ldots, f_n)\colon U \to \mathbb{C}^n$ is said to be *holomorphic* if each component $f_k = f_k(z_1, \ldots, z_n)$ is holomorphic. If we identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$, and set $z_j = x_j + iy_j$, $f_k = u_k + iv_k$, $j, k = 1, \ldots, n$, then the functions u_k, v_k are C^∞ functions of the variables $x_1, y_1, \ldots, x_n, y_n$ and satisfy the Cauchy–Riemann equations:

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}; \quad \frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j}. \tag{1.1.1}$$

Conversely, if $(u_1, v_1, \ldots, u_n, v_n)\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a C^∞ map satisfying the Cauchy–Riemann equations (1.1.1), then the map $(u_1 + iv_1, \ldots, u_n + iv_n)$ is holomorphic. In other words, a C^∞ map $F\colon U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defines a holomorphic map $\mathbb{C}^n \to \mathbb{C}^n$ if and only if the differential DF of F, written in terms of the basis

$$\{\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n\} \tag{1.1.2}$$

of the tangent space $T_p(\mathbb{R}^{2n})$ and the basis $\{\partial/\partial u_1, \ldots, \partial/\partial u_n, \partial/\partial v_1, \ldots, \partial/\partial v_n\}$ of $T_{F(p)}(\mathbb{R}^{2n})$ is of the form

$$DF(p) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{1.1.3}$$

for all $p \in U$. Thus, it follows from Exercise A.1.4 in Appendix A that F is holomorphic if and only if $DF(p)$ defines a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^n$.

Exercise 1.1.1 Prove that a $2n \times 2n$ matrix is of the form (1.1.3) if and only if it commutes with the matrix J:

$$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{1.1.4}$$

where I_n denotes the $n \times n$ identity matrix.

Definition 1.1.2 *A* ***complex structure*** *on a topological manifold M consists of a collection of* coordinate charts (U_α, ϕ_α) *satisfying the following conditions:*

(1) The sets U_α form an open covering of M.

(2) There is an integer n such that each $\phi_\alpha\colon U_\alpha \to \mathbb{C}^n$ is a homeomorphism of U_α onto an open subset of $\mathbb{C}^n$. We call n the complex dimension of M.

(3) If $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\phi_\beta \circ \phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1.5}$$

is holomorphic.

EXAMPLE 1.1.3 The simplest example of a complex manifold is $\mathbb{C}^n$ or any open subset of $\mathbb{C}^n$. For any $p \in \mathbb{C}^n$, the tangent space $T_p(\mathbb{C}^n) \cong \mathbb{R}^{2n}$ is identified, in the natural way, with $\mathbb{C}^n$ itself.

EXAMPLE 1.1.4 Since $\mathrm{GL}(n, \mathbb{C})$, the set of nonsingular $n \times n$ matrices with complex coefficients, is an open set in $\mathbb{C}^{n^2}$, we may view $\mathrm{GL}(n, \mathbb{C})$ as a complex manifold.

EXAMPLE 1.1.5 The basic example of a compact complex manifold is complex projective space which we will denote by $\mathbb{P}^n$. Recall that

$$\mathbb{P}^n \;:=\; \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*\,,$$

where $\mathbb{C}^*$ acts by componentwise multiplication. Given $z \in \mathbb{C}^{n+1} \setminus \{0\}$, let $[z]$ be its equivalence class in $\mathbb{P}^n$. The sets

$$U_i \;:= \{[z] \in \mathbb{P}^n : z_i \neq 0\} \tag{1.1.6}$$

are open and the maps

$$\phi_i \colon U_i \to \mathbb{C}^n\,; \quad \phi_i([z]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \tag{1.1.7}$$

define local coordinates such that the maps

$$\phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \tag{1.1.8}$$

are holomorphic.

In particular, if $n = 1$, $\mathbb{P}^1$ is covered by two coordinate neighborhoods (U_0, ϕ_0), (U_1, ϕ_1) with $\phi_i(U_i) = \mathbb{C}$. The coordinate change $\phi_1 \circ \phi_0^{-1} \colon \mathbb{C}^* \to \mathbb{C}^*$ is given by

$$\phi_1 \circ \phi_0^{-1}(z) = \phi_1([(1, z)]) = \frac{1}{z}.$$

Thus, this is the usual presentation of the sphere S^2 as the Riemann sphere, where we identify U_0 with $\mathbb{C}$ and denote the point $[(0, 1)]$ by ∞.

EXERCISE 1.1.6 Verify that the map (1.1.8) is holomorphic.

EXAMPLE 1.1.7 To each point $[z] \in \mathbb{P}^n$ we may associate the line spanned by z in $\mathbb{C}^{n+1}$; hence, we may regard $\mathbb{P}^n$ as the space of lines through the origin in $\mathbb{C}^{n+1}$. This construction may then be generalized by considering k-dimensional subspaces in $\mathbb{C}^n$. In this way one obtains the Grassmann manifold $\mathcal{G}(k, n)$. To define a complex manifold structure on $\mathcal{G}(k, n)$, we consider first of all the open set in $\mathbb{C}^{nk}$,

$$V(k, n) = \{W \in \mathcal{M}(n \times k, \mathbb{C}) : \mathrm{rank}(W) = k\}.$$

The Grassmann manifold $\mathcal{G}(k, n)$ may then be viewed as the quotient space

$$\mathcal{G}(k, n) \;:=\; V(k, n)/\,\mathrm{GL}(k, \mathbb{C})\,,$$

where $\mathrm{GL}(k,\mathbb{C})$ acts by right multiplication. Thus, $W, W' \in V(k,n)$ are in the same $\mathrm{GL}(k,\mathbb{C})$-orbit if and only if the column vectors of W and W' span the same k-dimensional subspace $\Omega \subset \mathbb{C}^n$.

Given an index set $I = \{1 \le i_1 < \cdots < i_k \le n\}$ and $W \in V(k,n)$, we consider the $k \times k$ matrix W_I consisting of the I-rows of W and note that if $W \sim W'$ then, for every index set I, $\det(W_I) \neq 0$ if and only if $\det(W'_I) \neq 0$. We then define

$$U_I := \{[W] \in \mathcal{G}(k,n) : \det(W_I) \neq 0\}. \tag{1.1.9}$$

This is clearly an open set in $\mathcal{G}(k,n)$ and the map

$$\phi_I \colon U_I \to \mathbb{C}^{(n-k)k}\,; \quad \phi_I([W]) = W_{I^c} \cdot W_I^{-1}\,,$$

where I^c denotes the $(n-k)$-tuple of indices complementary to I. The map ϕ_I defines coordinates in U_I and one can easily verify that given index sets I and J, the maps

$$\phi_I \circ \phi_J^{-1} \colon \phi_J(U_I \cap U_J) \to \phi_I(U_I \cap U_J) \tag{1.1.10}$$

are holomorphic.

EXERCISE 1.1.8 Verify that the map (1.1.10) is holomorphic.

EXERCISE 1.1.9 Prove that both $\mathbb{P}^n$ and $\mathcal{G}(k,n)$ are compact.

The notion of a holomorphic map between complex manifolds is defined in a way completely analogous to that of a smooth map between C^∞ manifolds; i.e., if M and N are complex manifolds of dimension m and n respectively, a map $F \colon M \to N$ is said to be *holomorphic* if for each $p \in M$ there exist local coordinate systems (U,ϕ), (V,ψ) around p and $q = F(p)$, respectively, such that $F(U) \subset V$ and the map

$$\psi \circ F \circ \phi^{-1} \colon \phi(U) \subset \mathbb{C}^m \to \psi(V) \subset \mathbb{C}^n$$

is holomorphic. Given an open set $U \subset M$ we will denote by $\mathcal{O}(U)$ the ring of holomorphic functions $f \colon U \to \mathbb{C}$ and by $\mathcal{O}^*(U)$ the nowhere-zero holomorphic functions on U. A map between complex manifolds is said to be *biholomorphic* if it is holomorphic and has a holomorphic inverse.

The following result shows a striking difference between C^∞ and complex manifolds:

THEOREM 1.1.10 *If M is a compact, connected, complex manifold and $f \colon M \to \mathbb{C}$ is holomorphic, then f is constant.*

PROOF. The proof uses the fact that the maximum principle[1] holds for holomorphic functions of several complex variables (cf. [30, Theorem 1.21]) as well as the principle of analytic continuation[2] [30, Theorem 1.22]. □

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[1] If $f \in \mathcal{O}(U)$, where $U \subset \mathbb{C}^n$ is open, has a local maximum at $p \in U$, then f is constant in a neighborhood of p.

[2] If $U \subset \mathbb{C}^n$ is a connected, open subset and $f \in \mathcal{O}(U)$ is constant on an open subset $V \subset U$, then f is constant on U.

Given a holomorphic map $F = (f_1, \dots, f_n)\colon U \subset \mathbb{C}^n \to \mathbb{C}^n$ and $p \in U$, we can associate to F the $\mathbb{C}$-linear map

$$DF(p)\colon \mathbb{C}^n \to \mathbb{C}^n\,; \quad DF(p)(v) = \left(\frac{\partial f_i}{\partial z_j}(p)\right) \cdot v,$$

where v is the column vector $(v_1, \dots, v_n)^T \in \mathbb{C}^n$. The Cauchy–Riemann equations imply that if we regard F as a smooth map $\tilde{F}\colon U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ then the matrix of the differential $D\tilde{F}(p)$ is of the form (1.1.3) and, clearly, $DF(p)$ is nonsingular if and only if $D\tilde{F}(p)$ is nonsingular. In that case, by the inverse function theorem, $\tilde{F}$ has a local inverse $\tilde{G}$ whose differential is given by $(D\tilde{F}(p))^{-1}$. By Exercise 1.1.1, the inverse of a nonsingular matrix of the form (1.1.3) is of the same form. Hence, it follows that $\tilde{G}$ is holomorphic and, consequently, F has a local holomorphic inverse. Thus we have:

THEOREM 1.1.11 (Holomorphic inverse function theorem) *Let $F\colon U \to V$ be a holomorphic map between open subsets $U, V \subset \mathbb{C}^n$. If $DF(p)$ is nonsingular for $p \in U$ then there exist open sets U', V' such that $p \in U' \subset U$ and $F(p) \in V' \subset V$ and such that $F\colon U' \to V'$ is a biholomorphic map.*

The fact that we have a holomorphic version of the inverse function theorem means that we may also extend the implicit function theorem or, more generally, the rank theorem:

THEOREM 1.1.12 (Rank theorem) *Let $F\colon U \to V$ be a holomorphic map between open subsets $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$. If $DF(q)$ has rank k for all $q \in U$ then, given $p \in U$, there exist open sets U', V' such that $p \in U' \subset U$, $F(p) \in V' \subset V$, $F(U') \subset V'$, and biholomorphic maps $\phi\colon U' \to A$, $\psi\colon V' \to B$, where A and B are open sets of the origin in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, so that the composition*

$$\psi \circ F \circ \phi^{-1}\colon A \to B$$

is the map $(z_1, \dots, z_n) \in A \mapsto (z_1, \dots, z_k, 0, \dots, 0)$.

PROOF. We refer to [1, Theorem 7.1] or [28] for a proof in the C^∞ case which can easily be generalized to the holomorphic case. □

Given a holomorphic map $F\colon M \to N$ between complex manifolds and $p \in M$, we may define the rank of F at p as

$$\operatorname{rank}_p(F) \;:=\; \operatorname{rank}(D(\psi \circ F \circ \phi^{-1})(\phi(p)))\,, \tag{1.1.11}$$

for any local-coordinates expression of F around p.

EXERCISE 1.1.13 Prove that $\operatorname{rank}_p(F)$ is well defined by (1.1.11); i.e., it is independent of the choices of local coordinates.

We then have the following consequence of the rank theorem:

THEOREM 1.1.14 *Let $F\colon M \to N$ be a holomorphic map, let $q \in F(M)$ and let $X = F^{-1}(q)$. Suppose $\operatorname{rank}_x(F) = k$ for all x in an open set U containing X. Then, X is a complex manifold and*

$$\operatorname{codim}(X) \;:=\; \dim M - \dim X \;=\; k\,.$$

PROOF. The rank theorem implies that given $p \in X$ there exist local coordinates (U, ϕ) and (V, ψ) around p and q, respectively, such that $\psi(q) = 0$ and

$$\psi \circ F \circ \phi^{-1}(z_1, \ldots, z_n) = (z_1, \ldots, z_k, 0, \ldots, 0).$$

Hence

$$\phi(U \cap X) = \{z \in \phi(U) : z_1 = \cdots = z_k = 0\}.$$

Hence $(U \cap X, p \circ \phi)$, where p denotes the projection onto the last $n - k$ coordinates in $\mathbb{C}^n$, defines local coordinates on X. It is easy to check that these coordinates are holomorphically compatible. □

DEFINITION 1.1.15 *We will say that $N \subset M$ is a* ***complex submanifold*** *if we may cover M with coordinate patches (U_α, ϕ_α) such that*

$$\phi_\alpha(N \cap U_\alpha) \;=\; \{z \in \phi_\alpha(U) : z_1 = \cdots = z_k = 0\},$$

for some fixed k. In this case, as we saw above, N has the structure of an $(n - k)$-dimensional complex manifold.

PROPOSITION 1.1.16 *There are no compact complex submanifolds of $\mathbb{C}^n$ of dimension greater than 0.*

PROOF. Suppose $M \subset \mathbb{C}^n$ is a submanifold. Then, each of the coordinate functions z_i restricts to a holomorphic function on M. But, if M is compact, it follows from Theorem 1.1.10 that z_i must be locally constant. Hence, $\dim M = 0$. □

Remark. The above result means that there is no chance for a Whitney embedding theorem in the holomorphic category. One of the major results of the theory of complex manifolds is the Kodaira embedding theorem (Theorem 1.3.14) which gives necessary and sufficient conditions for a compact complex manifold to embed in $\mathbb{P}^n$.

EXAMPLE 1.1.17 Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and suppose $Z = f^{-1}(0) \neq \emptyset$. Then we say that 0 is a *regular value* for f if $\operatorname{rank}_p(f) = 1$ for all $p \in Z$; i.e., for each $p \in X$ there exists some $i = 1, \ldots, n$, such that $\partial f/\partial z_i(p) \neq 0$. In this case, Z is a complex submanifold of $\mathbb{C}^n$ and $\operatorname{codim}(Z) = 1$. We call Z an *affine hypersurface*. More generally, given $F\colon \mathbb{C}^n \to \mathbb{C}^m$, we say that 0 is a *regular value* if $\operatorname{rank}_p(F) = m$ for all $p \in F^{-1}(0)$. In this case $F^{-1}(0)$ is either empty or is a submanifold of $\mathbb{C}^n$ of codimension m.

EXAMPLE 1.1.18 Let $P(z_0, \ldots, z_n)$ be a homogeneous polynomial of degree d. We set

$$X \ := \ \{[z] \in \mathbb{P}^n : P(z_0, \ldots, z_n) = 0\}.$$

We note that while P does not define a function on $\mathbb{P}^n$, the zero locus X is still well defined since P is a homogeneous polynomial. We assume now that the following regularity condition holds:

$$\left\{ z \in \mathbb{C}^{n+1} : \frac{\partial P}{\partial z_0}(z) = \cdots = \frac{\partial P}{\partial z_n}(z) = 0 \right\} = \{0\}; \tag{1.1.12}$$

i.e., 0 is a regular value of the map $P|_{\mathbb{C}^{n+1}\setminus\{0\}}$. Then X is a hypersurface in $\mathbb{P}^n$.

To prove this we note that the requirements of Definition 1.1.15 are local. Hence, it is enough to check that $X \cap U_i$ is a submanifold of U_i for each i, in fact, that it is an affine hypersurface. Consider the case $i = 0$ and let $f\colon U_0 \cong \mathbb{C}^n \to \mathbb{C}$ be the function $f(u_1, \ldots, u_n) = P(1, u_1, \ldots, u_n)$. Set $u = (u_1, \ldots, u_n)$ and $\tilde{u} = (1, u_1, \ldots, u_n)$. Suppose $[\tilde{u}] \in U_0 \cap X$ and

$$\frac{\partial f}{\partial u_1}(u) = \cdots = \frac{\partial f}{\partial u_n}(u) = 0.$$

Then

$$\frac{\partial P}{\partial z_1}(\tilde{u}) = \cdots = \frac{\partial P}{\partial z_n}(\tilde{u}) = 0.$$

But, since P is a homogeneous polynomial of degree d, it follows from the Euler identity that

$$0 \ = \ d \cdot P(\tilde{u}) \ = \ \frac{\partial P}{\partial z_0}(\tilde{u}).$$

Hence, by (1.1.12), we would have $\tilde{u} = 0$, which is impossible. Hence 0 is a regular value of f and $X \cap U_0$ is an affine hypersurface.

EXERCISE 1.1.19 Let $P_1(z_0, \ldots, z_n), \ldots, P_m(z_0, \ldots, z_n)$ be homogeneous polynomials. Suppose that 0 is a regular value of the map

$$(P_1, \ldots, P_m)\colon \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}^m.$$

Prove that

$$X = \{[z] \in \mathbb{P}^n : P_1([z]) = \cdots = P_m([z]) = 0\}$$

is a codimension-m submanifold of $\mathbb{P}^n$. X is called a complete intersection submanifold.

EXAMPLE 1.1.20 Consider the Grassmann manifold $\mathcal{G}(k, n)$ and let $I_1, \ldots, I_{\binom{n}{k}}$ denote all strictly increasing k-tuples $I \subset \{1, \ldots, n\}$. We then define

$$\mathfrak{p}\colon \mathcal{G}(k, n) \to \mathbb{P}^{N-1}; \quad \mathfrak{p}([W]) = [(\det(W_{I_1}), \ldots, \det(W_{I_N}))].$$

Note that the map $\mathfrak{p}$ is well defined since $W \sim W'$ implies that $W' = W \cdot M$ with $M \in \mathrm{GL}(k, \mathbb{C})$, and then for any index set I, $\det(W'_I) = \det(M)\det(W_I)$. We leave it to the reader to verify that the map $\mathfrak{p}$, which is usually called the *Plücker map*, is holomorphic.

EXERCISE 1.1.21 Consider the Plücker map $\mathfrak{p}\colon \mathcal{G}(2,4) \to \mathbb{P}^5$ and suppose that the index sets $I_1, \dots, I_6$ are ordered lexicographically. Show that $\mathfrak{p}$ is a 1:1 holomorphic map from $\mathcal{G}(2,4)$ onto the subset

$$X = \{[z_0, \dots, z_5] : z_0 z_5 - z_1 z_4 + z_2 z_3 = 0\}. \tag{1.1.13}$$

Prove that X is a hypersurface in $\mathbb{P}^5$. Compute $\operatorname{rank}_{[W]} \mathfrak{p}$ for $[W] \in \mathcal{G}(2,4)$.

EXAMPLE 1.1.22 We may define complex Lie groups in a manner completely analogous to the real, smooth case. A *complex Lie group* is a complex manifold G with a group structure such that the group operations are holomorphic. The basic example of a complex Lie group is $\mathrm{GL}(n,\mathbb{C})$. We have already observed that $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n^2}$ and the product of matrices is given by polynomial functions, while the inverse of a matrix is given by rational functions on the entries of the matrix. Other classical examples include the *special linear group* $\mathrm{SL}(n,\mathbb{C})$ and the *symplectic group* $\mathrm{Sp}(n,\mathbb{C})$. We recall the definition of the latter. Let Q be a symplectic form (cf. Definition B.1.1) on the $2n$-dimensional real vector space V, then

$$\mathrm{Sp}(V_{\mathbb{C}}, Q) \;:=\; \{X \in \mathrm{End}(V_{\mathbb{C}}) : Q(Xu, Xv) = Q(u,v)\}. \tag{1.1.14}$$

We define $\mathrm{Sp}(V,Q)$ analogously. When $V = \mathbb{R}^{2n}$ and Q is defined by the matrix (1.1.4) we will denote these groups by $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{R})$. The choice of a symplectic basis for Q, as in (B.1.2), establishes isomorphisms $\mathrm{Sp}(V_{\mathbb{C}}, Q) \cong \mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{Sp}(V,Q) \cong \mathrm{Sp}(n,\mathbb{R})$.

EXAMPLE 1.1.23 Let Q be a symplectic structure on a $2n$-dimensional, real vector space V. Consider the space

$$M = \{\Omega \in \mathcal{G}(n, V_{\mathbb{C}}) : Q(u,v) = 0 \text{ for all } u, v \in \Omega\}.$$

Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ be a basis of V in which the matrix of Q is as in (1.1.4). Then if

$$\Omega = [W] \;=\; \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

where W_1 and W_2 are $n \times n$ matrices, we have that $\Omega \in M$ if and only if

$$[W_1^T, W_2^T] \cdot \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \;=\; W_2^T \cdot W_1 - W_1^T \cdot W_2 \;=\; 0.$$

Set $I_0 = \{1, \dots, n\}$. Every element $\Omega \in M \cap U_{I_0}$, where U_{I_0} is as in (1.1.9), may be represented by a matrix of the form $\Omega = [I_n, Z]^T$ with $Z^T = Z$. It follows that $M \cap U_{I_0}$ is an $(n(n+1)/2)$-dimensional submanifold. Now, given an arbitrary $\Omega \in M$, there exists an element $X \in \mathrm{Sp}(V_{\mathbb{C}}, Q)$ such that $X \cdot \Omega = \Omega_0$, where $\Omega_0 = \mathrm{span}(e_1, \dots, e_n)$. Since the elements of $\mathrm{Sp}(V_{\mathbb{C}}, Q)$ act by biholomorphisms on $\mathcal{G}(n, V_{\mathbb{C}})$, it follows that M is an $(n(n+1)/2)$-dimensional submanifold of $\mathcal{G}(n, V_{\mathbb{C}})$. Moreover, since M is a closed submanifold of the compact manifold $\mathcal{G}(k,n)$, M is also compact.

We will also be interested in considering the open set $D \subset M$ consisting of

$$D = D(V, Q) \;:=\; \{\Omega \in M : i\,Q(w, \bar{w}) > 0 \text{ for all } 0 \neq w \in \Omega\}. \tag{1.1.15}$$

It follows that $\Omega \in D$ if and only if the Hermitian matrix

$$i \cdot \left[\bar{W}_1^T, \bar{W}_2^T\right] \cdot \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \;=\; i(\bar{W}_2^T \cdot W_1 - \bar{W}_1^T \cdot W_2)$$

is positive definite. Note that, in particular, $D \subset U_{I_0}$ and that

$$D \;\cong\; \{Z \in \mathcal{M}(n, \mathbb{C}) : Z^T = Z\,;\; \mathrm{Im}(Z) = (1/2i)(Z - \bar{Z}) > 0\}, \tag{1.1.16}$$

where $\mathcal{M}(n, \mathbb{C})$ denotes the $n \times n$ complex matrices. If $n = 1$ then $M \cong \mathbb{C}$ and D is the upper half plane. We will call D the generalized *Siegel upper half-space*.

The elements of the complex lie group $\mathrm{Sp}(V_{\mathbb{C}}, Q) \cong \mathrm{Sp}(n, \mathbb{C})$ define biholomorphisms of $\mathcal{G}(n, V_{\mathbb{C}})$ preserving M. The subgroup

$$\mathrm{Sp}(V, Q) = \mathrm{Sp}(V_{\mathbb{C}}, Q) \cap \mathrm{GL}(V) \cong \mathrm{Sp}(n, \mathbb{R})$$

preserves D.

EXERCISE 1.1.24 Prove that relative to the description of D as in (1.1.16), the action of $\mathrm{Sp}(V, Q)$ is given by generalized fractional linear transformations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z \;=\; (A \cdot Z + B) \cdot (C \cdot Z + D)^{-1}.$$

EXERCISE 1.1.25 Prove that the action of $\mathrm{Sp}(V, Q)$ on D is *transitive* in the sense that given any two points $\Omega, \Omega' \in D$, there exists $X \in \mathrm{Sp}(V, Q)$ such that $X \cdot \Omega = \Omega'$.

EXERCISE 1.1.26 Compute the *isotropy subgroup*

$$K \;:=\; \{X \in \mathrm{Sp}(V, Q) : X \cdot \Omega_0 = \Omega_0\},$$

where $\Omega_0 = [I_n, i\,I_n]^T$.

EXAMPLE 1.1.27 Let $T_\Lambda := \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{Z}^2$ is a rank-2 lattice in $\mathbb{C}$; i.e.,

$$\Lambda = \{m\,\omega_1 + n\,\omega_2\;;\; m, n \in \mathbb{Z}\},$$

where ω_1, ω_2 are complex numbers linearly independent over $\mathbb{R}$. T_Λ is locally diffeomorphic to $\mathbb{C}$ and since the translations by elements in Λ are biholomorphisms of $\mathbb{C}$, T_Λ inherits a complex structure relative to which the natural projection

$$\pi_\Lambda \colon \mathbb{C} \to T_\Lambda$$

is a local biholomorphic map.

It is natural to ask whether, for different lattices Λ, Λ', the *complex tori* T_Λ, $T_{\Lambda'}$ are biholomorphic. Suppose $F\colon T_\Lambda \to T_{\Lambda'}$ is a biholomorphism. Then, since $\mathbb{C}$ is the universal covering of T_Λ, there exists a map $\tilde{F}\colon \mathbb{C} \to \mathbb{C}$ such that the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{F}"] \arrow[d, "\pi_\Lambda"'] & \mathbb{C} \arrow[d, "\pi_{\Lambda'}"] \\ \mathbb{C}/\Lambda \arrow[r, "F"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

commutes. In particular, given $z \in \mathbb{C}$, $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda'$ such that

$$\tilde{F}(z+\lambda) \;=\; \tilde{F}(z) + \lambda'.$$

This means that the derivative $\tilde{F}'$ must be Λ-periodic and, hence, it defines a holomorphic function on $\mathbb{C}/\Lambda$ which, by Theorem 1.1.10, must be constant. This implies that $\tilde{F}$ must be a linear map and, after translation if necessary, we may assume that $\tilde{F}(z) = \mu \cdot z$, $\mu = a + ib \in \mathbb{C}$. Conversely, any such linear map $\tilde{F}$ induces a biholomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\tilde{F}(\Lambda)$. In particular, if $\{\omega_1, \omega_2\}$ is a $\mathbb{Z}$-basis of Λ then $\operatorname{Im}(\omega_2/\omega_1) \neq 0$ and we may assume without loss of generality that $\operatorname{Im}(\omega_2/\omega_1) > 0$. Setting $\tau = \omega_2/\omega_1$ we see that T_Λ is always biholomorphic to a torus T_τ associated with a lattice

$$\{m + n\tau \;;\; m, n \in \mathbb{Z}\}$$

with $\operatorname{Im}(\tau) > 0$.

Now, suppose the tori T_Λ, $T_{\Lambda'}$ are biholomorphic and let $\{\omega_1, \omega_2\}$ (resp. $\{\omega_1', \omega_2'\}$) be a $\mathbb{Z}$-basis of Λ (resp. Λ') as above. We have

$$\mu \cdot \omega_1 = m_{11}\omega_1' + m_{21}\omega_2' \;; \quad \mu \cdot \omega_2 = m_{12}\omega_1' + m_{22}\omega_2' \,, \quad m_{ij} \in \mathbb{Z}.$$

Moreover, $m_{11}m_{22} - m_{12}m_{21} = 1$, since F is biholomorphic and therefore $\tilde{F}(\Lambda) = \Lambda'$. Hence

$$\tau \;=\; \frac{\omega_1}{\omega_2} \;=\; \frac{m_{11}\omega_1' + m_{21}\omega_2'}{m_{12}\omega_1' + m_{22}\omega_2'} \;=\; \frac{m_{11} + m_{21}\tau'}{m_{12} + m_{22}\tau'}.$$

Consequently, $T_\tau \cong T_{\tau'}$ if and only if τ and τ' are points in the upper half plane congruent under the action of the group $\mathrm{SL}(2, \mathbb{Z})$ by fractional linear transformations. We refer to Section 4.2 for a fuller discussion of this example.

Remark. Note that while all differentiable structures on the torus $S^1 \times S^1$ are equivalent, there is a continuous *moduli space* of different complex structures. This is one of the key differences between real and complex geometry and one which we will study using Hodge theory.

1.1.2 Holomorphic Vector Bundles

We may extend the notion of a smooth vector bundle to complex manifolds and holomorphic maps.

Definition 1.1.28 *A* ***holomorphic vector bundle*** E *over a complex manifold* M *is a complex manifold* E *together with a holomorphic map* $\pi\colon E \to M$ *such that*

(1) for each $x \in M$*, the* fiber $E_x = \pi^{-1}(x)$ *is a complex vector space of dimension* d *(the* rank *of* E*);*

(2) there exist an open covering $\{U_\alpha\}$ *of* M *and biholomorphic maps*

$$\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^d$$

such that

(a) $p_1(\Phi_\alpha(x)) = x$ *for all* $x \in U$*, where* $p_1\colon U_\alpha \times \mathbb{C}^d \to U_\alpha$ *denotes projection on the first factor; and*

(b) for every $x \in U_\alpha$*, the map* $p_2 \circ \Phi|_{E_x}\colon E_x \to \mathbb{C}^d$ *is an isomorphism of complex vector spaces.*

We call E the *total space* of the bundle and M its *base*. The covering $\{U_\alpha\}$ is called a *trivializing cover* of M and the biholomorphisms $\{\Phi_\alpha\}$ *local trivializations*. When $d = 1$ we often refer to E as a *line bundle*.

We note that as in the case of smooth vector bundles, a holomorphic vector bundle may be described by *transition functions*, i.e., by a covering of M by open sets U_α together with holomorphic maps

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(d, \mathbb{C})$$

such that

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \tag{1.1.17}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. The maps $g_{\alpha\beta}$ are defined by the following commutative diagram:

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "\Phi_\beta"'] \arrow[dr, "\Phi_\alpha"] & \\ (U_\alpha \cap U_\beta) \times \mathbb{C}^d \arrow[rr, "(\mathrm{id}, g_{\alpha\beta})"] & & (U_\alpha \cap U_\beta) \times \mathbb{C}^d. \end{tikzcd} \tag{1.1.18}$$

In particular, a holomorphic line bundle over M is given by a collection $\{U_\alpha, g_{\alpha\beta}\}$, where U_α is an open cover of M and the $\{g_{\alpha\beta}\}$ are nowhere-zero holomorphic functions defined on $U_\alpha \cap U_\beta$, i.e., $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying the *cocycle condition* (1.1.17).

Example 1.1.29 The product $M \times \mathbb{C}^d$ with the natural projection may be viewed as vector bundle of rank d over the complex manifold M. It is called the *trivial bundle* over M.

EXAMPLE 1.1.30 We consider the *tautological* line bundle over $\mathbb{P}^n$. This is the bundle whose fiber over a point in $\mathbb{P}^n$ is the line in $\mathbb{C}^{n+1}$ defined by that point. More precisely, let

$$\mathcal{T} := \{([z], v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v = \lambda z, \lambda \in \mathbb{C}\},$$

and let $\pi\colon \mathcal{T} \to \mathbb{P}^n$ be the projection to the first factor. Let U_i be as in (1.1.6). Then we can define

$$\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$

by

$$\Phi_i([z], v) = ([z], v_i).$$

The transition functions g_{ij} are defined by the diagram (1.1.18) and we have

$$\Phi_i \circ \Phi_j^{-1}([z], 1) = \Phi_i([z], (z_0/z_j, \ldots, 1, \ldots, z_n/z_j)) = ([z], z_i/z_j),$$

with the 1 in the jth position. Hence,

$$g_{ij}\colon U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$$

is the map $[z] \mapsto z_i/z_j$. It is common to denote the tautological bundle as $\mathcal{O}(-1)$.

EXERCISE 1.1.31 Generalize the construction of the tautological bundle over projective space to obtain the universal rank-k bundle over the Grassmann manifold $\mathcal{G}(k, n)$. Consider the space

$$\mathcal{U} := \{(\Omega, v) \in \mathcal{G}(k, n) \times \mathbb{C}^n : v \in \Omega\}, \tag{1.1.19}$$

where we regard $\Omega \in \mathcal{G}(k, n)$ as a k-dimensional subspace of $\mathbb{C}^n$. Prove that $\mathcal{U}$ may be trivialized over the open sets U_I defined in Example 1.1.7 and compute the transition functions relative to these trivializations.

Let $\pi\colon E \to M$ be a holomorphic vector bundle and suppose $F\colon N \to M$ is a holomorphic map. Given a trivializing cover $\{(U_\alpha, \Phi_a)\}$ of E with transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(d, \mathbb{C})$, we define

$$h_{\alpha\beta}\colon F^{-1}(U_\alpha) \cap F^{-1}(U_\beta) \to \mathrm{GL}(d, \mathbb{C})\,; \quad h_{\alpha\beta} := g_{\alpha\beta} \circ F. \tag{1.1.20}$$

It is easy to check that the functions $h_{\alpha\beta}$ satisfy the *cocycle condition* (1.1.17) and, therefore, define a holomorphic vector bundle over N denoted by $F^*(E)$, and called the *pull-back* bundle. Note that we have a commutative diagram:

$$\begin{array}{ccc} F^*(E) & \xrightarrow{\bar{\mathrm{F}}} & E \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\mathrm{F}} & M. \end{array} \tag{1.1.21}$$

If L and L' are line bundles and $g^L_{\alpha\beta}$, $g^{L'}_{\alpha\beta}$ are their transition functions relative to a common trivializing cover, then the functions

$$h_{\alpha\beta} = g^L_{\alpha\beta} \cdot g^{L'}_{\alpha\beta}$$

satisfy (1.1.17) and define a new line bundle which we denote by $L \otimes L'$.

Similarly, the functions

$$h_{\alpha\beta} = (g^L_{\alpha\beta})^{-1}$$

also satisfy (1.1.17) and define a bundle, called the *dual bundle* of L and denoted by L^* or L^{-1}. Clearly $L \otimes L^*$ is the trivial line bundle over M. The dual bundle of the tautological bundle is called the *hyperplane bundle* over $\mathbb{P}^n$ and denoted by H or $\mathcal{O}(1)$. Note that the transition functions of H are $g^H_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ defined by

$$g^H_{ij}([z]) \;:=\; z_j/z_i. \tag{1.1.22}$$

We may also extend the notion of sections to holomorphic vector bundles:

DEFINITION 1.1.32 *A* ***holomorphic section*** *of a holomorphic vector bundle* $\pi\colon E \to M$ *over an open set* $U \subset M$ *is a holomorphic map*

$$\sigma\colon U \to E$$

such that

$$\pi \circ \sigma = \mathrm{id}|_U\,. \tag{1.1.23}$$

The sections of E *over* U *form an* $\mathcal{O}(U)$*-module which will be denoted by* $\mathcal{O}(U, E)$*. The local sections over* U *of the trivial line bundle are precisely the ring* $\mathcal{O}(U)$*.*

If $\pi\colon L \to M$ is a holomorphic line bundle and $g_{\alpha\beta}$ are the transition functions associated to a trivializing covering (U_α, Φ_α), then a section $\sigma\colon M \to L$ may be described by a collection of holomorphic functions $f_\alpha \in \mathcal{O}(U_\alpha)$ defined by

$$\sigma(x) = f_\alpha(x)\Phi_\alpha^{-1}(x, 1).$$

Hence, for $x \in U_\alpha \cap U_\beta$ we must have

$$f_\alpha(x) = g_{\alpha\beta}(x) \cdot f_\beta(x). \tag{1.1.24}$$

EXAMPLE 1.1.33 Let $M = \mathbb{P}^n$ and let $U_i = \{[z] \in \mathbb{P}^n : z_i \neq 0\}$. Let $P \in \mathbb{C}[z_0, \ldots, z_n]$ be a homogeneous polynomial of degree d. For each $i = 0, \ldots, n$ define

$$f_i([z]) = \frac{P(z)}{z_i^d} \in \mathcal{O}(U_i).$$

In $U_i \cap U_j$, we then have

$$z_i^d \cdot f_i([z]) = P(z) = z_j^d \cdot f_j([z]),$$

and therefore

$$f_i([z]) \;=\; (z_j/z_i)^d \cdot f_j([z]).$$

This means that we can consider the polynomial $P(z)$ as defining a section of the line bundle over $\mathbb{P}^n$ with transition functions

$$g_{ij} \;=\; (z_j/z_i)^d\,,$$

that is, of the bundle $H^d = \mathcal{O}(d)$. In fact, it is possible to prove that every global holomorphic section of the bundle $\mathcal{O}(d)$ is defined, as above, by a homogeneous polynomial of degree d. The proof of this fact requires Hartogs' theorem [13, Proposition 1.1.14] from the theory of holomorphic functions of several complex variables. We refer to [13, Proposition 2.4.1].

We note that, on the other hand, the tautological bundle has no nontrivial global holomorphic sections. Indeed, suppose $\sigma \in \mathcal{O}(\mathbb{P}^n, \mathcal{O}(-1))$ and let ℓ denote the global section of $\mathcal{O}(1)$ associated to a nonzero linear form ℓ. Then, the map

$$[z] \in \mathbb{P}^n \mapsto \ell([z])\sigma([z])$$

defines a global holomorphic function on the compact complex manifold $\mathbb{P}^n$, hence it must be constant. If that constant is nonzero then both σ and ℓ are nowhere zero which would imply that both $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ are trivial bundles. Hence σ must be identically zero.

Note that given a section $\sigma\colon M \to E$ of a vector bundle E, the zero locus $\{x \in M : \sigma(x) = 0\}$ is a well-defined subset of M. Thus, we may view the projective hypersurface defined in Example 1.1.18 by a homogeneous polynomial of degree d as the zero locus of a section of $\mathcal{O}(d)$.

Remark. The discussion above means that one should think of sections of line bundles as locally defined holomorphic functions satisfying a suitable compatibility condition. Given a compact, connected, complex manifold, global sections of holomorphic line bundles (when they exist) often play the role that global smooth functions play in the study of smooth manifolds. In particular, one uses sections of line bundles to define embeddings of compact complex manifolds into projective space. This vague observation will be made precise later in the chapter.

Given a holomorphic vector bundle $\pi\colon E \to M$ and a local trivialization

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^d,$$

we may define a basis of local sections of E over U (a *local frame*) as follows. Let $e_1, \dots, e_d$ denote the standard basis of $\mathbb{C}^d$ and for $x \in U$ set

$$\sigma_j(x) \ := \ \Phi^{-1}(x, e_j);\ j = 1, \dots, d.$$

Then $\sigma_j(x) \in \mathcal{O}(U, E)$ and for each $x \in U$, the vectors $\sigma_1(x), \dots, \sigma_d(x)$ are a basis of the d-dimensional vector space E_x (they are the image of the basis $e_1, \dots, e_d$ by a linear isomorphism). In particular, if $\tau\colon U \to M$ is a map satisfying (1.1.23) we can write

$$\tau(x) \ = \ \sum_{j=1}^{d} f_j(x)\sigma_j(x)$$

and τ is holomorphic (resp. smooth) if and only if the functions $f_j \in \mathcal{O}(U)$ (resp. $f_j \in C^\infty(U)$).

Conversely, suppose $U \subset M$ is an open set and let $\sigma_1, \ldots, \sigma_d \in \mathcal{O}(U, E)$ be a *local frame*, i.e., holomorphic sections such that for each $x \in U$, we have $\sigma_1(x), \ldots, \sigma_d(x)$ is a basis of E_x. Then we may define a local trivialization

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^d$$

by

$$\Phi(v) \;:=\; (\pi(v), (\lambda_1, \ldots, \lambda_d)),$$

where $v \in \pi^{-1}(U)$ and

$$v \;=\; \sum_{j=1}^{d} \lambda_j\, \sigma_j(\pi(v)).$$

1.2 DIFFERENTIAL FORMS ON COMPLEX MANIFOLDS

1.2.1 Almost Complex Manifolds

Let M be a complex manifold and (U_α, ϕ_α) coordinate charts covering M. Since the change-of-coordinate maps (1.1.5) are holomorphic, the matrix of the differential $D(\phi_\beta \circ \phi_\alpha^{-1})$ is of the form (1.1.3). This means that the map

$$J_p\colon T_p(M) \to T_p(M)$$

defined by

$$J\left(\frac{\partial}{\partial x_j}\right) \;:=\; \frac{\partial}{\partial y_j}\,; \quad J\left(\frac{\partial}{\partial y_j}\right) \;:=\; -\frac{\partial}{\partial x_j} \tag{1.2.1}$$

is well defined. We note that J is a smooth $(1,1)$ tensor on M such that $J^2 = -I$ and therefore, for each $p \in M$, then J_p defines a complex structure on the real vector space $T_p(M)$ (cf. (A.1.4)).

Definition 1.2.1 *An **almost complex structure** on a C^∞ (real) manifold M is a $(1,1)$ tensor J such that $J^2 = -I$. An* almost complex manifold *is a pair (M, J) where J is an almost complex structure on M. The almost complex structure J is said to be* integrable *if M has a complex structure inducing J.*

If (M, J) is an almost complex manifold then J_p is a complex structure on T_pM and therefore by Proposition A.1.2, M must be even-dimensional. Note also that (A.1.10) implies that if M has an almost complex structure then M is orientable.

Exercise 1.2.2 Let M be an orientable (and oriented) two-dimensional manifold and let $\langle\ ,\ \rangle$ be a Riemannian metric on M. Given $p \in M$ let $v_1, v_2 \in T_p(M)$ be a positively oriented orthonormal basis. Prove that $J_p\colon T_p(M) \to T_p(M)$ defined by

$$J_p(v_1) = v_2\,; \quad J_p(v_2) = -v_1$$

defines an almost complex structure on M. Show, moreover, that if $\langle\langle\ ,\ \rangle\rangle$ is a Riemannian metric conformally equivalent to $\langle\ ,\ \rangle$, then the two metrics define the same almost complex structure.

The discussion above shows that if M is a complex manifold then the operator (1.2.1) defines an almost complex structure. Conversely, the Newlander–Nirenberg theorem gives a necessary and sufficient condition for an almost complex structure J to arise from a complex structure. This is given in terms of the *Nijenhuis* torsion of J:

EXERCISE 1.2.3 Let J be an almost complex structure on M. Prove that

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y] \tag{1.2.2}$$

is a $(1, 2)$ tensor satisfying $N(X, Y) = -N(Y, X)$. The tensor N is called the *torsion* of J.

EXERCISE 1.2.4 Let J be an almost complex structure on a two-dimensional manifold M. Prove that $N(X, Y) = 0$ for all vector fields X and Y on M.

THEOREM 1.2.5 (Newlander–Nirenberg [20]) *Let (M, J) be an almost complex manifold, then M has a complex structure inducing the almost complex structure J if and only if $N(X, Y) = 0$ for all vector fields X and Y on M.*

PROOF. We refer to [34, Proposition 2], [30, Section 2.2.3] for a proof in the special case when M is a real analytic manifold. □

Remark. Note that assuming the Newlander–Nirenberg theorem, it follows from Exercise 1.2.4 that the almost complex structure constructed in Exercise 1.2.2 is integrable. We may explicitly construct the complex structure on M by using local isothermal coordinates. Thus, a complex structure on an oriented, two-dimensional manifold M is equivalent to a Riemannian metric up to conformal equivalence.

In what follows we will be interested in studying complex manifolds; however, the notion of almost complex structures gives a very convenient way to distinguish those properties of complex manifolds that depend only on having a (smoothly varying) complex structure on each tangent space. Thus, we will not explore in depth the theory of almost complex manifolds except to note that there are many examples of almost complex structures which are not integrable, that is, do not come from a complex structure. One may also ask which even-dimensional orientable manifolds admit almost complex structures. For example, in the case of a sphere S^{2n}, it was shown by Borel and Serre that only S^2 and S^6 admit almost complex structures. This is related to the fact that S^1, S^3, and S^7 are the only parallelizable spheres. We point out that while it is easy to show that S^6 has a nonintegrable almost complex structure, it is still unknown whether S^6 has a complex structure.

1.2.2 Tangent and Cotangent Space

Let (M, J) be an almost complex manifold and $p \in M$. Let $T_p(M)$ denote the tangent space of M. Then J_p defines a complex structure on $T_p(M)$ and therefore, by Proposition A.1.2, the complexification $T_{p,\mathbb{C}}(M) := T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as

$$T_{p,\mathbb{C}}(M) = T_p'(M) \oplus T_p''(M),$$

where $T_p''(M) = \overline{T_p'(M)}$ and $T_p'(M)$ is the i-eigenspace of J_p acting on $T_{p,\mathbb{C}}(M)$. Moreover, by Proposition A.1.3, the map $v \in T_p(M) \mapsto v - iJ_p(v)$ defines an isomorphism of complex vector spaces $(T_p(M), J_p) \cong T_p'(M)$.

If J is integrable, then given holomorphic local coordinates $\{z_1, \dots, z_n\}$ around p, we may consider the local coordinate frame (1.1.2) and, given (1.2.1), we have that the above isomorphism maps

$$\partial/\partial x_j \mapsto \partial/\partial x_j - i\,\partial/\partial y_j .$$

We set

$$\frac{\partial}{\partial z_j} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right) \;;\quad \frac{\partial}{\partial \bar z_j} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right) . \tag{1.2.3}$$

Then, the vectors $\partial/\partial z_j$ are a basis of the complex subspace $T_p'(M)$.

Remark. Given local coordinates $(U, \{z_1, \dots, z_n\})$ on M, a function $f\colon U \to \mathbb{C}$ is holomorphic if the local coordinates expression $f(z_1, \dots, z_n)$ satisfies the Cauchy–Riemann equations. This is equivalent to the condition

$$\frac{\partial}{\partial \bar z_j}(f) = 0$$

for all j. Moreover, in this case, $\frac{\partial}{\partial z_j}(f)$ coincides with the partial derivative of f with respect to z_j. This justifies the choice of notation. However, we point out that it makes sense to consider $\frac{\partial}{\partial z_j}(f)$ even if f is only a C^∞ function.

We will refer to $T_p'(M)$ as the holomorphic tangent space[3] of M at p. We note that if $\{z_1, \dots, z_n\}$ and $\{w_1, \dots, w_n\}$ are local complex coordinates around p then the change of basis matrix from the basis $\{\partial/\partial z_j\}$ to the basis $\{\partial/\partial w_k\}$ is given by the matrix of holomorphic functions

$$\left(\frac{\partial w_k}{\partial z_j}\right).$$

Thus, the complex vector spaces $T_p'(M)$ define a holomorphic vector bundle $T^h(M)$ over M, the *holomorphic tangent bundle*.

EXAMPLE 1.2.6 Let M be an oriented real surface with a Riemannian metric. Let (U, x, y) be positively oriented, local isothermal coordinates on M; i.e., the coordinate vector fields $\partial/\partial x$, $\partial/\partial y$ are orthogonal and of the same length. Then $z = x + iy$ defines complex coordinates on M and the vector field $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$ is a local holomorphic section of the holomorphic tangent bundle of M.

We can now characterize the tangent bundle and the holomorphic tangent bundle of $\mathbb{P}^n$:

[3] This construction makes sense even if J is not integrable. In that case, we may replace the coordinate frame (1.1.2) by a local frame $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ such that $J(X_j) = Y_j$ and $J(Y_j) = -X_j$.

THEOREM 1.2.7 *The tangent bundle $T\mathbb{P}^n$ is equivalent to the bundle*

$$\mathrm{Hom}(\mathcal{T}, E/\mathcal{T}),$$

where $E = \mathbb{P}^n \times \mathbb{C}^{n+1}$ is the trivial bundle of rank $n+1$ on $\mathbb{P}^n$ and $\mathcal{T}$ is the tautological bundle defined in Example 1.1.30. The holomorphic tangent bundle may be identified with the subbundle

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{T}, E/\mathcal{T}).$$

PROOF. We work in the holomorphic case; the statement about the smooth case follows identically. Consider the projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Given $\lambda \in \mathbb{C}^*$, let M_λ denote *multiplication by* λ in $\mathbb{C}^{n+1} \setminus \{0\}$. Then, for every $v \in \mathbb{C}^{n+1} \setminus \{0\}$ we may identify $T'(\mathbb{C}^{n+1} \setminus \{0\}) \cong \mathbb{C}^{n+1}$ and we have the following commutative diagram of $\mathbb{C}$-linear maps

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[rr, "M_\lambda"] \arrow[dr, "\pi_{*,v}"'] & & \mathbb{C}^{n+1} \arrow[dl, "\pi_{*,\lambda v}"] \\ & T_{[v]}(\mathbb{P}^n). & \end{tikzcd}$$

Now, the map $\pi_{*,v}\colon \mathbb{C}^{n+1} \to T'_{[v]}(\mathbb{P}^n)$ is surjective and its kernel is the line $L = \mathbb{C} \cdot v$. Hence we get a family of $\mathbb{C}$-linear isomorphisms

$$\mathfrak{p}_v\colon \mathbb{C}^{n+1}/L \to T'_{[v]}(\mathbb{P}^n); \quad v \in L,\ v \neq 0$$

with the relation

$$\mathfrak{p}_v \;=\; \lambda\, \mathfrak{p}_{\lambda v}.$$

We can now define a map

$$\Theta : \mathrm{Hom}_{\mathbb{C}}(\mathcal{T}, E/\mathcal{T}) \to T^h\mathbb{P}^n.$$

Let

$$\xi \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{T}, E/\mathcal{T})_{[z]} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{T}_{[z]}, (E/\mathcal{T})_{[z]}) \cong \mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C}^{n+1}/L);$$

then we set

$$\Theta(\xi) \;:=\; \mathfrak{p}_v(\xi(v)) \text{ for any } v \in L, v \neq 0.$$

Note that this is well defined since

$$\mathfrak{p}_{\lambda v}(\xi(v)) \;=\; \lambda^{-1}\mathfrak{p}_v(\lambda\, \xi(v)) \;=\; \mathfrak{p}_v(\xi(v)).$$

Alternatively one may define

$$\Theta(\xi) \;=\; \frac{d}{dt}\Big|_{t=0}(\gamma(t)),$$

where $\gamma(t)$ is the holomorphic curve through $[z]$ in $\mathbb{P}^n$ defined by

$$\gamma(t) \;:=\; [v + t\xi(v)].$$

One then has to show that this map is well defined. It is straightforward, though tedious, to verify that Θ is an isomorphism of vector bundles. $\square$

EXERCISE 1.2.8 Prove that

$$T(\mathcal{G}(k,n)) \cong \mathrm{Hom}(\mathcal{U}, E/\mathcal{U}),$$
$$T^h(\mathcal{G}(k,n)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{U}, E/\mathcal{U}),$$

where $\mathcal{U}$ is the universal bundle over $\mathcal{G}(k,n)$ defined in Exercise 1.1.31 and E is the trivial bundle $E = \mathcal{G}(k,n) \times \mathbb{C}^n$.

As seen in Appendix A, a complex structure on a vector space induces a complex structure on the dual vector space. Thus, the complexification of the cotangent space $T^*_{p,\mathbb{C}}(M)$ decomposes as

$$T^*_{p,\mathbb{C}}(M) := T^{1,0}_p(M) \oplus T^{0,1}_p(M)\,; \quad T^{0,1}_p(M) = \overline{T^{1,0}_p(M)}.$$

Given local holomorphic coordinates $\{z_1, \ldots, z_n\}$, $z_j = x_j + iy_j$, the 1-forms $dz_j := dx_j + i\,dy_j$, $d\bar{z}_j = dx_j - i\,dy_j$ are the dual coframe to $\partial/\partial z_1, \ldots, \partial/\partial \bar{z}_n$ and consequently, $dz_1, \ldots, dz_n$ are a local holomorphic frame of the holomorphic bundle $T^{1,0}(M)$.

The complex structure on $T^*_p(M)$ induces a decomposition of the kth exterior product (cf. (A.1.12)):

$$\textstyle\bigwedge^k(T^*_{p,\mathbb{C}}(M)) = \bigoplus_{a+b=k} \bigwedge^{a,b}_p(M),$$

where

$$\textstyle\bigwedge^{a,b}_p(M) = \overbrace{T^{1,0}_p(M) \wedge \cdots \wedge T^{1,0}_p(M)}^{a \text{ times}} \wedge \overbrace{T^{0,1}_p(M) \wedge \cdots \wedge T^{0,1}_p(M)}^{b \text{ times}}. \qquad (1.2.4)$$

In this way, the smooth vector bundle $\bigwedge^k(T^*_{\mathbb{C}}(M))$ decomposes as a direct sum of C^∞ vector bundles

$$\textstyle\bigwedge^k(T^*_{\mathbb{C}}(M)) = \bigoplus_{a+b=k} \bigwedge^{a,b}(M). \qquad (1.2.5)$$

We will denote by $\mathcal{A}^k(U)$ (resp. $\mathcal{A}^{a,b}(U)$) the $C^\infty(U)$-module of local sections of the bundle $\bigwedge^k(T^*_{\mathbb{C}}(M))$ (resp. $\bigwedge^{a,b}(M)$) over U. We then have

$$\mathcal{A}^k(U) = \bigoplus_{a+b=k} \mathcal{A}^{a,b}(U). \qquad (1.2.6)$$

Note that given holomorphic coordinates $\{z_1, \ldots, z_n\}$, the local differential forms

$$dz_I \wedge d\bar{z}_J := dz_{i_1} \wedge \cdots \wedge dz_{i_a} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_b}\,,$$

where I (resp. J) runs over all strictly increasing index sets $1 \le i_1 < \cdots < i_a \le n$ of length a (resp. $1 \le j_1 < \cdots < j_b \le n$ of length b) are a local frame for the bundle $\bigwedge^{a,b}(M)$.

We note that the bundles $\bigwedge^{k,0}(M)$ are holomorphic bundles of rank $\binom{n}{k}$. We denote them by Ω^k_M to emphasize that we are viewing them as holomorphic, rather than smooth, bundles. We denote the $\mathcal{O}(U)$-module of holomorphic sections by $\Omega^k(U)$. In particular, Ω^n_M is a holomorphic line bundle over M called the *canonical bundle* and usually denoted by K_M.

EXAMPLE 1.2.9 Let $M = \mathbb{P}^1$. Then as we saw in Example 1.1.5, M is covered by two coordinate neighborhoods (U_0, ϕ_0), (U_1, ϕ_1). The coordinate change is given by the map $\phi_1 \circ \phi_0^{-1}\colon \mathbb{C}^* \to \mathbb{C}^*$:

$$w = \phi_1 \circ \phi_0^{-1}(z) = \phi_1([(1, z)]) = 1/z.$$

This means that the local sections dz, dw of the holomorphic cotangent bundle are related by

$$dz \;=\; -(1/w)^2\, dw.$$

It follows from (1.1.18) that $g_{01}[(z_0, z_1)] \;=\; -(z_0/z_1)^2$. Hence $K_{\mathbb{P}^1} \cong \mathcal{O}(-2) = \mathcal{T}^2$.

EXERCISE 1.2.10 Find the transition functions for the holomorphic cotangent bundle of $\mathbb{P}^n$. Prove that $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1) = \mathcal{T}^{n+1}$.

1.2.3 De Rham and Dolbeault Cohomologies

We recall that if $U \subset M$ is an open set in a smooth manifold M and $\mathcal{A}^k(U)$ denotes the space of $\mathbb{C}$-valued differential k-forms on U, then there exists a unique operator, the *exterior differential*,

$$d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)\,; \quad k \geq 0$$

satisfying the following properties:

(1) d is $\mathbb{C}$-linear.

(2) For $f \in \mathcal{A}^0(U) = C^\infty(U)$, df is the 1-form on U which acts on a vector field X by $df(X) := X(f)$.

(3) Given $\alpha \in \mathcal{A}^r(U)$, $\beta \in \mathcal{A}^s(U)$, the *Leibniz property* holds:

$$d(\alpha \wedge \beta) \;=\; d\alpha \wedge \beta + (-1)^r\, \alpha \wedge d\beta. \tag{1.2.7}$$

(4) $d \circ d = 0$.

It follows from (2) above that if $\{X_1, \ldots, X_m\}$ is a local frame on $U \subset M$ and $\{\xi_1, \ldots, \xi_m\}$ is the dual coframe, then given $f \in C^\infty(U)$ we have

$$df \;=\; \sum_{i=1}^m X_i(f)\, \xi_i\,.$$

In particular, if M is a complex manifold and $(U, \{z_1, \ldots, z_n\})$ are local coordinates, then for a function $f \in C^\infty(U)$ we have

$$df \;=\; \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j} dy_j \;=\; \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j. \tag{1.2.8}$$

The properties of the operator d imply that for each open set U in M we have a complex:

$$\mathbb{C} \hookrightarrow C^\infty(U) \xrightarrow{d} \mathcal{A}^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{2n-1}(U) \xrightarrow{d} \mathcal{A}^{2n}(U). \tag{1.2.9}$$

The quotients

$$H^k_{dR}(U, \mathbb{C}) \ := \ \frac{\ker\{d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)\}}{d(\mathcal{A}^{k-1}(U))} \tag{1.2.10}$$

are called the *de Rham cohomology groups* of U. The elements in

$$\mathcal{Z}^k(U) := \ker\{d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)\}$$

are called *closed k-forms* and the elements in $\mathcal{B}^k(U) := d(\mathcal{A}^{k-1}(U))$ *exact k-forms*. We note that if U is connected then $H^0_{dR}(U, \mathbb{C}) \cong \mathbb{C}$. Unless there is the possibility of confusion we will drop the subscript since, in this chapter, we will only consider de Rham cohomology.

EXERCISE 1.2.11 Prove that the set of closed forms is a subring of the ring of differential forms and that the set of exact forms is an ideal in the ring of closed forms. Deduce that the de Rham cohomology

$$H^*(U, \mathbb{C}) \ := \ \bigoplus_{k \geq 0} H^k(U, \mathbb{C}) \tag{1.2.11}$$

inherits a ring structure

$$[\alpha] \cup [\beta] \ := \ [\alpha \wedge \beta]. \tag{1.2.12}$$

This is called the *cup product* on cohomology.

If $F\colon M \to N$ is a smooth map, then given an open set $V \subset N$, F induces maps

$$F^*\colon \mathcal{A}^k(V) \to \mathcal{A}^k(F^{-1}(V))$$

which commute with the exterior differential; i.e., F^* is a map of complexes. This implies that F^* defines a map between de Rham cohomology groups,

$$F^*\colon H^k(V, \mathbb{C}) \to H^k(F^{-1}(V), \mathbb{C}),$$

which satisfies the *chain rule* $(F \circ G)^* = G^* \circ F^*$. Since $(\mathrm{id})^* = \mathrm{id}$, it follows that if $F\colon M \to N$ is a diffeomorphism then $F^*\colon H^k(N, \mathbb{C}) \to H^k(M, \mathbb{C})$ is an isomorphism. In fact, the de Rham cohomology groups are a (smooth) homotopy invariant:

DEFINITION 1.2.12 *Let $f_0, f_1\colon M \to N$ be smooth maps. Then f_0 is (smoothly)* ***homotopic*** *to f_1 if there exists a smooth map*

$$H\colon \mathbb{R} \times M \to N$$

such that $H(0, x) = f_0(x)$ and $H(1, x) = f_1(x)$ for all $x \in M$.

THEOREM 1.2.13 *Let $f_0, f_1 \colon M \to N$ be smoothly homotopic maps. Then*

$$f_0^* = f_1^* \colon H^k(N,\mathbb{C}) \to H^k(M,\mathbb{C}).$$

PROOF. We refer to [2, Section 4] for a proof of this important result. □

COROLLARY 1.2.14 (Poincaré lemma) *Let $U \subset M$ be a contractible open subset. Then $H^k(U,\mathbb{C})$ vanishes for all $k \geq 1$.*

PROOF. The result follows from Theorem 1.2.13 since in a contractible open set the identity map is homotopic to a constant map. □

Hence, if U is contractible, the sequence

$$0 \to \mathbb{C} \hookrightarrow C^\infty(U) \xrightarrow{d} \mathcal{A}^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{2n-1}(U) \xrightarrow{d} \mathcal{A}^{2n}(U) \to 0 \qquad (1.2.13)$$

is exact.

The exterior differential operator is not of *pure bidegree* relative to the decomposition (1.2.6). Indeed, it follows from (1.2.8) that

$$d(\mathcal{A}^{a,b}(U)) \subset \mathcal{A}^{a+1,b}(U) \oplus \mathcal{A}^{a,b+1}(U). \qquad (1.2.14)$$

We remark that statement (1.2.14) makes sense for an almost complex manifold (M,J) and, indeed, its validity is equivalent to the integrability of the almost complex structure J; see [16, Theorem 2.8]. We write $d = \partial + \bar\partial$, where ∂ (resp. $\bar\partial$) is the component of d of bidegree $(1,0)$ (resp. $(0,1)$). From $d^2 = 0$ we obtain

$$\partial^2 = \bar\partial^2 = 0\ ; \quad \partial \circ \bar\partial + \bar\partial \circ \partial = 0. \qquad (1.2.15)$$

EXERCISE 1.2.15 Generalize the Leibniz property to the operators ∂ and $\bar\partial$.

PROPOSITION 1.2.16 *Let M be a complex manifold and $U \subset M$ an open subset. Then*

$$\ker\{\bar\partial \colon \mathcal{A}^{p,0}(U) \to \mathcal{A}^{p,1}(U)\} = \Omega^p(U). \qquad (1.2.16)$$

PROOF. We may assume that $(U,\{z_1,\dots,z_n\}$ is a coordinate neighborhood. Let $\alpha \in \mathcal{A}^{p,0}(U)$ and write $\alpha = \sum_I f_I\, dz_I$, where I runs over all increasing index sets $\{1 \leq i_1 < \cdots < i_p \leq n\}$. Then

$$\bar\partial\alpha = \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial \bar z_j} d\bar z_j \wedge dz_I = 0\,.$$

This implies that $\partial f_I / \partial \bar z_j = 0$ for all I and all j. Hence $f_I \in \mathcal{O}(U)$ for all I and α is a holomorphic p-form. □

It follows then from (1.2.15) and (1.2.16) that, for each p, $0 \leq p \leq n$, we get a complex

$$0 \to \Omega^p(U) \hookrightarrow \mathcal{A}^{p,0}(U) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(U) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n-1}(U) \xrightarrow{\bar\partial} \mathcal{A}^{p,n}(U) \to 0$$

called the *Dolbeault complex*. Its cohomology spaces are denoted by $H^{p,q}_{\bar\partial}(U)$ and called the *Dolbeault cohomology groups*.

EXERCISE 1.2.17 Let $\alpha \in \mathcal{A}^{p,q}(U)$. Prove that $\overline{\partial\alpha} = \bar\partial\bar\alpha$. Deduce that a form α is ∂-closed if and only if $\bar\alpha$ is $\bar\partial$-closed. Similarly for ∂-exact forms. Conclude that via conjugation, the study of ∂-cohomology reduces to the study of Dolbeault cohomology.

Given $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{R}_{>0} \cup \infty)^n$, we denote by

$$\Delta_\varepsilon(a) = \{z \in \mathbb{C}^n : |z_i - a_i| < \varepsilon_i\}$$

the n-dimensional *polydisk*. For $n = 1$, $a = 0$, $\varepsilon = 1$ we set $\Delta = \Delta_1(0)$, the *unit disk*, and $\Delta^* = \Delta \setminus \{0\}$, the *punctured unit disk*. The following result is known as the $\bar\partial$-Poincaré lemma:

THEOREM 1.2.18 *If $q \geq 1$ and α is a $\bar\partial$-closed (p,q)-form on a polydisk $\Delta_\varepsilon(a)$, then α is $\bar\partial$-exact; i.e.,*

$$H^{p,q}_{\bar\partial}(\Delta_\varepsilon(a)) = 0\,;\; q \geq 1\,.$$

PROOF. We refer to [10, Chapter 0] or [13, Corollary 1.3.9] for a proof. □

Hence, if $U = \Delta_\varepsilon(a)$ is a polydisk we have exact sequences:

$$0 \to \Omega^p(U) \hookrightarrow \mathcal{A}^{p,0}(U) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(U) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n}(U) \to 0. \tag{1.2.17}$$

Remark. Both the De Rham and Dolbeault cohomology groups may be realized as the sheaf cohomology groups of a constant sheaf. This is discussed in detail in Chapter 2. This will show, in particular, that even though our definition of the de Rham cohomology uses the differentiable structure, it is, in fact, a topological invariant. On the other hand, the Dolbeault cohomology groups depend essentially on the complex structure. This observation is at the core of Hodge theory.

1.3 SYMPLECTIC, HERMITIAN, AND KÄHLER STRUCTURES

In this section we will review the basic notions of Hermitian and Kähler metrics on complex manifolds. We begin by recalling the notion of a *symplectic* structure:

DEFINITION 1.3.1 *A* ***symplectic structure*** *on a $2d$-dimensional manifold M is a closed 2-form $\omega \in \bigwedge^2(M)$ such that $\Omega = \omega^d/d!$ is nowhere vanishing.*

Thus, if ω is a symplectic structure on M, at each $p \in M$, the form ω_p defines a symplectic structure Q_p on $T_p(M)$ (see Definition B.1.1). A *symplectic manifold* is a manifold M endowed with a symplectic structure ω.

The simplest example of a symplectic manifold is given by $\mathbb{R}^{2d}$ with coordinates denoted by $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ and the 2-form

$$\omega_0 = \sum_{j=1}^{d} dx_j \wedge dy_j .$$

The classical Darboux theorem asserts that, locally, every symplectic manifold is *symplectomorphic* to $(\mathbb{R}^{2d}, \omega_0)$:

THEOREM 1.3.2 (Darboux theorem) *Let (M, ω) be a symplectic manifold. Then for each $p \in M$ there exists an open neighborhood U and local coordinates $\varphi\colon U \to \mathbb{R}^{2d}$ such that $\omega|_U = \varphi^*(\omega_0)$.*

PROOF. We refer to [23, Theorem 6.1] for a proof. □

In what follows we will be particularly interested in symplectic structures on a complex manifold M compatible with the complex structure J:

DEFINITION 1.3.3 *Let M be a complex manifold and J its complex structure. A Riemannian metric g on M is said to be a **Hermitian metric** if and only if for each $p \in M$, the bilinear form g_p on the tangent space $T_p(M)$ is compatible with the complex structure J_p (cf.* (B.2.1)*).*

We recall from (B.2.2) in the second appendix to this chapter that given a symmetric bilinear form compatible with the complex structure we may define a J-invariant alternating form. Thus, given a Hermitian metric on M we may define a differential 2-form $\omega \in \mathcal{A}^2(M, \mathbb{C})$ by

$$\omega(X, Y) \;:=\; g(JX, Y), \tag{1.3.1}$$

where we also denote by g the bilinear extension of g to the complexified tangent space. By Theorem B.2.1, we have

$$\omega \in \mathcal{A}^{1,1}(M) \quad \text{and} \quad \bar{\omega} = \omega. \tag{1.3.2}$$

We also recall that Theorem B.2.1 implies that every form ω as in (1.3.2) defines a symmetric $(1, 1)$ tensor on M compatible with J and a Hermitian form H on the complex vector space $(T_p(M), J)$.

We express these objects in local coordinates: let $(U, \{z_1, \ldots, z_d\})$ be local complex coordinates on M. Then (1.3.2) implies that we may write

$$\omega \;:=\; \frac{i}{2} \sum_{j,k=1}^{d} h_{jk}\, dz_j \wedge d\bar{z}_k \;; \quad h_{kj} = \bar{h}_{jk}. \tag{1.3.3}$$

Hence $\omega(\partial/\partial z_j, \partial/\partial \bar{z}_k) = (i/2)\, h_{jk}$ from which it follows that

$$\omega(\partial/\partial x_j, \partial/\partial x_k) \;=\; -\operatorname{Im}(h_{jk}).$$

Moreover, we have

$$g(\partial/\partial x_j, \partial/\partial x_k) = \omega(\partial/\partial x_j, \partial/\partial y_k) \;=\; \operatorname{Re}(h_{jk}).$$

Hence g is positive definite if and only if the Hermitian matrix (h_{jk}) is positive definite. We may then restate Definition 1.3.3 by saying that a Hermitian structure is a $(1,1)$ real form ω as in (1.3.3) such that the matrix (h_{jk}) is positive definite. By abuse of notation we will say that, in this case, the 2-form ω is positive.

1.3.1 Kähler Manifolds

DEFINITION 1.3.4 *A Hermitian metric on a manifold M is said to be a **Kähler metric** if and only if the 2-form ω is closed. We will say that a complex manifold is Kähler if and only if it admits a Kähler structure and refer to ω as a Kähler form.*

EXERCISE 1.3.5 Let (M,ω) be a Kähler manifold. Show that there are local coframes $\chi_1, \dots, \chi_d$ in $\mathcal{A}^{1,0}(U)$ such that

$$\omega \;=\; \frac{i}{2}\sum_{j=1}^{d} \chi_j \wedge \bar{\chi}_j.$$

Clearly, every Kähler manifold M is symplectic. Moreover, if $\{z_1, \dots, z_d\}$ are local coordinates on M and ω is a Kähler form on M then

$$\begin{aligned} \omega^d &= d!\left(\frac{i}{2}\right)^n \det((h_{ij}))\bigwedge_{j=1}^{d}(dz_j \wedge d\bar{z}_j) \\ &= d!\,\det((h_{ij}))\bigwedge_{j=1}^{d}(dx_j \wedge dy_j), \end{aligned}$$

since $dz_j \wedge d\bar{z}_j = (2/i)dx_j \wedge dy_j$.

EXERCISE 1.3.6 Prove that $\omega^d/d!$ is the volume element of the Riemannian metric g defined by the Kähler form ω (see Exercise 1.4.3).

Thus we have a necessary condition for a compact complex manifold to be Kähler:

PROPOSITION 1.3.7 *If M is a compact Kähler manifold, then*

$$\dim H^{2k}(M,\mathbb{R}) > 0$$

for all $k = 0, \dots, d$.

PROOF. Indeed, this is true of all compact symplectic manifolds as the forms ω^k, $k = 1, \ldots, d$ induce nonzero de Rham cohomology classes. Suppose, otherwise, that $\omega^k = d\alpha$. Then

$$\omega^d = d(\omega^{d-k} \wedge \alpha).$$

But then it would follow from the Stokes theorem that

$$\int_M \omega^d = 0,$$

which contradicts the fact that ω^d is a nonzero multiple of the volume element. □

Remark. As we will see below, the existence of a Kähler metric on a manifold imposes many other topological restrictions beyond those satisfied by symplectic manifolds. The earliest examples of compact symplectic manifolds with no Kähler structure are due to Thurston [25]. We refer to [32] for further details.

EXAMPLE 1.3.8 The affine space $\mathbb{C}^d$ with the form

$$\omega = \frac{i}{2} \sum_{j=1}^{d} dz_j \wedge d\bar{z}_j$$

is a Kähler manifold. The form ω gives the usual symplectic structure on $\mathbb{R}^{2d}$.

The following theorem may be seen as a generalization of Darboux's theorem to Kähler manifolds:

THEOREM 1.3.9 *Let M be a complex manifold and g a Kähler metric on M. Then, given $p \in M$, there exist local coordinates $(U, \{z_1, \ldots, z_d\})$ around p such that $z_j(p) = 0$ and*

$$\omega = \frac{i}{2} \sum_{j=1}^{d} h_{jk}\, dz_j \wedge d\bar{z}_k\,,$$

where the coefficients h_{jk} are of the form

$$h_{jk}(z) = \delta_{jk} + O(||z||^2). \tag{1.3.4}$$

PROOF. We refer to [30, Proposition 3.14] for a proof. □

EXAMPLE 1.3.10 We will construct a Kähler form on $\mathbb{P}^n$. We will do this by exhibiting a positive, real, closed $(1,1)$-form on $\mathbb{P}^n$. The resulting metric is called the Fubini–Study metric on $\mathbb{P}^n$.

Given $z \in \mathbb{C}^{n+1}$ we denote by

$$||z||^2 = |z_0|^2 + \cdots + |z_n|^2.$$

Let $U_j \subset \mathbb{P}^n$ be the open set (1.1.6) and let $\rho_j \in C^\infty(U_j)$ be the positive function

$$\rho_j([z]) \;:=\; \frac{||z||^2}{|z_j|^2}, \tag{1.3.5}$$

and define $\omega_j \in \mathcal{A}^{1,1}(U_j)$ by

$$\omega_j \;:=\; \frac{-1}{2\pi i}\,\partial\bar\partial \log(\rho_j). \tag{1.3.6}$$

Clearly, ω_j is a real, closed $(1,1)$-form. Moreover, on $U_j \cap U_k$ we have

$$\log(\rho_j) - \log(\rho_k) = \log|z_k|^2 - \log|z_j|^2 = \log(z_k\bar z_k) - \log(z_j\bar z_j).$$

Hence, since $\partial\bar\partial(\log(z_j\bar z_j)) = 0$, we have $\omega_j = \omega_k$ on $U_j \cap U_k$. Thus, the forms ω_j piece together to give a global, real, closed $(1,1)$-form ω on $\mathbb{P}^n$. We write

$$\omega \;=\; \frac{-1}{2\pi i}\,\partial\bar\partial \log(||z||^2). \tag{1.3.7}$$

It remains to show that ω is positive. We observe first of all that the expression (1.3.7) shows that if A is a unitary matrix and $\mu_A\colon \mathbb{P}^n \to \mathbb{P}^n$ is the biholomorphic map $\mu_A([z]) := [A\cdot z]$, then $\mu_A^*(\omega) = \omega$. Hence, since given any two points $[z], [z'] \in \mathbb{P}^n$ there exists a unitary matrix such that $\mu_a([z]) = [z']$, it suffices to prove that ω is positive definite at just one point, say $[1, 0, \ldots, 0] \in U_0$. In the coordinates $\{u_1, \ldots, u_n\}$ in U_0, we have $\rho_0(u) = 1 + ||u||^2$ and therefore

$$\bar\partial(\log\rho_0(u)) \;=\; \rho_0^{-1}(u)\sum_{k=1}^n u_k\,\bar\partial\bar u_k \;=\; \rho_0^{-1}(u)\sum_{k=1}^n u_k\,d\bar u_k\,,$$

$$\omega \;=\; \frac{i}{2\pi}\,\rho_0^{-2}(u)\left(\rho_0(u)\sum_{j=1}^n du_j\wedge d\bar u_j + \left(\sum_{j=1}^n \bar u_j du_j\right)\wedge\left(\sum_{j=1}^n u_k d\bar u_k\right)\right).$$

Hence, at the origin, we have

$$\omega = \frac{i}{2\pi}\sum_{j=1}^n du_j\wedge d\bar u_j,$$

which is a positive form.

The function $\log(\rho_j)$ in the above proof is called a Kähler *potential*. As the following result shows, every Kähler metric may be described by a (local) potential.

PROPOSITION 1.3.11 *Let M be a complex manifold and ω a Kähler form on M. Then for every $p \in M$ there exists an open set $U \subset M$ and a real function $v \in C^\infty(U)$ such that $\omega = i\,\partial\bar\partial(v)$.*

PROOF. Since $d\omega = 0$, it follows from the Poincaré lemma that in a neighborhood U' of p, we have $\omega = d\alpha$, where $\alpha \in \mathcal{A}^1(U', \mathbb{R})$. Hence, we may write $\alpha = \beta + \bar{\beta}$, where $\beta \in \mathcal{A}^{1,0}(U', \mathbb{R})$. Now, we can write

$$\omega \;=\; d\alpha \;=\; \partial\beta + \bar{\partial}\beta + \partial\bar{\beta} + \bar{\partial}\bar{\beta}\,,$$

but, since ω is of type $(1,1)$ it follows that

$$\omega \;=\; \bar{\partial}\beta + \partial\bar{\beta} \quad \text{and} \quad \partial\beta \;=\; \bar{\partial}\bar{\beta} \;=\; 0.$$

We may now apply the $\bar{\partial}$-Poincaré lemma to conclude that there exists a neighborhood $U \subset U'$ around p where $\bar{\beta} = \bar{\partial} f$ for some ($\mathbb{C}$-valued) C^∞ function f on U. Hence

$$\omega \;=\; \bar{\partial}\partial\bar{f} + \partial\bar{\partial} f \;=\; \partial\bar{\partial}(f - \bar{f}) \;=\; 2i\,\partial\bar{\partial}(\mathrm{Im}(f)).$$

□

THEOREM 1.3.12 *Let (M, ω) be a Kähler manifold and suppose $N \subset M$ is a complex submanifold. Then $(N, \omega|_N)$ is a Kähler manifold.*

PROOF. Let g denote the J-compatible Riemannian metric on M associated with ω. Then g restricts to a Riemannian metric on N, compatible with the complex structure on N, and whose associated 2-form is $\omega|_N$. Since $d(\omega|_N) = (d\omega)|_N = 0$, it follows that N is a Kähler manifold as well. □

It follows from Theorem 1.3.12 that a necessary condition for a compact complex manifold M to have an embedding in $\mathbb{P}^n$ is that there exists a Kähler metric on M. Moreover, as we shall see below, for a submanifold of projective space, there exists a Kähler metric whose associated cohomology class satisfies a suitable integrality condition.

1.3.2 The Chern Class of a Holomorphic Line Bundle

The construction of the Kähler metric in $\mathbb{P}^n$ may be further understood in the context of Hermitian metrics on (line) bundles. We recall that a Hermitian metric on a $\mathbb{C}$-vector bundle $\pi\colon E \to M$ is given by a positive-definite Hermitian form

$$H_p\colon E_p \times E_p \to \mathbb{C}$$

on each fiber E_p, which is smooth in the sense that given sections $\sigma, \tau \in \Gamma(U, E)$, the function

$$H(\sigma, \tau)(p) \;:=\; H_p(\sigma(p), \tau(p))$$

is C^∞ on U. Using partitions of unity, one can prove that every smooth vector bundle E has a Hermitian metric H.

In the case of a line bundle L, the Hermitian form H_p is completely determined by the value $H_p(v, v)$ on a nonzero element $v \in L_p$. In particular, if $\{(U_\alpha, \Phi_\alpha)\}$ is a cover of M by trivializing neighborhoods of L and $\sigma_\alpha \in \mathcal{O}(U_\alpha, L)$ is the local frame

$$\sigma_\alpha(x) \;=\; \Phi_\alpha^{-1}(x, 1)\,; \quad x \in U_\alpha,$$

then a Hermitian metric H on L is determined by the collection of positive functions

$$\rho_\alpha \;:=\; H(\sigma_\alpha, \sigma_\alpha) \in C^\infty(U_\alpha).$$

We note that if $U_\alpha \cap U_\beta \neq \emptyset$ then we have $\sigma_\beta \;=\; g_{\alpha\beta} \cdot \sigma_\alpha$ and, consequently, the functions ρ_α satisfy the compatibility condition

$$\rho_\beta \;=\; |g_{\alpha\beta}|^2 \rho_\alpha. \tag{1.3.8}$$

In particular, if L is a holomorphic line bundle then the transition functions $g_{\alpha\beta}$ are holomorphic and as in Example 1.3.10, we have

$$\partial\bar\partial \log(\rho_\alpha) \;=\; \partial\bar\partial \log(\rho_\beta)$$

on $U_\alpha \cap U_\beta$ and therefore the form

$$\frac{1}{2\pi i}\,\partial\bar\partial \log(\rho_\alpha) \tag{1.3.9}$$

is a global, real, closed $(1,1)$-form on M. The cohomology class

$$[(1/2\pi i)\,\partial\bar\partial \log(\rho_\alpha)] \in H^2(M, \mathbb{R}) \tag{1.3.10}$$

is called the *Chern class* of the vector bundle L and denoted by $c(L)$. The factor $1/2\pi$ is chosen so that the Chern class is actually an integral cohomology class:

$$c(L) \in H^2(M, \mathbb{Z}). \tag{1.3.11}$$

Recall that if $g_{\alpha\beta}$ are the transition functions for a bundle L then the functions $g_{\alpha\beta}^{-1}$ are the transition functions of the dual bundle L^*. In particular, if ρ_α are a collection of positive C^∞ functions defining a Hermitian metric on L then the functions ρ_α^{-1} define a Hermitian metric H^* on L^*. We call H^* the dual Hermitian metric. We then have

$$c(L^*) \;=\; -c(L). \tag{1.3.12}$$

Definition 1.3.13 *A holomorphic line bundle $L \to M$ over a compact Kähler manifold is said to be* ***positive*** *if and only if there exists a Hermitian metric H on L for which the $(1,1)$-form* (1.3.10) *is positive. We say that L is* negative *if its dual bundle L^* is positive.*

We note that in Example 1.3.10 we have

$$|z_k|^2\, \rho_k([z]) \;=\; |z_j|^2\, \rho_j([z])$$

on $U_j \cap U_k$. Hence

$$\rho_k([z]) \;=\; \left|\frac{z_j}{z_k}\right|^2 \rho_j([z])$$

and, by (1.3.8), it follows that the functions ρ_j define a Hermitian metric on the tautological bundle $\mathcal{O}(-1)$. Hence, taking into account the sign change in (1.3.6), it follows that the Kähler class of the Fubini–Study metric agrees with the Chern class of the hyperplane bundle $\mathcal{O}(1)$. Thus,

$$c(\mathcal{O}(1)) \;=\; [\omega] \;=\; \left[\frac{i}{2\pi}\,\partial\bar{\partial}\log(||z||^2)\right] \tag{1.3.13}$$

and the hyperplane bundle $\mathcal{O}(1)$ is a positive line bundle. Moreover, if $M \subset \mathbb{P}^n$ is a complex submanifold then the restriction of $\mathcal{O}(1)$ to M is a positive line bundle over M. We can now state the following theorem:

THEOREM 1.3.14 (Kodaira embedding theorem) *A compact complex manifold M may be embedded in $\mathbb{P}^n$ if and only if there exists a positive holomorphic line bundle $\pi\colon L \to M$.*

We refer to [30, Theorem 7.11], [19, Theorem 8.1], [10], [35, Theorem 4.1], and [13, Section 5.3] for various proofs of this theorem.

Remark. The existence of a positive holomorphic line bundle $\pi\colon L \to M$ implies that M admits a Kähler metric whose Kähler class is integral. Conversely, any integral cohomology class represented by a closed $(1,1)$-form is the Chern class of a line bundle (cf. [6, Section 6]), hence a compact complex manifold M may be embedded in $\mathbb{P}^n$ if and only if it admits a Kähler metric whose Kähler class is integral.

Recall (see [10, Section 1.3]) that Chow's theorem asserts that every analytic subvariety of $\mathbb{P}^n$ is algebraic. When this result is combined with the Kodaira embedding theorem, we obtain a characterization of complex projective varieties as those compact Kähler manifolds admitting a Kähler metric whose Kähler class is integral.

1.4 HARMONIC FORMS—HODGE THEOREM

1.4.1 Compact Real Manifolds

Unless otherwise specified, throughout Section 1.4.1 we will let M denote a compact, oriented, real, n-dimensional manifold with a Riemannian metric g. We recall that the metric on the tangent bundle TM induces a dual metric on the cotangent bundle T^*M such that the dual coframe of a local orthonormal frame $X_1, \ldots, X_n$ in $\Gamma(U, TM)$ is also orthonormal. We will denote the dual inner product by $\langle\ ,\ \rangle$.

EXERCISE 1.4.1 Verify that this metric on T^*M is well defined; i.e., it is independent of the choice of local orthonormal frames.

We extend the inner product to the exterior bundles $\bigwedge^r(T^*M)$ by the specification that the local frame

$$\xi_I \;:=\; \xi_{i_1} \wedge \cdots \wedge \xi_{i_r},$$

where I runs over all strictly increasing index sets $\{1 \le i_1 < \cdots < i_r \le n\}$, is orthonormal.

EXERCISE 1.4.2 Verify that this metric on $\bigwedge^r T^*M$ is well defined; i.e., it is independent of the choice of local orthonormal frames, by proving that

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_r, \beta_1 \wedge \cdots \wedge \beta_r \rangle \;=\; \det(\langle \alpha_i, \beta_j \rangle),$$

where $\alpha_i, \beta_j \in \mathcal{A}^1(U)$.
Hint: use the Cauchy–Binet formula for determinants.

Recall that given an oriented Riemannian manifold, the volume element is defined as the unique n-form $\Omega \in \mathcal{A}^n(M)$ such that

$$\Omega(p)(v_1, \ldots, v_n) \;=\; 1$$

for any positively oriented orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$. If $\xi_1, \ldots, \xi_n \in \mathcal{A}^1(U)$ is a positively oriented orthonormal coframe then

$$\Omega|_U \;=\; \xi_1 \wedge \cdots \wedge \xi_n.$$

EXERCISE 1.4.3 Prove that the volume element may be written as

$$\Omega \;=\; \sqrt{G}\; dx_1 \wedge \cdots \wedge dx_n\,,$$

where $\{x_1, \ldots, x_n\}$ are positively oriented local coordinates, $G = \det(g_{ij})$, and

$$g_{ij} \;:=\; g(\partial/\partial x_i, \partial/\partial x_j).$$

We now define the *Hodge $*$-operator*. Let $\beta \in \mathcal{A}^r(M)$. Then $*\beta \in \mathcal{A}^{n-r}(M)$ is given by $(*\beta)(p) = *(\beta(p))$, where the $*$ operator on $\bigwedge^r T_p^*$ is defined as in (B.1.3). Therefore, for every $\alpha \in \mathcal{A}^r(M)$,

$$\alpha \wedge *\beta \;=\; \langle \alpha, \beta \rangle\, \Omega. \tag{1.4.1}$$

We extend the definition to $\mathcal{A}^r(M, \mathbb{C})$ by linearity.

EXERCISE 1.4.4 Suppose $\alpha_1, \ldots, \alpha_n \in T_p^*(M)$ is a positively oriented orthonormal basis. Let $I = \{1 \le i_1 < \cdots < i_r \le n\}$ be an index set and I^c its complement. Prove that

$$*(\alpha_I) = \operatorname{sign}(I, I^c)\, \alpha_{I^c}\,, \tag{1.4.2}$$

where $\operatorname{sign}(I, I^c)$ is the sign of the permutation $\{I, I^c\}$.

EXERCISE 1.4.5 Prove that $*$ is an isometry and that $*^2$ acting on $\mathcal{A}^r(M)$ equals $(-1)^{r(n-r)}\, I$.

Suppose now that M is compact. We can then define an L^2 inner product on the space of r-forms on M by

$$(\alpha, \beta) \;:=\; \int_M \alpha \wedge *\beta \;=\; \int_M \langle \alpha(p), \beta(p) \rangle\; \Omega\,; \quad \alpha, \beta \in \mathcal{A}^r(M). \tag{1.4.3}$$

PROPOSITION 1.4.6 *The bilinear form $(\bullet, \bullet)$ is a positive-definite inner product on $\mathcal{A}^r(M)$.*

PROOF. First of all we check that $(\bullet, \bullet)$ is symmetric:

$$(\beta, \alpha) = \int_M \beta \wedge *\alpha = (-1)^{r(n-r)} \int_M *(*\beta) \wedge *\alpha = \int_M *\alpha \wedge *(*\beta) = (\alpha, \beta).$$

Now, given $0 \neq \alpha \in \mathcal{A}^r(M)$, we have

$$(\alpha, \alpha) \;=\; \int_M \alpha \wedge *\alpha \;=\; \int_M \langle \alpha, \alpha \rangle \, \Omega > 0$$

since $\langle \alpha, \alpha \rangle$ is a nonnegative function which is not identically zero. $\square$

PROPOSITION 1.4.7 *The operator $\delta \colon \mathcal{A}^{r+1}(M) \to \mathcal{A}^r(M)$, defined by*

$$\delta \;:=\; (-1)^{nr+1} * d\, *, \tag{1.4.4}$$

is the formal adjoint of d; that is,

$$(d\alpha, \beta) \;=\; (\alpha, \delta\beta) \quad \textit{for all} \;\; \alpha \in \mathcal{A}^r(M), \beta \in \mathcal{A}^{r+1}(M). \tag{1.4.5}$$

PROOF.

$$\begin{aligned} (d\alpha, \beta) \;&=\; \int_M d\alpha \wedge *\beta \;=\; \int_M d(\alpha \wedge *\beta) - (-1)^r \int_M \alpha \wedge d * \beta \\ &=\; -(-1)^r(-1)^{r(n-r)} \int_M \alpha \wedge *(* \, d * \beta) \;=\; \int_M \alpha \wedge *\delta\beta \\ &=\; (\alpha, \delta\beta). \end{aligned}$$

$\square$

Remark. Note that if $\dim M$ is even then $\delta = - * d\, *$ independently of the degree of the form. Since we will be interested in applying these results in the case of complex manifolds which, as real manifolds, are even-dimensional, we will make that assumption from now on.

We now define the *Laplace–Beltrami operator* of (M, g) by

$$\Delta \colon \mathcal{A}^r(M) \to \mathcal{A}^r(M)\,; \quad \Delta\alpha \;:=\; d\delta\alpha + \delta d\alpha.$$

PROPOSITION 1.4.8 *The operators d, δ, $*$ and Δ satisfy the following properties:*

(1) Δ *is self-adjoint; i.e.,* $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$.

(2) $[\Delta, d] \;=\; [\Delta, \delta] \;=\; [\Delta, *] \;=\; 0$.

(3) $\Delta\alpha = 0$ *if and only if* $d\alpha \;=\; \delta\alpha \;=\; 0$.

PROOF. We leave the first two items as exercises. Note that given operators D_1, D_2, the bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. Thus, (2) states that the Laplacian Δ commutes with d, δ, and $*$.

Clearly, if $d\alpha = \delta\alpha = 0$ then we have $\Delta\alpha = 0$. Conversely, suppose $\alpha \in \mathcal{A}^r(M)$ and $\Delta\alpha = 0$. Then

$$0 \;=\; (\Delta\alpha, \alpha) \;=\; (d\delta\alpha + \delta d\alpha, \alpha) = (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha).$$

Hence $d\alpha = \delta\alpha = 0$. □

DEFINITION 1.4.9 *A form $\alpha \in \mathcal{A}^r(M)$ is said to be **harmonic** if $\Delta\alpha = 0$ or, equivalently, if α is closed and coclosed, i.e., $d\alpha = \delta\alpha = 0$.*

EXERCISE 1.4.10 Let M be a compact, connected, oriented, Riemannian manifold. Show that the only harmonic functions on M are the constant functions.

EXERCISE 1.4.11 Let $\alpha \in \mathcal{A}^r(M)$ be closed. Show that $*\alpha$ is closed if and only if α is harmonic.

The following result shows that harmonic forms are very special within a given de Rham cohomology class:

PROPOSITION 1.4.12 *A closed r-form α is harmonic if and only if $||\alpha||^2$ is a local minimum within the de Rham cohomology class of α. Moreover, in any given de Rham cohomology class there is at most one harmonic form.*

PROOF. Let $\alpha \in \mathcal{A}^r(M)$ be such that $||\alpha||^2$ is a local minimum within the de Rham cohomology class of α. Then, for every $\beta \in \mathcal{A}^{r-1}(M)$, the function $\nu(t) := ||\alpha + t\, d\beta||^2$ has a local minimum at $t = 0$. In particular,

$$\nu'(0) = 2(\alpha, d\beta) = 2(\delta\alpha, \beta) = 0 \quad \text{for all} \quad \beta \in \mathcal{A}^{r-1}(M).$$

Hence, $\delta\alpha = 0$ and α is harmonic. Now, if α is harmonic, then

$$||\alpha + d\beta||^2 = ||\alpha||^2 + ||d\beta||^2 + 2(\alpha, d\beta) = ||\alpha||^2 + ||d\beta||^2 \geq ||\alpha||^2$$

and equality holds only if $d\beta = 0$. This proves the uniqueness statement. □

Hodge's theorem asserts that, in fact, every de Rham cohomology class contains a (unique) harmonic form. More precisely, we have the following theorem:

THEOREM 1.4.13 (Hodge theorem) *Let M be a compact Riemannian manifold and let $\mathcal{H}^r(M)$ denote the vector space of harmonic r-forms on M. Then*

(1) $\mathcal{H}^r(M)$ is finite-dimensional for all r;

(2) we have the following decomposition of the space of r-forms:

$$\begin{aligned}\mathcal{A}^r(M) &= \Delta(\mathcal{A}^r(M)) \oplus \mathcal{H}^r(M)\\ &= d\delta(\mathcal{A}^r(M)) \oplus \delta d(\mathcal{A}^r(M)) \oplus \mathcal{H}^r(M)\\ &= d(\mathcal{A}^{r-1}(M)) \oplus \delta(\mathcal{A}^{r+1}(M)) \oplus \mathcal{H}^r(M).\end{aligned}$$

The proof of this fundamental result involves the theory of elliptic differential operators on a manifold. We refer to [10, Chapter 0], [33, Chapter 6], and [35, Chapter 4].

Since d and δ are formal adjoints of each other it follows that

$$(\ker(d), \mathrm{Im}(\delta)) \;=\; (\ker(\delta), \mathrm{Im}(d)) \;=\; 0$$

and, consequently, if $\alpha \in \mathcal{Z}^r(M)$ and we write

$$\alpha = d\beta + \delta\gamma + \mu\,; \quad \beta \in \mathcal{A}^{r-1}(M), \gamma \in \mathcal{A}^{r+1}(M), \mu \in \mathcal{H}^r(M),$$

then

$$0 = (\alpha, \delta\gamma) = (\delta\gamma, \delta\gamma)$$

and therefore $\delta\gamma = 0$. Hence, $[\alpha] = [\mu]$. By the uniqueness statement in Proposition 1.4.12 we get

$$H^r(M, \mathbb{R}) \;\cong\; \mathcal{H}^r(M). \tag{1.4.6}$$

COROLLARY 1.4.14 *Let M be a compact, oriented, n-dimensional manifold. Then $H^r(M, \mathbb{R})$ is finite-dimensional for all r.*

COROLLARY 1.4.15 (Poincaré duality) *Let M be a compact, oriented, n-dimensional manifold. Then the bilinear pairing*

$$\int_M : H^r(M, \mathbb{R}) \times H^{n-r}(M, \mathbb{R}) \to \mathbb{R} \tag{1.4.7}$$

that maps $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$ is nondegenerate. Hence

$$\big(H^{n-r}(M, \mathbb{R})\big)^* \;\cong\; H^r(M, \mathbb{R}).$$

PROOF. We may assume without loss of generality that M is a Riemannian manifold. Then, the Hodge star operator commutes with the Laplacian and defines an isomorphism:

$$\mathcal{H}^r(M) \;\cong\; \mathcal{H}^{n-r}(M).$$

Hence if $0 \neq \alpha \in \mathcal{H}^r(M)$ we have $*\alpha \in \mathcal{H}^{n-r}(M)$ and

$$\int_M \alpha \wedge *\alpha = (\alpha, \alpha) \neq 0.$$

□

EXERCISE 1.4.16 Prove that the pairing (1.4.7) is well defined.

1.4.2 The $\bar\partial$-Laplacian

Let (M, J, ω) be a compact Kähler manifold and, as before, let g denote the associated Riemannian metric. Consider the L^2 inner product $(\bullet, \bullet)$ on $\mathcal{A}^*(M)$ defined in (1.4.3). Let $*$ be the corresponding star operator and $\delta = - * d *$ the adjoint of d. We extend these operators linearly to $\mathcal{A}^*(M, \mathbb{C})$. It follows from (B.2.4) that

$$*(\mathcal{A}^{p,q}(M)) \subset \mathcal{A}^{n-q,n-p}(M). \tag{1.4.8}$$

We write

$$\delta = - * d * = - * \bar\partial * - * \partial *,$$

and set

$$\partial^* := - * \bar\partial * ; \quad \bar\partial^* := - * \partial * . \tag{1.4.9}$$

Note that $\bar\partial^*$ is indeed the conjugate of ∂^* and that ∂^* is pure of type $(-1, 0)$ and that $\bar\partial^*$ is pure of type $(0, -1)$ (see Exercise B.2.3).

EXERCISE 1.4.17 Let M be a compact, complex, n-dimensional manifold and $\alpha \in \mathcal{A}^{2n-1}(M, \mathbb{C})$. Prove that

$$\int_M \partial\alpha = \int_M \bar\partial\alpha = 0.$$

PROPOSITION 1.4.18 *The operator $\partial^* := - * \bar\partial *$ (resp. $\bar\partial^* := - * \partial *$) is the formal adjoint of ∂ (resp. $\bar\partial$) relative to the Hermitian extension $(\bullet, \bullet)^h$ of $(\bullet, \bullet)$ to $\mathcal{A}^*(M, \mathbb{C})$.*

PROOF. Given Exercise 1.4.17 and the Leibniz property for the operators ∂, $\bar\partial$, the proof of the first statement is analogous to that of Proposition 1.4.7. The details are left as an exercise. □

We can now define Laplace–Beltrami operators:

$$\Delta_\partial = \partial\partial^* + \partial^*\partial ; \quad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial. \tag{1.4.10}$$

The operators Δ_∂ and $\Delta_{\bar\partial}$ are of bidegree $(0, 0)$; i.e., they map forms of bidegree (p, q) to forms of the same bidegree. In particular, if $\alpha \in \mathcal{A}^k(U)$ is decomposed according to (1.2.6) as

$$\alpha = \alpha^{k,0} + \alpha^{k-1,1} + \cdots + \alpha^{0,k},$$

then $\Delta_{\bar\partial}(\alpha) = 0$ if and only if $\Delta_{\bar\partial}(\alpha^{p,q}) = 0$ for all p, q. The operators Δ_∂ and $\Delta_{\bar\partial}$ are elliptic and, consequently, the Hodge theorem remains valid for them. Thus if we set

$$\mathcal{H}^{p,q}_{\bar\partial}(M) := \{\alpha \in \mathcal{A}^{p,q}(M) : \Delta_{\bar\partial}(\alpha) = 0\}, \tag{1.4.11}$$

we have

$$H^{p,q}_{\bar\partial}(M) \cong \mathcal{H}^{p,q}_{\bar\partial}(M). \tag{1.4.12}$$

1.5 COHOMOLOGY OF COMPACT KÄHLER MANIFOLDS

1.5.1 The Kähler Identities

DEFINITION 1.5.1 *Let (M,ω) be an n-dimensional, compact, Kähler manifold. We define*

$$L_\omega\colon \mathcal{A}^k(M) \to \mathcal{A}^{k+2}(M)\,; \quad L_\omega(\alpha) = \omega\wedge\alpha\,. \tag{1.5.1}$$

Let Λ_ω be the adjoint of L_ω relative to the inner product on $(\bullet,\bullet)$.

EXERCISE 1.5.2 Prove that for $\alpha\in\mathcal{A}^k(M)$, then $\Lambda_\omega\alpha = (-1)^k * L_\omega * \alpha$.

If there is no chance of confusion we will drop the subscript ω. It is clear, however, that the *Lefschetz operators* L and Λ depend on the choice of a Kähler form ω. We extend these operators linearly to $\mathcal{A}^k(M,\mathbb{C})$. It is easy to check that Λ is then the adjoint of L relative to the Hermitian extension of $(\bullet,\bullet)$ to $\mathcal{A}^k(M,\mathbb{C})$.

The following result describes the Kähler identities which describe the commutation relations among the differential operators $d,\partial,\bar\partial$ and the Lefschetz operators.

THEOREM 1.5.3 (Kähler identities) *Let (M,ω) be a compact, Kähler manifold. Then the following identities hold:*

(1) $[\partial,L] \;=\; [\bar\partial,L] \;=\; [\partial^*,\Lambda] \;=\; [\bar\partial^*,\Lambda] \;=\; 0.$

(2) $[\bar\partial^*,L] = i\partial\,;\;\; [\partial^*,L] = -i\bar\partial\,;\;\; [\bar\partial,\Lambda] = i\partial^*\,;\;\; [\partial,\Lambda] = -i\bar\partial^*$

One of the standard ways to prove these identities makes use of the fact that they are of a local nature and only involve the coefficients of the Kähler metric up to first order. On the other hand, Theorem 1.3.9 asserts that a Kähler metric agrees with the standard Hermitian metric on $\mathbb{C}^n$ up to order two. Thus, it suffices to verify the identities in that case. This is done by a direct computation. This is the approach in [10] and [30, Proposition 6.5]. In Appendix B we describe a conceptually simpler proof due to Phillip Griffiths that reduces Theorem 1.5.3 to similar statements in the symplectic case. Since, by Darboux's theorem, a symplectic manifold is locally symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic structure, the proof reduces to that case.

A remarkable consequence of the Kähler identities is the fact that on a compact Kähler manifold, the Laplacians Δ and $\Delta_{\bar\partial}$ are multiples of each other:

THEOREM 1.5.4 *Let M be a compact Kähler manifold. Then*

$$\Delta \;=\; 2\Delta_{\bar\partial}. \tag{1.5.2}$$

PROOF. Note first of all that Theorem 1.5.3(2) yields

$$i(\partial\bar\partial^* + \bar\partial^*\partial) \;=\; \partial[\Lambda,\partial] + [\Lambda,\partial]\partial \;=\; \partial\Lambda\partial - \partial\Lambda\partial \;=\; 0.$$

Therefore,

$$\begin{aligned}
\Delta_\partial &= \partial\partial^* + \partial^*\partial \;=\; i\,\partial[\Lambda,\bar\partial] + i\,[\Lambda,\bar\partial]\partial \\
&= i\,(\partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial) \\
&= i\,\big(([\partial,\Lambda]\bar\partial + \Lambda\partial\bar\partial) - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - (\bar\partial[\Lambda,\partial] + \bar\partial\partial\Lambda)\big) \\
&= i\,\big(\Lambda(\partial\bar\partial + \bar\partial\partial) + (\partial\bar\partial + \bar\partial\partial)\Lambda - i\,(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\big) \\
&= \Delta_{\bar\partial}.
\end{aligned}$$

These two identities together yield (1.5.2). □

1.5.2 The Hodge Decomposition Theorem

Theorem 1.5.4 has a remarkable consequence: suppose $\alpha \in \mathcal{H}^k(M,\mathbb{C})$ is decomposed according to (1.2.6) as

$$\alpha \;=\; \alpha^{k,0} + \alpha^{k-1,1} + \cdots + \alpha^{0,k}\,;$$

then since $\Delta \;=\; 2\Delta_{\bar\partial}$, the form α is $\Delta_{\bar\partial}$-harmonic and consequently, the components $\alpha^{p,q}$ are $\Delta_{\bar\partial}$-harmonic and hence, Δ-harmonic as well. Therefore, if we set for $p+q = k$,

$$\mathcal{H}^{p,q}(M) \;:=\; \mathcal{H}^k(M,\mathbb{C}) \cap \mathcal{A}^{p,q}(M), \tag{1.5.3}$$

we get

$$\mathcal{H}^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M). \tag{1.5.4}$$

Moreover, since Δ is a real operator, it follows that

$$\mathcal{H}^{q,p}(M) \;=\; \overline{\mathcal{H}^{p,q}(M)}. \tag{1.5.5}$$

If we combine these results with the Hodge theorem we get the following result:

THEOREM 1.5.5 (Hodge decomposition theorem) *Let M be a compact Kähler manifold and let $H^{p,q}(M)$ be the space of de Rham cohomology classes in $H^{p+q}(M,\mathbb{C})$ that have a representative of bidegree (p,q). Then,*

$$H^{p,q}(M) \;\cong\; H^{p,q}_{\bar\partial}(M) \;\cong\; \mathcal{H}^{p,q}(M) \tag{1.5.6}$$

and

$$H^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M). \tag{1.5.7}$$

Moreover, $H^{q,p}(M) = \overline{H^{p,q}(M)}$.

Remark. In view of Definition A.4.1, Theorem 1.5.5 may be restated as follows: the subspaces $(H(M,\mathbb{C}))^{p,q} \;\cong\; H^{p,q}_{\bar\partial}(M)$ define a Hodge structure of weight k on de Rham cohomology groups $H^k(M,\mathbb{R})$.

We will define $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(M)$. These are the so-called *Hodge numbers* of M. Note that the *Betti numbers* b^k, that is, the dimensions of the kth cohomology space, are given by

$$b^k = \sum_{p+q=k} h^{p,q}. \tag{1.5.8}$$

In particular, the Hodge decomposition theorem implies a new restriction on the cohomology of a compact Kähler manifold:

COROLLARY 1.5.6 *The odd Betti numbers of a compact Kähler manifold are even.*

PROOF. This assertion follows from (1.5.8) together with the fact that $h^{p,q} = h^{q,p}$. □

Remark. The examples constructed by Thurston in [25] of complex symplectic manifolds with no Kähler structure are manifolds which do not satisfy Corollary 1.5.6.

Remark. As pointed out in Exercise 1.2.11, the de Rham cohomology $H^*(M, \mathbb{C})$ is an algebra under the cup product. We note that the Hodge decomposition (1.5.7) is compatible with the algebra structure in the sense that

$$H^{p,q} \cup H^{p',q'} \subset H^{p+p',q+q'}. \tag{1.5.9}$$

This additional topological restriction for a compact, complex, symplectic manifold to have a Kähler metric has been successfully exploited by Voisin [32] to obtain remarkable examples of non-Kähler, symplectic manifolds.

Let M be a compact, n-dimensional Kähler manifold and $X \subset M$ a complex submanifold of codimension k. We may define a linear map:

$$\int_X : H^{2(n-k)}(M, \mathbb{C}) \to \mathbb{C}\,; \quad [\alpha] \mapsto \int_X \alpha|_X. \tag{1.5.10}$$

This map defines an element in $(H^{2(n-k)}(M, \mathbb{C}))^*$ and, therefore, by Corollary 1.4.15, a cohomology class $\eta_X \in H^{2k}(M, \mathbb{C})$ defined by the property that for all $[\alpha] \in H^{2(n-k)}(M, \mathbb{C})$,

$$\int_M \alpha \wedge \eta_X = \int_X \alpha|_X. \tag{1.5.11}$$

The class η_X is called the *Poincaré dual* of X and one can show that

$$\eta_X \in H^{k,k}(M) \cap H^{2k}(M, \mathbb{Z}). \tag{1.5.12}$$

One can also prove that the construction of the Poincaré dual may be extended to singular analytic subvarieties (cf. [10, 13]).

The following establishes a deep connection between the algebraic and analytic aspects of a smooth projective variety[12].

HODGE CONJECTURE *Let M be a smooth, projective manifold. Then*

$$H^{k,k}(M,\mathbb{Q}) \ := \ H^{k,k}(M) \cap H^{2k}(M,\mathbb{Q})$$

is generated, as a $\mathbb{Q}$-vector space, by the Poincaré duals of analytic subvarieties of M.

The Hodge conjecture is one of the remaining six Clay millennium problems [7]. It should be pointed out that all natural generalizations of the Hodge conjecture to compact Kähler manifolds fail; see [36, 29].

1.5.3 Lefschetz Theorems and Hodge–Riemann Bilinear Relations

Let (M,ω) be a compact Kähler manifold and let

$$\mathcal{A}^*(M,\mathbb{C}) \ := \ \bigoplus_{k=0}^{2n} \mathcal{A}^k(M,\mathbb{C}).$$

We can consider the operators L and Λ acting on $\mathcal{A}^*(M,\mathbb{C})$ and define a semisimple linear map $Y\colon \mathcal{A}^*(M,\mathbb{C}) \to \mathcal{A}^*(M,\mathbb{C})$ by

$$Y := \sum_{k=0}^{2n} (k-n)\pi_k,$$

where $\pi_k\colon \mathcal{A}^*(M,\mathbb{C}) \to \mathcal{A}^k(M,\mathbb{C})$ is the natural projection. Clearly L and Y are defined pointwise and, because of Exercise 1.5.2, so is Λ. Thus, it follows from Corollary B.2.5 that the operators $\{L,\Lambda,Y\}$ define an $\mathfrak{sl}_2$-triple.

We will now show how the Kähler identities imply that the Laplace–Beltrami operator Δ commutes with these operators and consequently, we get a (finite-dimensional) $\mathfrak{sl}_2$-representation on the space of harmonic forms $\mathcal{H}^*(M)$.

THEOREM 1.5.7 *Let (M,ω) be a Kähler manifold. Then, Δ commutes with L, Λ, and Y.*

PROOF. Clearly $[\Delta,L]=0$ if and only if $[\Delta_\partial,L]=0$. We have

$$\begin{aligned}
[\Delta_\partial, L] &= [\partial\partial^* + \partial^*\partial, L] \\
&= \partial\partial^* L - L\partial\partial^* + \partial^*\partial L - L\partial^*\partial \\
&= \partial\left([\partial^*,L] + L\partial^*\right) - L\partial\partial^* + \left([\partial^*,L] + L\partial^*\right)\partial - L\partial^*\partial \\
&= -i\partial\bar\partial - i\bar\partial\partial \\
&= 0.
\end{aligned}$$

The identity $[\Delta,\Lambda]=0$ follows by taking adjoints and $[\Delta,Y]=0$ since Δ preserves the degree of a form. □

We can now define an $\mathfrak{sl}_2$-representation on the de Rham cohomology of a compact Kähler manifold:

THEOREM 1.5.8 *The operators L, Y, and Λ define a real representation of $\mathfrak{sl}(2,\mathbb{C})$ on the de Rham cohomology $H^*(M,\mathbb{C})$. Moreover, these operators commute with the Weil operators of the Hodge structures on the subspaces $H^k(M,\mathbb{R})$.*

PROOF. This is a direct consequence of Theorem 1.5.7. The last statement follows from the fact that L, Y, and Λ are of bidegree $(1,1)$, $(0,0)$ and $(-1,-1)$, respectively. □

COROLLARY 1.5.9 (Hard Lefschetz theorem) *Let (M,ω) be an n-dimensional, compact Kähler manifold. For each $k \le n$ the map*

$$L_\omega^k \colon H^{n-k}(M,\mathbb{C}) \to H^{n+k}(M,\mathbb{C}) \tag{1.5.13}$$

is an isomorphism.

PROOF. This follows from the results in Section A.3, in particular, Exercise A.3.7. □

We note, in particular, that for $j \le k \le n$, the maps

$$L^j \colon H^{n-k}(M,\mathbb{C}) \to H^{n-k+2j}(M,\mathbb{C})$$

are injective. This observation together with the hard Lefschetz theorem imply further cohomological restrictions on a compact Kähler manifold:

THEOREM 1.5.10 *The Betti and Hodge numbers of a compact Kähler manifold satisfy*

(1) $b^{n-k} = b^{n+k};\ h^{p,q} = h^{q,p} = h^{n-q,n-p} = h^{n-p,n-q};$

(2) $b^0 \le b^2 \le b^4 \le \cdots;$

(3) $b^1 \le b^3 \le b^5 \le \cdots.$

In both cases the inequalities continue up to, at most, the middle degree.

DEFINITION 1.5.11 *Let (M,ω) be a compact, n-dimensional Kähler manifold. For each index $k = p + q \le n$, we define the* ***primitive cohomology*** *spaces*

$$H_0^{p,q}(M) := \ker\{L_\omega^{n-k+1} \colon H^{p,q}(M) \to H^{n-q+1,n-p+1}(M)\}, \tag{1.5.14}$$

$$H_0^k(M,\mathbb{C}) := \bigoplus_{p+q=k} H_0^{p,q}(M). \tag{1.5.15}$$

From Proposition A.3.9, we now have the following theorem:

THEOREM 1.5.12 (Lefschetz decomposition) *Let (M,ω) be an n-dimensional, compact Kähler manifold. For each $k = p + q \le n$, we have*

$$H^k(M,\mathbb{C}) = H_0^k(M,\mathbb{C}) \oplus L_\omega(H^{k-2}(M,\mathbb{C})), \tag{1.5.16}$$

$$H^{p,q}(M) = H_0^{p,q}(M) \oplus L_\omega(H^{p-1,q-1}(M)). \tag{1.5.17}$$

The following result, whose proof may be found in [13, Proposition 1.2.31], relates the Hodge star operator with the $\mathfrak{sl}_2$-action.

PROPOSITION 1.5.13 *Let $\alpha \in \mathcal{P}^k(M, \mathbb{C})$. Then*

$$*L^j(\alpha) \;=\; (-1)^{k(k+1)/2} \, \frac{j!}{(n-k-j)!} \cdot L^{n-k-j}(C(\alpha)), \tag{1.5.18}$$

where C is the Weil operator in $\mathcal{A}^k(M, \mathbb{C})$.

DEFINITION 1.5.14 *Let (M, ω) be an n-dimensional, compact, Kähler manifold. Let k be such that $0 \le k \le n$. We define a bilinear form*

$$Q_k = Q\colon H^k(M, \mathbb{C}) \times H^k(M, \mathbb{C}) \to \mathbb{C},$$

$$Q_k(\alpha, \beta) \;:=\; (-1)^{k(k-1)/2} \int_M \alpha \wedge \beta \wedge \omega^{n-k}. \tag{1.5.19}$$

EXERCISE 1.5.15 Prove that Q is well defined; i.e., it is independent of our choice of representative in the cohomology class.

THEOREM 1.5.16 (Hodge–Riemann bilinear relations) *The bilinear form Q satisfies the following properties:*

(1) Q_k is symmetric if k is even and skew symmetric if k is odd.

(2) $Q(L_\omega \alpha, \beta) + Q(\alpha, L_\omega \beta) = 0$; we say that L_ω is an infinitesimal isomorphism of Q.

(3) $Q(H^{p,q}(M), H^{p',q'}(M)) = 0$ unless $p' = q$ and $q' = p$.

(4) If $0 \ne \alpha \in H_0^{p,q}(M)$ then

$$Q(C\alpha, \bar{\alpha}) > 0. \tag{1.5.20}$$

PROOF. The first statement is clear. For the second note that the difference between the two terms is the preceding sign which changes as we switch from $k+2$ to k. The third assertion follows from the fact that the integral vanishes unless the bidegree of the integrand is (n, n) and, for that to happen, we must have $p' = q$ and $q' = p$.

Therefore, we only need to show the positivity condition (4). Let $\alpha \in H_0^{p,q}(M)$. It follows from Proposition 1.5.13 that

$$(-1)^{k(k+1)/2} \, \omega^{n-k} \wedge \bar{\alpha} = *^{-1} (n-k)! \, C(\bar{\alpha}).$$

On the other hand, on $H^k(M)$, we have $C^2 = (-1)^k I = *^2$ and therefore

$$Q(C\alpha, \bar{\alpha}) \;=\; \int_M \alpha \wedge * \bar{\alpha} \;=\; (\alpha, \alpha)^h > 0.$$

□

Properties (3) and (4) in Theorem 1.5.16 are called the first and second *Hodge–Riemann bilinear relations*. In view of Definition A.4.7 we may say that the Hodge–Riemann bilinear relations amount to the statement that the Hodge structure in the primitive cohomology $H_0^k(M,\mathbb{R})$ is polarized by the *intersection form* Q defined by (1.5.19).

EXAMPLE 1.5.17 Let $X = X_g$ denote a compact Riemann surface of genus g. Then we know that $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. The Hodge decomposition in degree 1 is of the form

$$H^1(X,\mathbb{C}) \;=\; H^{1,0}(X) \oplus \overline{H^{1,0}(X)},$$

where $H^{1,0}(X)$ consists of the 1-forms on X which, locally, are of the form $f(z)\,dz$, with $f(z)$ holomorphic. The form Q on $H^1(X,\mathbb{C})$ is alternating and given by

$$Q(\alpha,\beta) \;=\; \int_X \alpha \wedge \beta.$$

The Hodge–Riemann bilinear relations then take the form $Q(H^{1,0}(X), H^{1,0}(X)) = 0$ and, since $H_0^{1,0}(X) = H^{1,0}(X)$,

$$iQ(\alpha,\bar\alpha) \;=\; i\int_X \alpha \wedge \bar\alpha > 0$$

if α is a nonzero form in $H^{1,0}(X)$. Note that, locally,

$$i\alpha \wedge \bar\alpha \;=\; i|f(z)|^2 dz \wedge d\bar z \;=\; 2|f(z)|^2 dx \wedge dy,$$

so both bilinear relations are clear in this case. We note that it follows that $H^{1,0}(X)$ defines a point in the complex manifold $D = D(H^1(X,\mathbb{R}),Q)$ defined in Example 1.1.23.

EXAMPLE 1.5.18 Suppose now that (M,ω) is a compact, connected, Kähler surface and let us consider the Hodge structure in the middle cohomology $H^2(X,\mathbb{R})$. We have the Hodge decomposition

$$H^2(X,\mathbb{C}) = \; H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)\,; \quad H^{0,2}(X) = \overline{H^{2,0}(X)}.$$

Moreover, $H_0^{2,0}(X) = H^{2,0}(X)$, while

$$H^{1,1}(X) \;=\; H_0^{1,1}(X) \oplus L_\omega H^{0,0}(X) \;=\; H_0^{1,1}(X) \oplus \mathbb{C}\cdot\omega$$

and

$$H_0^{1,1}(X) \;=\; \{\alpha \in H^{1,1}(X) : [\omega \wedge \alpha] = 0\}.$$

The polarization form on $H^2(X,\mathbb{R})$ is given by

$$Q(\alpha,\beta) \;=\; -\int_X \alpha \wedge \beta$$

and the second Hodge–Riemann bilinear relation is equivalent to the following statements

$$\int_X \alpha \wedge \bar{\alpha} > 0, \quad \text{if} \quad 0 \neq \alpha \in H^{2,0}(X),$$

$$\int_X \omega^2 > 0,$$

$$\int_X \beta \wedge \bar{\beta} < 0, \quad \text{if} \quad 0 \neq \beta \in H_0^{1,1}(X).$$

We note that the first two statements are easy to verify, but that is not the case with the last one. We point out that the integration form $\mathcal{I}(\alpha, \beta) = -Q(\alpha, \beta)$ has index $(+, \cdots, +, -)$ in $H^{1,1}(X) \cap H^2(X, \mathbb{R})$; i.e., $\mathcal{I}$ is a hyperbolic symmetric bilinear form. Such forms satisfy the reverse Cauchy–Schwarz inequality: if $\mathcal{I}(\alpha, \alpha) \geq 0$, then

$$\mathcal{I}(\alpha, \beta)^2 \geq \mathcal{I}(\alpha, \alpha) \cdot \mathcal{I}(\beta, \beta) \tag{1.5.21}$$

for all $\beta \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$.

The inequality (1.5.21) is called Hodge's inequality and plays a central role in the study of algebraic surfaces. Via Poincaré duals it may be interpreted as an inequality between intersection indexes of curves in an algebraic surface or, in other words, about the number of points where two curves intersect. If the ambient surface is an *algebraic torus,* $X = \mathbb{C}^* \times \mathbb{C}^*$, then a curve is the zero locus of a Laurent polynomial in two variables and a classical result of Bernstein–Kushnirenko–Khovanskii says that, generically on the coefficients of the polynomials, the intersection indexes may be computed combinatorially from the Newton polytope of the defining polynomials (see Khovanskii's appendix in [4] for a full account of this circle of ideas). This relationship between the Hodge inequality, and combinatorics led Khovanskii and Teissier [24] to give (independent) proofs of the classical Alexandrov–Fenchel inequality for mixed volumes of polytopes using the Hodge inequality, and set the basis for a fruitful interaction between algebraic geometry and combinatorics. In particular, motivated by problems in convex geometry, Gromov [11] stated a generalization of the hard Lefschetz theorem, Lefschetz decomposition and Hodge–Riemann bilinear relations to the case of *mixed* Kähler forms. We give a precise statement in the case of the hard Lefschetz theorem and refer to [26, 27, 9, 5] for further details.

Kähler classes are real, $(1, 1)$ cohomology classes satisfying a positivity condition and define a cone $\mathcal{K} \subset H^{1,1}(M) \cap H^2(M, \mathbb{R})$. We have the following theorem:

THEOREM 1.5.19 (Mixed hard Lefschetz theorem) *Let M be a compact Kähler manifold of dimension n. Let $\omega_1, \ldots, \omega_k \in \mathcal{K}$, $1 \leq k \leq n$. Then the map*

$$L_{\omega_1} \cdots L_{\omega_k} \colon H^{n-k}(M, \mathbb{C}) \to H^{n+k}(M, \mathbb{C})$$

is an isomorphism.

As mentioned above, this result was originally formulated by Gromov who proved it in the $(1,1)$ case (note that the operators involved preserve the Hodge decomposition). Later, Timorin [26, 27] proved it in the linear algebra case and in the case of simplicial toric varieties. Dinh and Nguyên [9] proved it in the form stated above. In [5] the author gave a proof in the context of variations of Hodge structure which unifies those previous results as well as similar results in other contexts [14, 3].