Chapter One

Introduction

These lectures are devoted to the interplay between cohomology and Chow groups, and also to the consequences, for the topology of a family of smooth projective varieties, of statements concerning Chow groups of the general or very general fiber.

A crucial notion is that of the *coniveau* of a cohomology. A Betti cohomology class has geometric coniveau $\geq c$ if it is supported on a closed algebraic subset of codimension $\geq c$. The coniveau of a class of degree k is $\leq \frac{k}{2}$. As a smooth projective variety X has nonzero cohomology in degrees $0, 2, 4, \ldots$ obtained by taking $c_1(L)$, with L an ample line bundle on X, and its powers $c_1(L)^i$, it is not expected that the whole cohomology of X has large coniveau. But it is quite possible that the "transcendental" cohomology $H^*_B(X)^{\perp \mathrm{alg}}$, consisting of classes orthogonal (with respect to the Poincaré pairing) to cycle classes on X, has large coniveau.

There is another notion of coniveau: the *Hodge coniveau*, which is computed by looking at the shape of the Hodge structures on $H^*_B(X, \mathbb{Q})$. Classes of algebraic cycles are conjecturally detected by Hodge theory as Hodge classes, which are the degree $2k$ rational cohomology classes of Hodge coniveau k. The generalized Hodge conjecture due to Grothendieck [50] more generally identifies the coniveau above (or geometric coniveau) to the Hodge coniveau.

The next crucial idea goes back to Mumford [71], who observed that for a smooth projective surface S, there is a strong correlation between the structure of the group $\mathrm{CH}_0(S)$ of 0-cycles on S modulo rational equivalence and the spaces of holomorphic forms on S. The degree 1 holomorphic forms govern the Albanese map, which itself provides us with a certain natural quotient of the group $\mathrm{CH}_0(S)_{\mathrm{hom}}$ of 0-cycles homologous to 0 (that is, of degree 0 if S is connected), which is in fact an abelian variety. This part of $\mathrm{CH}_0(S)_{\mathrm{hom}}$ is small in different (but equivalent) senses, first of all because it is parametrized by an algebraic group, and second because, for any ample curve $C \subset S$, the composite map

$$\mathrm{CH}_0(C)_{\mathrm{hom}} \to \mathrm{CH}_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$$

is surjective. Thus 0-cycles supported on a given ample curve are sufficient to exhaust this part of $\mathrm{CH}_0(S)_{\mathrm{hom}}$.

Mumford's theorem [71] says the following.

THEOREM 1.1 (Mumford 1968). *If $H^{2,0}(S) \neq 0$, no curve $C \stackrel{j}{\hookrightarrow} S$ satisfies the property that $j_* : \mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective.*

The parallel with geometric coniveau in cohomology is obvious in this case; indeed, the assumption that $H^{2,0}(S) \neq 0$ is equivalent (by the Lefschetz theorem on $(1,1)$-classes) to the fact that the cohomology $H^2_B(S, \mathbb{Q})$ is not supported on a divisor of S. Thus Mumford's theorem exactly says that if the degree 2 cohomology of S is not supported on any divisor, then its Chow group $\mathrm{CH}_0(S)$ is not supported on any divisor.

The converse to such a statement is the famous Bloch conjecture [13]. The Bloch conjecture has been generalized in various forms, one involving filtrations on Chow groups, the graded pieces of the filtration being governed by the coniveau of Hodge structures of adequate degree (see [58], [89], and Section 2.1.4). The crucial properties of this conjectural filtration are functoriality under correspondences, finiteness, and the fact that correspondences homologous to 0 shift the filtration.

We will focus in these notes on a more specific higher-dimensional generalization of the Bloch conjecture, "the generalized Bloch conjecture," which says that if the cohomology $H^*_B(X, \mathbb{Q})^{\perp \mathrm{alg}}$ has coniveau $\geq c$, then the cycle class map $\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H^{2n-2i}_B(X, \mathbb{Q})$ is injective for $i \leq c - 1$. In fact, if the variety X has dimension > 2, there are two versions of this conjecture, according to whether we consider the geometric or the Hodge coniveau. Of course, the two versions are equivalent assuming the generalized Hodge conjecture. In Section 4.3 we will prove this conjecture, following [114], for the geometric coniveau and for very general complete intersections of ample hypersurfaces in a smooth projective variety X with "trivial" Chow groups, that is, having the property that the cycle class map

$$\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}_B(X, \mathbb{Q})$$

is injective (hence an isomorphism according to [67]).

A completely different approach to such statements was initiated by Kimura [59], and it works concretely for those varieties that are dominated by products of curves. It should be mentioned here that all we have said before works as well in the case of motives (see Section 2.1.3). In the above-mentioned work of Kimura, one can replace "varieties that are dominated by products of curves" by "motives that are a direct summand of the motive of a product of curves." In our paper [114], we can work with a variety X endowed with the action of a finite group G and consider the submotives of G-invariant complete intersections obtained by considering the projectors $\Gamma_\pi \in \mathrm{CH}(Y \times Y)_{\mathbb{Q}}$ associated via the action of G on Y to projectors $\pi \in \mathbb{Q}[G]$.

An important tool introduced by Bloch and Srinivas in [15] is the so-called decomposition of the diagonal. It relates information concerning Chow groups $\mathrm{CH}_i(X)$, for small i, to the geometric coniveau of X. Bloch and Srinivas initially considered the decomposition of the diagonal in its simplest form, starting from information on $\mathrm{CH}_0(X)$, and this has subsequently been generalized in [66], [80] to a generalized decomposition of the diagonal. This leads to an elegant proof of the generalized Mumford–Roitman theorem, stating that if the cycle class map

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2n-2i}(X, \mathbb{Q})$$

is injective for $i \leq c-1$, then the transcendental cohomology $H_B^*(X,\mathbb{Q})^{\perp \mathrm{alg}}$ has geometric coniveau $\geq c$. (The generalized Bloch conjecture is thus the converse to this statement.)

The study of the diagonal will play a crucial role in our proof of the generalized Bloch conjecture for very general complete intersections. The diagonal will appear in a rather different context in Chapter 5, where we will describe our joint work with Beauville and further developments concerning the Chow rings of $K3$ surfaces and hyper-Kähler manifolds. Here we will be concerned not with the diagonal $\Delta_X \subset X \times X$ but with the small diagonal $\Delta \cong X \subset X \times X \times X$. The reason is that if we consider Δ as a correspondence from $X \times X$ to X, we immediately see that it governs, among other things, the ring structure of $\mathrm{CH}^*(X)$. In [11] we obtained for $K3$ surfaces X a decomposition of Δ involving the large diagonals, and a certain canonical 0-cycle o canonically attached to X.

We will show in Section 5.3 an unexpected consequence, obtained in [110], of this study combined with the basic spreading principle described in Section 3.1, concerning the topology of families of $K3$ surfaces.

In a rather different direction, in the final chapter we present recent results concerning Chow groups and Hodge classes with integral coefficients. Playing on the defect of the Hodge conjecture for integral Hodge classes (see [5]), we exhibit a number of birational invariants which vanish for rational projective varieties and are of torsion for unirational varieties. Among them is precisely the failure of the Bloch–Srinivas diagonal decomposition with integral coefficients: in general, under the assumption that $\mathrm{CH}_0(X)$ is small, only a multiple of the diagonal of X can be decomposed as a cycle in $X \times X$. The minimal such multiple appears to be an interesting birational invariant of X.

In the rest of this introduction, we survey the main ideas and results presented in this monograph a little more precisely. Background material is to be found in Chapter 2.

1.1 DECOMPOSITION OF THE DIAGONAL AND SPREAD

1.1.1 Spread

The notion of the spread of a cycle is very important in the geometric study of algebraic cycles. The first place where it appears explicitly is Nori's paper [76], where it is shown that the cohomology class of the spread cycle governs many invariants of the cycle restricted to general fibers. The idea is the following (see also [47]): Assume that we have a family of smooth algebraic varieties, that is, a smooth surjective morphism

$$\pi : \mathcal{X} \to B,$$

with geometric generic fiber $\mathcal{X}_{\overline{\eta}}$ and closed fiber $\mathcal{X}_s$. If we have a cycle $Z \in \mathcal{Z}^k(\mathcal{X}_{\overline{\eta}})$, then we can find a finite cover $\widetilde{U} \to U$ of a Zariski open set U of B such that Z is the restriction to the geometric generic fiber of a cycle $\mathcal{Z}_{\widetilde{U}} \in \mathcal{Z}^k(\mathcal{X}_{\widetilde{U}})$.

If we are over $\mathbb{C}$, we can speak of the spread of a cycle $Z_s \in \mathcal{Z}^k(X_s)$, where $s \in B$ is a very general point. Indeed, we may assume that π is projective. We know that there are countably many relative Hilbert schemes $M_i \to B$ parametrizing all subschemes in fibers of π. Cycles $Z = \sum_i n_i Z_i$ in the fibers of π are similarly parametrized by countably many varieties $\pi_J : N_J \to B$, where the π_J's are proper, and the indices J also encode the multiplicities n_i.

Let $B' \subset B$ be the complement of the union $\cup_{J \in E} \operatorname{Im} \pi_J$, where E is the set of indices J for which π_J is not surjective. A point of B' is a very general point of B, and by construction of B', for any $s \in B'$, and any cycle $Z_s \in \mathcal{Z}^k(X_s)$, there exist an index J such that $M_J \to B$ is surjective, and a point $s' \in M_J$ such that $\pi_J(s') = s$, and the fiber $\mathcal{Z}_{M_J,s'}$ at s' of the universal cycle

$$\mathcal{Z}_{M_J} \subset \mathcal{X}_{M_J} = \mathcal{X} \times_B M_J,$$

parametrized by M_J, is the cycle Z_s. By taking linear sections, we can then find $M'_J \subset M_J$, with $s' \in M'_J$, such that the morphism $M'_J \to B$ is dominating and generically finite. The restriction $\mathcal{Z}_{M'_J}$ of the universal cycle $\mathcal{Z}_{M_J}$ to $\mathcal{X} \times_B M'_J$ is then a spread of Z_s.

1.1.2 Spreading out rational equivalence

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism, where B is smooth irreducible and quasi-projective, and let $\mathcal{Z} \subset \mathcal{X}$ be a codimension k cycle. Let us denote by $Z_t \subset X_t$ the restriction of $\mathcal{Z}$ to the fiber X_t. We refer to Chapter 2 for the basic notions concerning rational equivalence, Chow groups, and cycle classes.

An elementary but fundamental fact is the following result, proved in Section 3.1.

THEOREM 1.2 (See Theorem 3.1). *If for any $t \in B$ the cycle Z_t is rationally equivalent to 0, there exist a Zariski open set $U \subset B$ and a nonzero integer N such that $N\mathcal{Z}_{|\mathcal{X}_U}$ is rationally equivalent to 0, where $\mathcal{X}_U := \pi^{-1}(U)$.*

Note that the set of points $t \in B$ such that Z_t is rationally equivalent to 0 is a countable union of closed algebraic subsets of B, so that we could in the above statement, by a Baire category argument, make the a priori weaker (but in fact equivalent) assumption that Z_t is rationally equivalent to 0 for a very general point of B.

This statement is what we call the spreading-out phenomenon for rational equivalence. This phenomenon does not occur for weaker equivalence relations such as algebraic equivalence.

An immediate but quite important corollary is the following.

COROLLARY 1.3. *In the situation of Theorem 1.2, there exists a dense Zariski open set $U \subset B$ such that the Betti cycle class $[\mathcal{Z}] \in H^{2k}_B(\mathcal{X}, \mathbb{Q})$ vanishes on the open set $\mathcal{X}_U$.*

The general principle above applied to the case where the family $\mathcal{X} \to B$ is trivial, that is, $\mathcal{X} \cong X \times B$, leads to the so-called decomposition principle due to Bloch and Srinivas [15]. In this case, the cycle $\mathcal{Z} \subset B \times X$ can be seen as a family of cycles on X parametrized by B or as a correspondence between B and X. Then Theorem 1.2 says that if a correspondence $\mathcal{Z} \subset B \times X$ induces the trivial map

$$\mathrm{CH}_0(B) \to \mathrm{CH}^k(X), \quad b \mapsto Z_b,$$

then the cycle $\mathcal{Z}$ vanishes up to torsion on some open set of the form $U \times X$, where U is a dense Zariski open set of B.

The first instance of the diagonal decomposition principle appears in [15]. This is the case where $X = Y \setminus W$, with Y smooth and projective, and $W \subset Y$ is a closed algebraic subset, $B = Y$, and $\mathcal{Z}$ is the restriction to $Y \times (Y \setminus W)$ of the diagonal of Y. In this case, to say that the map

$$\mathrm{CH}_0(B) \to \mathrm{CH}_0(X), \quad b \mapsto Z_b,$$

is trivial is equivalent to saying, by the localization exact sequence (2.2), that any point of Y is rationally equivalent to a 0-cycle supported on W. The conclusion is then the fact that the restriction of the diagonal cycle Δ to a Zariski open set $U \times (Y \setminus W)$ of $Y \times Y$ is of torsion, for some dense Zariski open set $U \subset Y$. Using the localization exact sequence, one concludes that a multiple of the diagonal is rationally equivalent in $Y \times Y$ to the sum of a cycle supported on $Y \times W$ and a cycle supported on $D \times Y$, where $D := Y \setminus U$. Passing to cohomology, we get the following consequence.

Corollary 1.4. *If Y is smooth projective of dimension n and $\mathrm{CH}_0(Y)$ is supported on $W \subset Y$, the class $[\Delta_Y] \in H_B^{2n}(Y \times Y, \mathbb{Q})$ decomposes as*

$$[\Delta_Y] = [Z_1] + [Z_2],$$

where the cycles Z_i are cycles with $\mathbb{Q}$-coefficients on $Y \times Y$, Z_1 is supported on $D \times Y$ for some proper closed algebraic subset $D \subsetneq Y$, and Z_2 is supported on $Y \times W$.

1.1.3 Applications of Mumford-type theorems

In the paper [15] by Bloch and Srinivas, an elegant proof of Mumford's theorem (Theorem 1.1) is provided, together with the following important generalization.

Theorem 1.5 (Roitman 1980, Bloch and Srinivas 1983; see Theorem 3.13). *Let X be a smooth projective variety and $W \subset X$ be a closed algebraic subset of dimension $\leq k$ such that any point of X is rationally equivalent to a 0-cycle supported on W. Then $H^0(X, \Omega_X^l) = 0$ for $l > k$.*

This theorem, together with other very important precisions concerning the coniveau (see Section 2.2.5) of the cohomology of X, is obtained using only the cohomological decomposition of the diagonal of X, that is, Corollary 1.4.

Theorem 1.2 is also the only ingredient in the proof of the generalized decomposition of the diagonal (see [66], [80], [101, II, 10.3], and Section 3.2.1). The Bloch–Srinivas decomposition of the diagonal described in the previous subsection is a decomposition (up to torsion and modulo rational equivalence) involving two pieces, one supported via the first projection over a divisor of X, the other supported via the second projection over a subset $W \subset X$. It is obtained under the condition that $\mathrm{CH}_0(X)$ is supported on W. The generalized decomposition of the diagonal subsequently obtained independently by Laterveer and Paranjape is the following statement, where X is smooth projective of dimension n.

THEOREM 1.6 (See Theorem 3.18). *Assume that for $k < c$, the cycle class maps*

$$\mathrm{cl} : \mathrm{CH}_k(X) \otimes \mathbb{Q} \to H_B^{2n-2k}(X, \mathbb{Q})$$

are injective. Then there exists a decomposition

$$m\Delta_X = Z_0 + \cdots + Z_{c-1} + Z' \in \mathrm{CH}^n(X \times X), \tag{1.1}$$

where $m \neq 0$ is an integer, Z_i is supported in $W_i' \times W_i$ with $\dim W_i = i, \dim W_i' = n - i$, and Z' is supported in $T \times X$, where $T \subset X$ is a closed algebraic subset of codimension $\geq c$.

Note that a version of this theorem involving the Deligne cycle class instead of the Betti cycle class was established in [37]. We refer to Section 3.2.1 for applications of this theorem. In fact, the main application involves only the corresponding cohomological version of the decomposition (1.1), that is, the generalization of Corollary 1.4, which concerned the case $c = 1$. It implies that under the same assumptions, the transcendental cohomology $H_B^*(X, \mathbb{Q})^{\perp \mathrm{alg}}$ has geometric coniveau $\geq c$.

Other applications involve the decomposition (1.1) in the group $\mathrm{CH}(X \times X)/\mathrm{alg}$ of cycles modulo algebraic equivalence. This is the case of applications to the vanishing of positive degree unramified cohomology with $\mathbb{Q}$-coefficients (and in fact with $\mathbb{Z}$-coefficients (see [6], [15], [24]) thanks to the Bloch–Kato conjecture proved by Rost and Voevodsky; see [97]).

1.1.4 Another spreading principle

Although elementary, the following spreading result proved in Section 4.3.3 is crucial for our proof of the equivalence of the generalized Bloch and Hodge conjectures for very general complete intersections in varieties with trivial Chow groups (see [114] and Section 4.3). Its proof is based as usual on the countability of the relative Hilbert schemes for a smooth projective family $\mathcal{Y} \to B$.

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism and let $(\pi, \pi) : \mathcal{X} \times_B \mathcal{X} \to B$ be the fibered self-product of $\mathcal{X}$ over B. Let $\mathcal{Z} \subset \mathcal{X} \times_B \mathcal{X}$ be a codimension k algebraic cycle. We denote the fibers $\mathcal{X}_b := \pi^{-1}(b)$, $\mathcal{Z}_b := \mathcal{Z}_{|\mathcal{X}_b \times \mathcal{X}_b}$.

THEOREM 1.7 (See Proposition 4.25). *Assume that for a very general point $b \in B$, there exist a closed algebraic subset $Y_b \subset \mathcal{X}_b \times \mathcal{X}_b$ of codimension c, and an algebraic cycle $Z'_b \subset Y_b \times Y_b$ with $\mathbb{Q}$-coefficients, such that*

$$[Z'_b] = [\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

Then there exist a closed algebraic subset $\mathcal{Y} \subset \mathcal{X}$ of codimension c, and a codimension k algebraic cycle $\mathcal{Z}'$ with $\mathbb{Q}$-coefficients on $\mathcal{X} \times_B \mathcal{X}$, such that $\mathcal{Z}'$ is supported on $\mathcal{Y} \times_B \mathcal{Y}$, and for any $b \in B$,

$$[\mathcal{Z}'_b] = [\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

1.2 THE GENERALIZED BLOCH CONJECTURE

As we will explain in Section 3.2.1, the generalized decomposition of the diagonal (Theorem 1.6, or rather its cohomological version) leads to the following result.

THEOREM 1.8 (See Theorem 3.20). *Let X be a smooth projective variety of dimension m. Assume that the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$$

is injective for $i \leq c-1$. Then we have $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$).

The Hodge structures on $H_B^k(X, \mathbb{Q})^{\perp \mathrm{alg}}$ are thus all of Hodge coniveau $\geq c$; in fact they are even of geometric coniveau $\geq c$, that is, these Hodge structures satisfy the generalized Hodge conjecture (Conjecture 2.40) for coniveau c.

The generalized Bloch conjecture is the converse to this statement (it can also be generalized to motives). It generalizes the Bloch conjecture which concerned the case of 0-cycles on surfaces. One way to state it is the following.

CONJECTURE 1.9. *Assume conversely that $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$). Then the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$$

is injective for $i \leq c-1$.

However, as a consequence of Theorem 1.8 above, this formulation also contains a positive solution to the generalized Hodge conjecture, which predicts, under the above vanishing assumptions, that the transcendental part of the cohomology of X is supported on a closed algebraic subset of codimension c.

A slightly restricted version of the generalized Bloch conjecture (which is equivalent for surfaces or motives of surfaces) is thus the following (see [58]).

CONJECTURE 1.10. *Let X be a smooth projective complex variety of dimension m. Assume that the transcendental cohomology $H_B^*(X, \mathbb{Q})^{\perp \mathrm{alg}}$ is supported on a closed algebraic subset of codimension c. Then the cycle class map $\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$ is injective for $i \leq c-1$.*

Note that Conjecture 1.11 below is also an important generalization of the Bloch conjecture, concerning only 0-cycles and starting with rather different cohomological assumptions. There is no direct relation between Conjectures 1.9 and 1.11 except for the fact that they coincide for varieties X with $H^{i,0}(X) = 0$ for all $i > 0$. Of course they both fit within the general Bloch–Beilinson conjecture (Conjecture 2.19) on filtrations on Chow groups, and can be considered as concrete consequences of them.

CONJECTURE 1.11. *Let X be a smooth projective variety such that $H^{i,0}(X) = 0$ for $i > r$. Then there exists a dimension r closed algebraic subset $Z \overset{j}{\hookrightarrow} X$ such that $j_* : \mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$ is surjective.*

Our main result in [114] presented in Section 4.3 is obtained as a consequence of the spreading principle, Theorem 1.7. It proves Conjecture 1.10 for very general complete intersections in ambient varieties with "trivial Chow groups," assuming the Lefschetz standard conjecture. The situation is the following: X is a smooth projective variety of dimension n that satisfies the property that the cycle class map

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2n-2i}(X, \mathbb{Q})$$

is injective for all i. Let L_i, $i = 1, \ldots, r$ be very ample line bundles on X. We consider smooth complete intersections $X_t \subset X$ of hypersurfaces $X_i \in |L_i|$. They are parametrized by a quasi-projective base B.

THEOREM 1.12 (Voisin 2011; see Theorem 4.16). *Assume that for the very general point $t \in B$, the vanishing cohomology $H_B^{n-r}(X_t, \mathbb{Q})_{\mathrm{van}}$ is supported on a codimension c closed algebraic subset of X_t. Assume also the Lefschetz standard conjecture. Then the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X_t)_{\mathbb{Q}} \to H_B^{2n-2r-2i}(X_t, \mathbb{Q})$$

is injective for $i \leq c - 1$.

In dimension $(n - r) \geq 4$, the theorem above is conditional on the Lefschetz standard conjecture (or more precisely on Conjecture 2.29 for codimension $(n-r)$ cycles). It turns out that in dimensions $(n - r) \leq 3$, the precise instance of the conjecture we need will be satisfied, so that the result is unconditional for surfaces and threefolds.

In applications, this theorem is particularly interesting in the case where X is the projective space $\mathbb{P}^n$. In this case, the hypersurfaces X_i are characterized by their degrees d_i and we may assume $d_1 \leq \cdots \leq d_r$. When n is large compared to the d_i's, the Hodge coniveau (conjecturally, the geometric coniveau) of X_t is also large due to the following result established in [48] in the case of hypersurfaces (see [38] for the case of complete intersections), the proof of which will be sketched in Section 4.1.

THEOREM (See Theorem 4.1). *A smooth complete intersection $X_t \subset \mathbb{P}^n$ of r hypersurfaces of degrees $d_1 \leq \cdots \leq d_r$ has Hodge coniveau $\geq c$ if and only if*

$$n \geq \sum_i^r d_i + (c-1)d_r.$$

If we consider motives (see Section 2.1.3) and in particular those that are obtained starting from a variety X with an action by finite group G, and looking at invariant complete intersections $X_t \subset X$ and motives associated to projectors of G, we get many new examples where the adequate variant of Theorem 1.12 holds, because a submotive often has a larger coniveau. A typical example is the case of Godeaux quintic surfaces, which are free quotients of quintic surfaces in $\mathbb{P}^3$ invariant under a certain action of $G \cong \mathbb{Z}/5\mathbb{Z}$. The G-invariant part of $H^{2,0}(S)$ is 0 although the quotient surface S/G is of general type; the Bloch conjecture has already been proved for the quotient surfaces S/G in [98] but the proof we give here is much simpler and has a much wider range of applications. In fact, a much softer version of Theorem 1.12 for surfaces is established in [109], and it gives a proof of the Bloch conjecture for other surfaces with $p_g = q = 0$.

In Section 3.2.3, we will also describe the ideas of Kimura, which lead to results of a similar shape, namely the implication from the generalized Hodge conjecture (geometric coniveau = Hodge coniveau) to the generalized Bloch conjecture (the Chow groups CH_i are "trivial" for i smaller than the Hodge coniveau), but for a completely different class of varieties. More precisely, Kimura's method applies to all motives that are direct summands in the motive of a product of curves.

1.3 DECOMPOSITION OF THE SMALL DIAGONAL AND APPLICATION TO THE TOPOLOGY OF FAMILIES

In Chapter 5, we will exploit the spreading principle, Theorem 1.2(ii), or rather its cohomological version, to exhibit a rather special phenomenon satisfied by families of abelian varieties, $K3$ surfaces, and conjecturally also by families of Calabi–Yau hypersurfaces in projective space. Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. The decomposition theorem, proved by Deligne in [30] as a consequence of the hard Lefschetz theorem, is the following statement.

THEOREM (Deligne 1968). *In the derived category of sheaves of $\mathbb{Q}$-vector spaces on B, there is a decomposition*

$$R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]. \tag{1.2}$$

The question we consider in Chapter 5, following [110], is the following.

QUESTION 1.13. *Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, does there exist a decomposition as above that is multiplicative, that is, compatible with the morphism $\mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ given by the cup-product?*

As we will see quickly, the answer is generally negative, even for projective bundles $\mathbb{P}(E) \to B$. However, it is affirmative for families of abelian varieties (see Section 5.3) due to the group structure of the fibers.

The right formulation in order to get a larger range of families satisfying such a property is to ask whether there exists a decomposition isomorphism that is multiplicative over a Zariski dense open set U of the base B (or more optimistically, that is multiplicative locally on B for the Zariski topology).

One of our main results in this chapter is the following (see [110] and Section 5.3).

THEOREM 1.14 (Voisin 2011; see Theorem 5.35).

(i) For any smooth projective family $\pi : \mathcal{X} \to B$ of $K3$ surfaces, there exist a nonempty Zariski open subset B^0 of B and a multiplicative decomposition isomorphism as in (5.27) for the restricted family $\pi : \mathcal{X}^0 \to B^0$.

(ii) The class of the diagonal $[\Delta_{\mathcal{X}^0/B^0}] \in H^4_B(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ belongs to the direct summand $H^0(B^0, R^4(\pi,\pi)_*\mathbb{Q})$ of $H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0, \mathbb{Q})$ for the induced decomposition of $R(\pi,\pi)_*\mathbb{Q}$.

(iii) For any line bundle L on $\mathcal{X}$, there is a Zariski dense open set $B^0 \subset B$ such that its topological first Chern class $c_1^{\rm top}(L) \in H^2_B(\mathcal{X}, \mathbb{Q})$ restricted to $\mathcal{X}^0$ belongs to the direct summand $H^0(B^0, R^2\pi_*\mathbb{Q})$.

In the second statement, $(\pi,\pi) : \mathcal{X}^0 \times_{B^0} \mathcal{X}^0 \to B^0$ denotes the natural map. A decomposition $R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i]$ induces a decomposition

$$R(\pi,\pi)_*\mathbb{Q} = \oplus_i R^i(\pi,\pi)_*\mathbb{Q}[-i]$$

by the relative Künneth isomorphism

$$R(\pi,\pi)_*\mathbb{Q} \cong R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q}.$$

In statements (ii) and (iii), we use the fact that a decomposition isomorphism as in (1.2) for $\pi : \mathcal{X} \to B$ induces a decomposition

$$H^k(\mathcal{X}, \mathbb{Q}) \cong \oplus_{p+q=k} H^p(B, R^q\pi_*\mathbb{Q})$$

(which is compatible with cup-product if the given decomposition isomorphism is).

This result is in fact a formal consequence of the spreading principle (Theorem 1.2) and of the following result proved in [11].

THEOREM (Beauville and Voisin 2004; see Theorem 5.3). *Let S be a smooth projective $K3$ surface. There exists a 0-cycle $o \in \mathrm{CH}_0(S)$ such that we have the following equality:*

$$\Delta = \Delta_{12} \cdot o_3 + (\text{perm.}) - (o_1 \cdot o_2 + (\text{perm.})) \text{ in } \mathrm{CH}^4(S \times S \times S)_{\mathbb{Q}}. \qquad (1.3)$$

Here $\Delta \subset S \times S \times S$ is the small diagonal $\{(x, x, x),\ x \in S\}$ and the Δ_{ij}'s are the diagonals $x_i = x_j$. The class $o \in \mathrm{CH}_0(S)$ is the class of any point belonging to a rational curve in S and the o_i's are its pull-back in $\mathrm{CH}^2(S \times S \times S)_{\mathbb{Q}}$ via the various projections. The cycle $\Delta_{12} \cdot o_3$ is thus the algebraic subset $\{(x, x, o),\ x \in S\}$ of $S \times S \times S$. The term +(perm.) means that we sum over the permutations of $\{1, 2, 3\}$.

We will also prove in Section 5.2.1 a partial generalization of this result for Calabi–Yau hypersurfaces.

Our topological application (Theorem 1.14 above) involves the structure of the cup-product map

$$R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q},$$

which is not surprising since the small diagonal, seen as a correspondence from $S \times S$ to itself, governs the cup-product map on cohomology, as well as the intersection product on Chow groups. For the same reason, the decomposition (1.3) is related to properties of the Chow ring of $K3$ surfaces. These applications are described in Section 5.1. Our partial decomposition result concerning the small diagonal of Calabi–Yau hypersurfaces (Theorem 5.21) allows the following result to be proved in a similar way (see [110]).

THEOREM (Voisin 2011; see Theorem 5.25). *Let $X \subset \mathbb{P}^n$ be a Calabi–Yau hypersurface in projective space. Let Z_i, Z'_i be positive-dimensional cycles on X such that* $\mathrm{codim}\, Z_i + \mathrm{codim}\, Z'_i = n - 1$. *Then if we have a cohomological relation*

$$\sum_i n_i [Z_i] \cup [Z'_i] = 0 \ \textit{in} \ H_B^{2n-2}(X, \mathbb{Q}),$$

this relation already holds at the level of Chow groups:

$$\sum_i n_i Z_i \cdot Z'_i = 0 \ \textit{in} \ \mathrm{CH}^{n-1}(X)_{\mathbb{Q}}.$$

1.4 INTEGRAL COEFFICIENTS AND BIRATIONAL INVARIANTS

Everything that has been said before was with rational coefficients. This concerns Chow groups and cohomology. In fact it is known that the Hodge conjecture is wrong with integral coefficients and also that the diagonal decomposition principle (Theorem 3.10 or Corollary 3.12) is wrong with integral coefficients (that is, with an integer N set equal to 1). But it turns out that some birational invariants can be constructed out of this. For example, assume for simplicity that $\mathrm{CH}_0(X)$ is trivial. Then the smallest positive integer N such that there is a Chow-theoretic (respectively cohomological) decomposition of N times the diagonal in $\mathrm{CH}^n(X \times X)$ (respectively $H^{2n}(X \times X, \mathbb{Z})$), $n = \dim X$, that is,

$$N\Delta_X = Z_1 + Z_2, \ \text{(respectively, } N[\Delta_X] = [Z_1] + [Z_2]), \tag{1.4}$$

where Z_1, Z_2 are codimension n cycles in $X \times X$ with

$$\operatorname{Supp} Z_1 \subset T \times X, \quad T \subsetneq X, \qquad Z_2 = X \times x$$

for some $x \in X$, is a birational invariant of X. In the cohomological case, on which we will focus, we get in general a nontrivial invariant simply because, as we will show, it annihilates the torsion in $H^3_B(X,\mathbb{Z})$ and, at least when $\dim X \leq 3$, the torsion in the whole integral cohomology $H^*_B(X,\mathbb{Z})$. But we will see that it also controls much more subtle phenomena.

The groups $Z^{2i}(X)$ defined as

$$Z^{2i}(X) := \operatorname{Hdg}^{2i}(X,\mathbb{Z})/H^{2i}_B(X,\mathbb{Z})_{\mathrm{alg}}$$

themselves are birationally invariant for $i = 2$, $i = n-1$, $n = \dim X$, as remarked in [93], [105]. One part of Chapter 6 is devoted to describing recent results concerning the groups $Z^{2i}(X)$ for X a rationally connected smooth projective complex variety. On the one hand, it is proved in [24] that for such an X, the group $Z^4(X)$ is equal to the third unramified cohomology group $H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$ with torsion coefficients. This result, also proved in [6], is a consequence of the Bloch–Kato conjecture, which has now been proved by Voevodsky and Rost. We will sketch in Section 6.2.2 the basic facts from Bloch–Ogus theory (see [14]) that are needed to establish this result.

It follows then from the work of Colliot-Thélène and Ojanguren [23] that there are rationally connected sixfolds X for which $Z^4(X) \neq 0$. Such examples are not known in smaller dimensions, and we proved that they do not exist in dimension 3. We even have the following result [103].

Theorem (See Theorem 6.5). *Let X be a smooth projective threefold that is either uniruled or Calabi–Yau (that is, with trivial canonical bundle and $H^1(X,\mathcal{O}_X) = 0$). Then $Z^4(X) = 0$.*

The Calabi–Yau case has been extended and used by Höring and Voisin in [53] to show the following (see also [41] for the generalization to Fano n-folds of index $n-3$, $n \geq 8$).

Theorem (See Section 6.2.1). *Let X be a Fano fourfold or a Fano fivefold of index 2. Then the group $Z^{2n-2}(X)$ is trivial, $n = \dim X$. Hence the cohomology $H^{2n-2}_B(X,\mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of curves.*

It is in fact very likely that the group $Z^{2n-2}(X)$ is trivial for rationally connected varieties, as follows from its invariance under deformations and also specialization to characteristic p (see Theorem 6.10 and [112]).

The final section is devoted to the study of the existence of an integral cohomological decomposition of the diagonal, particularly for a rationally connected threefold. This is in fact closely related in this case to the integral Hodge conjecture, in the following way: If X is a rationally connected threefold (or more generally, any smooth projective threefold with $h^{3,0}(X) = 0$), then the intermediate Jacobian $J(X)$ is an abelian variety. There is a degree 4 integral Hodge class $\alpha \in \mathrm{Hdg}^4(X \times J(X), \mathbb{Z})$ built using the canonical isomorphism

$$H_1(J(X), \mathbb{Z}) \cong H^3_B(X, \mathbb{Z})/\text{torsion}.$$

As we will show in Theorem 6.38, if X admits an integral cohomological decomposition of the diagonal, that is, a decomposition as in (1.4) with $N = 1$, the above class α is algebraic, or equivalently, there is a universal codimension 2 cycle on X parametrized by $J(X)$. The existence of such a universal codimension 2 cycle is not known in general even for rationally connected threefolds.

1.5 ORGANIZATION OF THE TEXT

Chapter 2 is introductory. We will review Chow groups, correspondences and their cohomological and Hodge-theoretic counterpart. The emphasis will be put on the notion of coniveau and the generalized Hodge conjecture which states the equality of geometric and Hodge coniveau.

Chapter 3 is devoted to a description of various forms of the "decomposition of the diagonal" and applications of it. This mainly leads to one implication that is well understood now, namely the fact that for a smooth projective variety X, having "trivial" Chow groups of dimension $\leq c - 1$ implies having (geometric) coniveau $\geq c$. We state the converse conjecture (generalized Bloch conjecture).

In Chapter 4, we will first describe how to compute the Hodge coniveau of complete intersections. We will then explain a strategy to attack the generalized Hodge conjecture for complete intersections of coniveau 2. The guiding idea is that although the powerful method of the decomposition of the diagonal suggests that computing Chow groups of small dimension is the right way to solve the generalized Hodge conjecture, it might be better to invert the logic and try to compute the geometric coniveau directly. And indeed, this chapter culminates with the proof of the fact that for very general complete intersections, assuming Conjecture 2.29 or the Lefschetz standard conjecture, the generalized Hodge conjecture implies the generalized Bloch conjecture; in other words, for a very general complete intersection, the fact that its geometric coniveau is $\geq c$ implies the triviality of its Chow groups of dimension $\leq c - 1$.

In Chapter 5, we turn to the study of the Chow rings of $K3$ surfaces and other K-trivial varieties. This study is related to a decomposition of the small diagonal of the triple self-product. We finally show the consequences of this study for the topology of certain K-trivial varieties: $K3$ surfaces, abelian varieties, and Calabi–Yau hypersurfaces.

The final chapter is devoted in part to the study of the groups $Z^{2i}(X)$ measuring the failure of the Hodge conjecture with integral coefficients. Some vanishing and nonvanishing results are presented, together with a comparison of the group $Z^4(X)$ with the so-called unramified cohomology of X with torsion coefficients. We also consider various forms of the existence of an integral cohomological decomposition of the diagonal (see (1.1)) of a threefold X with trivial

CH_0 group. We show that an affirmative answer to this question is equivalent to the vanishing of numerous birational invariants of X.

Convention. Most of the time, we will work over $\mathbb{C}$ and we will write X for $X(\mathbb{C})$. Unless otherwise specified, cohomology of X with constant coefficients will be Betti cohomology of $X(\mathbb{C})$ endowed with the Euclidean topology (only in Chapter 6 will we discuss other cohomology theories). A general point of X is a complex point $x \in U(\mathbb{C})$, where $U \subset X$ is a Zariski dense open set. When we say that a property is satisfied by a general point, we thus mean that there exists a Zariski open set $U \subset X$ such that the property is satisfied for any point of $U(\mathbb{C})$. This is not always equivalent to being satisfied at the geometric generic point (assuming X is connected), since some properties are not Zariski open. A typical example is the following: Let $\phi : Y \to X$ be a smooth projective morphism with fiber Y_t, $t \in X$. Let L be a line bundle on Y. The property that $\mathrm{Pic}\, Y_t = \mathbb{Z}L_{|Y_t}$ is not Zariski open.

A very general point of X is a complex point $x \in X' \subset X(\mathbb{C})$, where $X' \subset X(\mathbb{C})$ is the complement of a countable union of proper closed algebraic subsets of X. When we say that a property is satisfied by a very general point, we mean that there exists a countable collection of dense Zariski open sets U_i such that the property is satisfied by any element of $\cap_i U_i(\mathbb{C})$. For example, in the situation above, we find that the property $\mathrm{Pic}\, Y_t = \mathbb{Z}L_{|Y_t}$ is satisfied at a very general point of X if it is satisfied at the geometric generic point of X.