Chapter One

The Calderón-Zygmund Theory I: Ellipticity

Our story begins with a classical situation: convolution with homogeneous, Calderón-Zygmund kernels on $\mathbb{R}^n$. Let $S^{n-1} \hookrightarrow \mathbb{R}^n$ denote the unit sphere in $\mathbb{R}^n$. We let $C_z^\infty\left(S^{n-1}\right) \subset C^\infty\left(S^{n-1}\right)$ denote those $k \in C^\infty\left(S^{n-1}\right)$ with $\int_{S^{n-1}} k(\omega)\, d\omega = 0$ (where ω denotes the surface area measure on S^{n-1}).

For a function $k \in C_z^\infty\left(S^{n-1}\right)$, and a complex number $c \in \mathbb{C}$, we define a distribution $K = K(k,c) \in C_0^\infty(\mathbb{R}^n)'$ by

$$\langle K, f\rangle = cf(0) + \lim_{\epsilon\to 0} \int_{|x|>\epsilon} k(x/|x|)\frac{1}{|x|^n} f(x)\, dx, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{1.1}$$

LEMMA 1.0.1. (1.1) *defines a distribution.*[1]

PROOF. Fix $M > 0$, and let $\Gamma := \overline{B^n(M)}$, where $B^n(M) \subset \mathbb{R}^n$ is the ball of radius M in $\mathbb{R}^n$, centered at 0. Note that Γ is compact. Let $C^\infty(\Gamma)$ denote the Fréchet space of those $f \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset \Gamma$. We wish to show that for $f \in C^\infty(\Gamma)$, the limit in (1.1) exists and that $\langle K,\cdot\rangle : C^\infty(\Gamma) \to \mathbb{C}$ is continuous. Consider, for $f \in C^\infty(\Gamma)$,

$$\begin{aligned}\langle K, f\rangle &= cf(0) + \lim_{\epsilon\to 0}\int_{\epsilon<|x|\le M} k(x/|x|)\frac{1}{|x|^n} f(x)\, dx\\ &= cf(0) + \lim_{\epsilon\to 0}\int_{\epsilon<|x|\le M} k(x/|x|)\frac{1}{|x|^{n-1}}\frac{(f(x)-f(0))}{|x|}\, dx,\end{aligned}$$

where we have used that $\int k(\omega)\, d\sigma(\omega) = 0$. Since $|f(x) - f(0)|/|x| \le 2\|f\|_{C^1}$, and since $1/|x|^{n-1} \in L^1(\Gamma)$, the result follows. □

DEFINITION 1.0.2. *We define* CZ $\subset C_0^\infty(\mathbb{R}^n)'$ *to be the vector space of distributions given by*

$$\mathrm{CZ} = \left\{K(k,c) \;\middle|\; k \in C_z^\infty\left(S^{n-1}\right), c \in \mathbb{C}\right\},$$

where $K(k,c)$ *is given by* (1.1).

Remark 1.0.3 Definition 1.0.2 uses the standard space $C_0^\infty(\mathbb{R}^n)'$–the space of distributions on $\mathbb{R}^n$. The reader wishing background on this and other standard spaces which appear in this text (e.g., $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)'$) is referred to Appendix A for more details and further references.

[1] See Appendix A for the basic definitions concerning distributions.

Remark 1.0.4 CZ stands for "Calderón-Zygmund," and the distributions in CZ are called "Calderón-Zygmund kernels."

Corresponding to $K \in \mathrm{CZ}$, we define an operator $\mathrm{Op}(K) : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$\mathrm{Op}(K) f(x) = K * f(x).$$

Operators of the form $\mathrm{Op}(K)$ are often referred to as "Calderón-Zygmund singular integral operators." The main properties of these operators are outlined in the following theorems.

THEOREM 1.0.5. *For $K \in \mathrm{CZ}$, $\mathrm{Op}(K)$ extends to a bounded operator $\mathrm{Op}(K) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

In light of Theorem 1.0.5, if $K_1, K_2 \in \mathrm{CZ}$, it makes sense to consider

$$\mathrm{Op}(K_1)\,\mathrm{Op}(K_2)$$

as a composition of bounded operators on L^p $(1 < p < \infty)$.

THEOREM 1.0.6. *Let $K_1, K_2 \in \mathrm{CZ}$. There is $K_3 \in \mathrm{CZ}$ with*

$$\mathrm{Op}(K_3) = \mathrm{Op}(K_1)\,\mathrm{Op}(K_2).$$

Formally, we may write $K_3 = K_1 * K_2$.

We discuss proofs of the above two theorems in the next section, where we proceed in a more general setting.

Remark 1.0.7 Theorem 1.0.6 can be restated as saying that the space of operators

$$\{\mathrm{Op}(K) \mid K \in \mathrm{CZ}\}$$

is a subalgebra of the algebra bounded operators on L^p $(1 < p < \infty)$.

The operators $\mathrm{Op}(K)$ are homogeneous in the following sense. For $r > 0$ define a dilation operator by $D_r f(x) = f(rx)$. Then, $D_{1/r}\mathrm{Op}(K) D_r = \mathrm{Op}(K)$, for $K \in \mathrm{CZ}$. It is this basic dilation invariance that is fundamental to many of the aspects of Calderón-Zygmund singular integral operators. We refer to this theory and generalizations of it as "single-parameter theories" where this refers to the "single-parameter" dilations D_r; in this simple case, this means that D_r depends on only one variable r. One main goal of this monograph is to develop a reasonable, useful, and somewhat general definition of a "multi-parameter singular integral."

In this chapter and the next, we discuss various generalizations and modifications of the above definitions. These ideas are well known, and we refer to all of these situations as being "single-parameter." After, we turn to the "multi-parameter" situation, where new theory is developed.

Remark 1.0.8 Even though all the main ideas of this chapter are well known, we present them with an eye toward the new situation covered in Chapter 5. Results like Theorem 1.1.26 and Theorem 1.2.10, below, provide a main motivation for the new definitions in Chapter 5.

1.1 NON-HOMOGENEOUS KERNELS

It is not essential that we consider only operators which are exactly homogeneous in the sense that $D_{1/r}\mathrm{Op}(K)\,D_r = \mathrm{Op}(K)$. We merely need the *class* of operators we consider to be scale invariant in an appropriate way, i.e., that $D_{1/r}\mathrm{Op}(K)\,D_r$ be "of the same form" as $\mathrm{Op}(K)$. In addition, the operators we introduce can be seen in three equivalent ways (Theorem 1.1.23). This trichotomy will be a running theme throughout this monograph. We now turn to making these ideas precise.

Let $K \in C_0^\infty(\mathbb{R}^n)'$ be a distribution; we abuse notation and for $\phi \in C_0^\infty(\mathbb{R}^n)$ we write $\langle K, \phi\rangle = \int K(x)\,\phi(x)\;dx$. If, for some set $U \subseteq \mathbb{R}^n$ and for $\phi \in C_0^\infty(U)$, we have that $\int K(x)\,\phi(x)\;dx$ agrees with integration against an L^1_{loc} function, then we identify $K\big|_U$ with this function.[2]

DEFINITION 1.1.1. *We say $K \in C_0^\infty(\mathbb{R}^n)'$ is a Calderón-Zygmund kernel if:*

(i) (Growth Condition) For every multi-index α, $|\partial_x^\alpha K(x)| \le C_\alpha |x|^{-n-|\alpha|}$. In particular, we assume that $K(x)$ is a C^∞ function for $x \neq 0$.

(ii) (Cancellation Condition) For every bounded set $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$, we assume

$$\sup_{\substack{\phi\in\mathcal{B}\\ R>0}} \left| \int K(x)\,\phi(Rx)\;dx \right| \le C_{\mathcal{B}}.$$

The reader interested in a simple characterization of bounded subsets of $C_0^\infty(\Omega)$ is referred to Corollary A.1.26.

For each multi-index α, we define a semi-norm on the space of Calderón-Zygmund kernels by taking the least C_α in the Growth Condition. For each bounded set $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$, we define a semi-norm to be the least $C_{\mathcal{B}}$ in the Cancellation Condition. We give the space of Calderón-Zygmund kernels the coarsest topology with respect to which all of these semi-norms are continuous.[3]

Given a Calderón-Zygmund kernel, we define an operator, $\mathrm{Op}(K) : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by

$$\mathrm{Op}(K)\,f = f * K = \int f(x-y)\,K(y)\;dy.$$

Three of the fundamental properties of Calderón-Zygmund kernels are contained in the following theorem.

THEOREM 1.1.2. *(a) If K is a Calderón-Zygmund kernel, then*

$$\mathrm{Op}(K) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p < \infty.$$

More precisely, $\mathrm{Op}(K)$ *extends to a bounded operator*[4] $L^p \to L^p$, $1 < p < \infty$.

[2] See Appendix A for more details on this notation.

[3] For the notion of defining a topology by giving a family of semi-norms, we refer the reader to Appendix A.

[4] In the rest of the monograph, when we are given an operator T, initially defined on some dense subspace of L^p, and we say $T : L^p \to L^p$, we mean that T extends to a bounded operator $L^p \to L^p$.

(b) *If* K *is a Calderón-Zygmund kernel, then* $\mathrm{Op}(K)^* = \mathrm{Op}(K')$, *where* K' *is a Calderón-Zygmund kernel and* $\mathrm{Op}(K)^*$ *denotes the* L^2 *adjoint of* $\mathrm{Op}(K)$.

(c) *If* K_1 *and* K_2 *are Calderón-Zygmund kernels, then it makes sense to consider* $\mathrm{Op}(K_1)\,\mathrm{Op}(K_2)$*–for instance, as a bounded operator on* L^2. *We have*

$$\mathrm{Op}(K_1)\,\mathrm{Op}(K_2) = \mathrm{Op}(K_3)$$

for a Calderón-Zygmund kernel K_3. *Formally, we have* $K_3 = K_1 * K_2$.

Part (c) of Theorem 1.1.2 can be restated as saying that

$$\{\mathrm{Op}(K) \mid K \text{ is a Calderón-Zygmund kernel}\} \tag{1.2}$$

is a subalgebra of the algebra of bounded operators on L^2. As motivation for the more complicated situations which arise later, we review some aspects of the proof of Theorem 1.1.2.

The easiest way to see that (1.2) forms an algebra involves the Fourier transform. To use the Fourier transform, we introduce Schwartz space.

DEFINITION 1.1.3. $\mathcal{S}(\mathbb{R}^n)$*–the space of Schwartz functions on* $\mathbb{R}^n$*–is defined to be*

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall \alpha, \beta \sup_{x \in \mathbb{R}^n} \left|\partial_x^\alpha x^\beta f(x)\right| < \infty \right\},$$

and we give $\mathcal{S}(\mathbb{R}^n)$ *the Fréchet topology given by the countable family of semi-norms*[5]

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left|\partial_x^\alpha x^\beta f(x)\right|.$$

For a function $f \in \mathcal{S}(\mathbb{R}^n)$ we define the Fourier transform

$$\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)\, dx,$$

so that $\hat{f} : \mathbb{R}^n \to \mathbb{C}$. A fundamental fact is that the Fourier transform is an automorphism of the Fréchet space $\mathcal{S}(\mathbb{R}^n)$.[6] We denote by $\vee : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ the inverse Fourier transform. Because the Fourier transform is an automorphism of Schwartz space, we may define the Fourier transform on the space of "tempered distributions," $\mathcal{S}(\mathbb{R}^n)'$, by duality. Namely, for $K \in \mathcal{S}(\mathbb{R}^n)'$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we define $\int \widehat{K}(x) f(x)\, dx = \int K(x) \hat{f}(x)\, dx$. This extends the Fourier transform to be an automorphism of $\mathcal{S}(\mathbb{R}^n)'$.

Remark 1.1.4 The Fourier transform is well-adapted to convolution operators. Indeed, $(K * f)^\wedge = \widehat{K}\hat{f}$. This makes the Fourier transform a useful tool when dealing with the convolution operators in this section. Later in the monograph, we work with

[5]For the definition of a Fréchet space see Definition A.1.9.

[6]By this we mean that the Fourier transform is a bijective, linear, homeomorphism $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

operators which are not translation invariant, and the Fourier transform is not as directly applicable. As motivation for these later applications, we somewhat limit our use of the Fourier transform in this section. Some arguments which follow may have shorter arguments by way of the Fourier transform, but we choose the ones that follow as these are the ones we generalize. We do still use the Fourier transform heavily in this section, but as we move to more generalize situations, it will be used less and less. For a further discussion of this, see the remarks following the proof of Theorem 1.1.23.

In light of the Growth Condition, even though Calderón-Zygmund kernels are a priori elements of $C_0^\infty(\mathbb{R}^n)'$, we see that they may be extended to be elements of $\mathcal{S}(\mathbb{R}^n)'$. This allows us to characterize Calderón-Zygmund kernels in terms of their Fourier transforms. Indeed, we have the following theorem.

THEOREM 1.1.5. *Let $K \in \mathcal{S}(\mathbb{R}^n)'$. The following are equivalent.*

(i) K is a Calderón-Zygmund kernel.

(ii) $\widehat{K}$ is given by a function which satisfies

$$\left|\partial_\xi^\alpha \widehat{K}(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0. \tag{1.3}$$

Before we prove Theorem 1.1.5, we need to introduce a new notation that will be used throughout the monograph. We write $A \lesssim B$ for $A \leq CB$, where C is a constant which is independent of any relevant parameters. Also, we write $A \approx B$ for $A \lesssim B$ and $B \lesssim A$.

PROOF OF THEOREM 1.1.5. We begin with (i)⇒(ii). A priori, $\widehat{K}$ is only a tempered distribution. We will show that $\widehat{K}(\xi)$ agrees with a C^∞ function away from $\xi = 0$, and that for every α

$$\sup_{1\leq|\xi|\leq 2} \left|\partial_\xi^\alpha \widehat{K}(\xi)\right| \leq C_\alpha, \tag{1.4}$$

where C_α can be chosen to depend only on α and $\|K\|$, and where $\|\cdot\|$ is a continuous semi-norm (depending on α) on the space of Calderón-Zygmund kernels.

First, we see why this yields (ii). Indeed, for $R > 0$ if we replace K by the distribution $K^R(x) := R^n K(Rx)$, then $\left\{K^R \mid R > 0\right\}$ is a bounded set[7] of Calderón-Zygmund kernels. If we take the Fourier transform of K^R, we obtain $\widehat{K}\left(R^{-1}\xi\right)$. From this homogeneity and (1.4), (1.3) follows. The only remaining issue is to show that $\widehat{K}$ is given by a function; i.e., we need to show that $\widehat{K}(\xi)$ does not have a part supported at $\xi = 0$. The only distributions supported at 0 are finite linear combinations of derivatives of $\delta_0(\xi)$ (where δ_0 denotes the δ function at 0), which are all homogeneous of degree $\geq n$. That is, we know that $\widehat{K}$ is a sum of a function satisfying (1.3) plus a finite linear combination of derivatives of δ_0–and we wish to show that this finite linear combination of derivatives of δ_0 is actually 0. Because $\left\{K^R \mid R > 0\right\}$ is a bounded set

[7] See Definition A.1.15 for the notion of a bounded set in a locally convex topological vector space.

of Calderón-Zygmund kernels, it is also a bounded set of tempered distributions. Thus, $\left\{\widehat{K}\left(R^{-1}\xi\right) \mid R>0\right\}$ is a bounded set of tempered distributions. Taking $R\to\infty$, we see that if there were any terms supported at $\xi=0$, then $\left\{\widehat{K}\left(R^{-1}\xi\right) \mid R>0\right\}$ would not be a bounded set of tempered distributions, which shows that $\widehat{K}$ does not have a part which is supported at 0.

Hence, to prove (i)⇒(ii), it suffices to prove (1.4). Now consider ξ with $1\leq|\xi|\leq 2$. We have

$$\partial_\xi^\alpha \widehat{K}(\xi)=\partial_\xi^\alpha\int K(x)\, e^{-2\pi i x\cdot\xi}\, dx=\int(-2\pi i x)^\alpha K(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

Let $\phi\in C_0^\infty(\mathbb{R}^n)$ be supported in $B^n(2)$ (the ball of radius 2, centered at 0), and equal to 1 on $B^n(3/2)$.

We decompose

$$\begin{aligned}\partial_\xi^\alpha \widehat{K}(\xi)=&\int(-2\pi i x)^\alpha \phi(x)\, e^{-2\pi i x\cdot\xi} K(x)\; dx\\ &+\int(1-\phi(x))(-2\pi i x)^\alpha e^{-2\pi i x\cdot\xi} K(x)\; dx,\end{aligned}$$

and we estimate the two terms separately. The estimate

$$\left|\int(-2\pi i x)^\alpha \phi(x)\, e^{-2\pi i x\cdot\xi} K(x)\; dx\right|\lesssim 1$$

follows immediately from the cancellation condition (recall $1\leq|\xi|\leq 2$). For the second term, we integrate by parts to see (for any $L\in\mathbb{N}$)

$$\begin{aligned}&\int(-2\pi i x)^\alpha(1-\phi(x))\, K(x)\, e^{i x\cdot\xi}\, dx\\ &=\left(-\left|2\pi\xi\right|^2\right)^L\int\triangle_x^L\left[(-2\pi i x)^\alpha(1-\phi(x))\, K(x)\right] e^{i x\cdot\xi}\, dx.\end{aligned}$$

We take $L=L(\alpha,m)$ large. If any of the derivatives land on $1-\phi(x)$, then the resulting function is supported in $B^n(2)\setminus B^n(3/2)$ and the growth condition shows that the integral converges absolutely proving the desired estimate. Otherwise, all but at most $|\alpha|$ derivatives land on K. In this case, the growth condition shows that the resulting distribution falls off like $|x|^{-n-2L+|\alpha|}$. Taking $2L\geq|\alpha|+1$, the integral converges absolutely, showing

$$\left|\int(-2\pi i x)^\alpha(1-\phi(x))\, K(x)\, e^{i x\cdot\xi}\, dx\right|\lesssim 1,$$

as desired. This completes the proof of (1.4) and therefore completes the proof of (i)⇒(ii).

We now turn to (ii)⇒(i), and we assume $\widehat{K}$ is a function satisfying (1.3). We wish to show K is a Calderón-Zygmund kernel. We begin with the growth condition. We show, for every multi-index β,

$$\sup_{1\leq|x|\leq 2}\left|\partial_x^\beta K(x)\right| \leq D_\beta, \tag{1.5}$$

where D_β is a constant which depends on only a finite number (depending on β) of the constants C_α in (1.3). Because of the relationship between $\widehat{K}\left(R^{-1}\xi\right)$ and K^R discussed in the first part of the proof, the growth condition follows immediately from (1.5). (1.5), in turn, follows just as in the estimates for the first part of this proof–indeed, we do not need a "cancellation condition" for $\widehat{K}$, since it is an element of $L^\infty \subset L^1_{\mathrm{loc}}$.

We turn to the cancellation condition. Let $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$ be a bounded set. Let $R > 0$ and $\phi \in \mathcal{B}$. We have, letting $\check{\phi}$ denote the inverse Fourier transform of ϕ,

$$\begin{aligned}\left|\int K(x)\,\phi(Rx)\;dx\right| &= \left|\int \widehat{K}(\xi)\,R^{-n}\check{\phi}\left(-R^{-1}\xi\right)\;d\xi\right| \\ &= \left|\int \widehat{K}(R\xi)\,\check{\phi}(\xi)\;d\xi\right| \\ &\leq \int\left|\widehat{K}(R\xi)\,\check{\phi}(\xi)\right|\;d\xi \\ &\lesssim 1,\end{aligned}$$

where in the last line we have used $\left|\widehat{K}(R\xi)\right| \lesssim 1$ and $\left\{\check{\phi}\mid\phi\in\mathcal{B}\right\} \subset \mathcal{S}(R^n)$ is a bounded set. This completes the proof. □

Remark 1.1.6 The proof of Theorem 1.1.5 yields something which might seem somewhat surprising. Indeed, it shows that we could have defined the topology on Calderón-Zygmund kernels in another way, by taking the semi-norms to be, for each multi-index α, the least possible C_α from (1.3). This shows that the space of Calderón-Zygmund kernels is, in fact, a Fréchet space, even though we originally defined it with an uncountable collection of semi-norms (indeed, we had one semi-norm for each bounded subset of $C_0^\infty(\mathbb{R}^n)$). We see examples of this sort of phenomenon several times in the sequel. See Remarks 2.8.2 and 5.7.6.

Theorem 1.1.5 immediately implies parts (b) and (c) of Theorem 1.1.2. Notice

$$\mathrm{Op}(K)\,f = \left(\widehat{K}\hat{f}\right)^\vee,$$

where $\vee$ denotes the inverse Fourier transform. Therefore $\mathrm{Op}(K)^* f = \left(\overline{\widehat{K}}\hat{f}\right)^\vee$, where $\overline{\widehat{K}}$ denotes the complex conjugate of $\widehat{K}$. Since $\widehat{K}$ satisfies (1.3) if and only if $\overline{\widehat{K}}$ does, (b) follows.[8] To see (c) notice

$$\mathrm{Op}(K_1)\,\mathrm{Op}(K_2)\,f = \left(\widehat{K}_1\widehat{K}_2\hat{f}\right)^\vee,$$

[8] (b) is also quite easy to see directly from the definitions.

and the result follows from Theorem 1.1.5. Finally, since $\widehat{K}$ is a bounded function, it follows that $\mathrm{Op}(K)$ is bounded on L^2. Thus, the remainder of Theorem 1.1.2 follows from the next, conditional, proposition.

PROPOSITION 1.1.7. *If $K \in C_0^\infty(\mathbb{R}^n)'$ is a distribution which satisfies the Growth Condition (i.e., (i) of Definition 1.1.1) and for which $\mathrm{Op}(K)$ extends to a bounded operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, then $\mathrm{Op}(K)$ extends to a bounded operator $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $(1 < p \leq 2)$.*

Notice that Proposition 1.1.7 completes the proof that $\mathrm{Op}(K)$ is bounded on L^p, $1 < p < \infty$. Indeed, for $p > 2$, we may apply Proposition 1.1.7 to $\mathrm{Op}(K)^*$, and the result follows by duality.

The key to proving Proposition 1.1.7 is the "Calderón-Zygmund decomposition," which we state without proof. See [Ste93] for a proof and further details.

LEMMA 1.1.8. *Fix $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. There is a countable family of closed cubes $\{Q_k\}$ whose interiors are disjoint, and functions g and b with $f = g + b$, such that*

(i) $\sum \mathrm{Vol}(Q_k) \leq \frac{1}{\alpha} \int |f|$.

(ii) *b is supported on $\bigcup Q_k$ and* $\int_{Q_k} b = 0$, $\frac{1}{\mathrm{Vol}(Q_k)} \int |b| \leq 2^{n+1}\alpha$.

(iii) $|g| \leq 2^n \alpha$.

PROOF SKETCH OF PROPOSITION 1.1.7. The goal is to show that $\mathrm{Op}(K)$ is weak-type $(1,1)$. That is, we want to show,

$$\mathrm{Vol}\left(\{x \mid |\mathrm{Op}(K) f(x)| > \alpha\}\right) \lesssim \frac{1}{\alpha} \int |f| . \tag{1.6}$$

The result then follows from the L^2 boundedness of $\mathrm{Op}(K)$ and the Marcinkiewicz interpolation theorem.

For a fixed α, we apply Lemma 1.1.8 to obtain $\{Q_k\}$ and a decomposition $f = g+b$ as in the statement of that lemma. (1.6) follows once we show

$$\mathrm{Vol}\left(\{x \mid |\mathrm{Op}(K) g(x)| > \alpha/2\}\right) + \mathrm{Vol}\left(\{x \mid |\mathrm{Op}(K) b(x)| > \alpha/2\}\right) \lesssim \frac{1}{\alpha} \int |f| .$$

First notice $g \in L^2$. Indeed, $g = f$ on $\left(\bigcup Q_k\right)^c$, and we therefore have

$$\int |g|^2 \leq \int_{\left(\bigcup Q_k\right)^c} |f|\,|g| + \int_{\bigcup Q_k} |g|^2 .$$

We use that $|g| \leq 2^n\alpha$, to see

$$\int_{\bigcup Q_k} |g|^2 \lesssim \alpha^2 \sum \mathrm{Vol}(Q_k) \lesssim \alpha \int |f| ,$$

and

$$\int_{(\bigcup Q_k)^c} |f|\,|g| \lesssim \alpha \int |f|\,.$$

It follows that

$$\int |g|^2 \lesssim \alpha \int |f|\,,$$

and therefore $g \in L^2$. Chebycheff's inequality then completes the proof for g:

$$\mathrm{Vol}\left(\{x \mid |\mathrm{Op}\,(K)\,g| > \alpha/2\}\right) \lesssim \alpha^{-2} \int |\mathrm{Op}\,(K)\,g|^2 \lesssim \alpha^{-2} \int |g|^2 \lesssim \frac{1}{\alpha} \int |f|\,.$$

We now turn to the estimate of $\mathrm{Op}\,(K)\,b$. Let B_k be the smallest ball containing the cube Q_k and let $2B_k$ denote the ball with the same center as B_k but with twice the radius. Notice,

$$\sum \mathrm{Vol}\,(2B_k) \approx \sum \mathrm{Vol}\,(Q_k) \lesssim \frac{1}{\alpha} \int |f|\,.$$

Thus, it suffices to show

$$\mathrm{Vol}\left(\left\{x \in \left(\bigcup 2B_k\right)^c \;\middle|\; |\mathrm{Op}\,(K)\,b\,(x)| > \alpha/2\right\}\right) \lesssim \frac{1}{\alpha} \int |f|\,.$$

We have

$$\int_{(\bigcup 2B_k)^c} |\mathrm{Op}\,(K)\,b| \leq \sum_k \int_{(2B_k)^c} |\mathrm{Op}\,(K)\,b_k|\,,$$

where $b_k = b$ on Q_k and is 0 on Q_k^c. Let y_k denote the center of the cube Q_k. Using that $\int b_k = 0$, we have

$$\int_{(2B_k)^c} |\mathrm{Op}\,(K)\,b_k| \leq \int_{(2B_k)^c} \int_{B_k} |K\,(x-y) - K\,(x-y_k)|\,|b_k\,(y)|\;dy\,dx. \quad (1.7)$$

For $y \in B_k$, $x \in (2B_k)^c$, it is easy to see that

$$|K\,(x-y) - K\,(x-y_k)| \lesssim |y-y_k|\,|x-y_k|^{-n-1}\,.$$

Using $\int |b_k| \lesssim \alpha \mathrm{Vol}\,(Q_k)$, we have from (1.7),

$$\int_{(2B_k)^c} |\mathrm{Op}\,(K)\,b_k| \lesssim \alpha \mathrm{Vol}\,(Q_k)\,,$$

and it follows that $\int_{(\bigcup 2B_k)^c} |\mathrm{Op}\,(K)\,b| \lesssim \alpha \sum_k \mathrm{Vol}\,(Q_k) \lesssim \int |f|$. That

$$\mathrm{Vol}\left(\{x \mid |\mathrm{Op}\,(K)\,b\,(x)| > \alpha/2\}\right) \lesssim \frac{1}{\alpha} \int |f|$$

now follows from another application of Chebycheff's inequality. This completes the proof. □

Remark 1.1.9 The proof of Proposition 1.1.7 generalizes to a number of settings; for instance see Chapter 2. In fact, these ideas work in the even more general setting of a "space of homogeneous type," as developed by Coifman and Weiss [CW71]. See [Ste93] for more details. These methods will be a useful tool in creating an appropriate Littlewood-Paley theory, which we will use to show the L^p boundedness of certain "multi-parameter" operators. This is discussed in Sections 2.15.4 and 3.4.

Let us now turn to characterizing operators of the form $\mathrm{Op}(K)$, where K is a Calderón-Zygmund kernel, in several ways. These ideas will be used as motivation for definitions in later chapters. For this we need to introduce a subspace of Schwartz space which plays a pivotal role.

DEFINITION 1.1.10. $\mathcal{S}_0(\mathbb{R}^n)$*–the space of Schwartz functions, all of whose moments vanish–is the closed subspace of* $\mathcal{S}(\mathbb{R}^n)$ *defined by*

$$\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n) \;\middle|\; \forall \alpha, \int x^\alpha f(x)\, dx = 0 \right\}.$$

Remark 1.1.11 As a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}_0(\mathbb{R}^n)$ has the induced subspace topology which turns $\mathcal{S}_0(\mathbb{R}^n)$ into a Fréchet space. Notice that $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set if and only if $\mathcal{B}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$; i.e., $\forall \alpha, \beta, \sup_{f\in\mathcal{B}} \|f\|_{\alpha,\beta} < \infty$.[9]

When working with $\mathcal{S}_0(\mathbb{R}^n)$ it is often more convenient to work on the Fourier transform side, as the next few results illustrate.

LEMMA 1.1.12. *Let* $f \in \mathcal{S}(\mathbb{R}^n)$. *Then,* $f \in \mathcal{S}_0(\mathbb{R}^n)$ *if and only if* $\forall \alpha$, $\partial_\xi^\alpha \hat{f}(0) = 0$.

PROOF. The result follows immediately from the definitions. □

COROLLARY 1.1.13. *Fix* $s \in \mathbb{R}$. *The map* $\triangle^s : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ *given by* $\triangle^s : f \mapsto \left(|2\pi\xi|^{2s} \hat{f}(\xi) \right)^\vee$ *is an automorphism.*

PROOF. It follows easily from Lemma 1.1.12 that for $f \in \mathcal{S}_0(\mathbb{R}^n)$, $\triangle^s f \in \mathcal{S}_0(\mathbb{R}^n)$. The closed graph theorem (Theorem A.1.14) shows that $\triangle^s$ is continuous. The continuous inverse of $\triangle^s$ is given by $\triangle^{-s}$. □

COROLLARY 1.1.14. $\mathcal{S}_0(\mathbb{R}^n)$ *is dense in* $L^2(\mathbb{R}^n)$.

PROOF. This follows easily from Lemma 1.1.12. □

Remark 1.1.15 Actually, $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, but it is dense in neither L^1, nor L^∞.

The next lemma offers a characterization of $\mathcal{S}_0(\mathbb{R}^n)$, which we do not use directly, but which motivates some future definitions.

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[9]For further information regarding bounded sets and their relationship with semi-norms, see Appendix A.

LEMMA 1.1.16. *Consider subsets $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^n)$ satisfying the following condition: $\forall f \in \mathcal{G}$, $\exists f_1, \ldots, f_n \in \mathcal{G}$, with $f = \sum_{j=1}^{n} \partial_{x_j} f_j$. $\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is the largest subset satisfying this condition.*

PROOF. Suppose $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^n)$ satisfies the above condition. Fix a multi-index α. By repeated applications of the above property, we may write $g = \sum_{|\beta|=|\alpha|+1} \partial_x^\beta g_\beta$, where $g_\beta \in \mathcal{G} \subset \mathcal{S}(\mathbb{R}^n)$. We have,

$$\int x^\alpha g(x)\, dx = \sum_{|\beta|=|\alpha|+1} \int x^\alpha \partial_x^\beta g_\beta(x)\, dx = 0,$$

where the last equality follows by integration by parts. Thus $\mathcal{G} \subset \mathcal{S}_0(\mathbb{R}^n)$.

Conversely, given $g \in \mathcal{S}_0(\mathbb{R}^n)$ we may write $g = \triangle g_0$, where $g_0 = \triangle^{-1} g \in \mathcal{S}_0(\mathbb{R}^n)$ (by Corollary 1.1.13). We have $g = \sum_{j=1}^{n} \partial_{x_j}\left(-\partial_{x_j} g_0\right)$. Since $-\partial_{x_j} g_0 \in \mathcal{S}_0(\mathbb{R}^n)$, the result follows. □

Remark 1.1.17 Suppose $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set. In light of Lemma 1.1.16, for $f \in \mathcal{B}$, we may write $f = \sum_{|\alpha|=1} \partial_x^\alpha f_\alpha$, where $f_\alpha \in \mathcal{S}_0(\mathbb{R}^n)$. The proof of Lemma 1.1.16 shows more: we may choose the f_α so that $\left\{ f_\alpha \,\middle|\, f \in \mathcal{B}, |\alpha| = 1 \right\} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set.

Remark 1.1.18 Let $K \in \mathcal{S}_0(\mathbb{R}^n)'$. It makes sense to define $\mathrm{Op}(K) : \mathcal{S}_0(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. Since $\mathcal{S}_0(\mathbb{R}^n)$ is dense in L^2, there is at most one continuous extension of $\mathrm{Op}(K) : L^2 \to L^2$ (but there may be no continuous extension). If $\widetilde{K} \in \mathcal{S}(\mathbb{R}^n)'$ is a Calderón-Zygmund kernel such that $\widetilde{K}\big|_{\mathcal{S}_0(\mathbb{R}^n)} = K$, then $\mathrm{Op}(K)$ and $\mathrm{Op}\left(\widetilde{K}\right)$ extend to the same bounded operator on L^2. Thus, the map $\widetilde{K} \mapsto \widetilde{K}\big|_{\mathcal{S}_0(\mathbb{R}^n)}$ from Calderón-Zygmund kernels to elements of $\mathcal{S}_0(\mathbb{R}^n)'$ is injective. It, therefore, makes sense to ask whether an element $K \in \mathcal{S}_0(\mathbb{R}^n)'$ is a Calderón-Zygmund kernel, and if so to identify it with a unique Calderón-Zygmund kernel in $C_0^\infty(\mathbb{R}^n)'$.

For a function $f \in \mathcal{S}(\mathbb{R}^n)$ and $R > 0$, we define $f^{(R)}(x) = R^n f(Rx)$. Note that this is defined in such a way that

$$\int f^{(R)}(x)\, dx = \int f(x)\, dx,$$

and, more generally,

$$\left(f^{(R)}\right)^\wedge(\xi) = \hat{f}(\xi/R).$$

LEMMA 1.1.19. *Let $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^n)$ be a bounded set. For $R_1, R_2 > 0$, $\phi_1, \phi_2 \in \mathcal{B}$, define a function $\psi = \psi(R_1, R_2, \phi_1, \phi_2)$ by*

$$\psi^{(R_1)} = \phi_1^{(R_1)} * \phi_2^{(R_2)}.$$

Then, for every N, the set

$$\left\{ \left(\frac{R_1 \vee R_2}{R_1 \wedge R_2} \right)^N \psi \,\middle|\, \phi_1, \phi_2 \in \mathcal{B}, R_1, R_2 > 0 \right\}$$

is a bounded subset of $\mathcal{S}_0\left(\mathbb{R}^n\right)$. Here $\vee$ denotes maximum and $\wedge$ denotes minimum.

PROOF. Fix M large, to be chosen later. By Corollary 1.1.13, each $\phi \in \mathcal{B}$ may be written as $\phi = \triangle^M \widetilde{\phi}$, where $\left\{ \widetilde{\phi} \mid \phi \in \mathcal{B} \right\}$ is a bounded set. Fix $\phi_1, \phi_2 \in \mathcal{B}$ and $R_1, R_2 > 0$. Suppose $R_1 \geq R_2$, we have,

$$\phi_1^{(R_1)} * \phi_2^{(R_2)} = \left(\frac{R_2}{R_1} \right)^{2M} \widetilde{\phi}_1^{(R_1)} * \left(\triangle^M \phi_2 \right)^{(R_2)}.$$

If $R_2 > R_1$, we instead have

$$\phi_1^{(R_1)} * \phi_2^{(R_2)} = \left(\frac{R_1}{R_2} \right)^{2M} \left(\triangle \phi_1 \right)^{(R_1)} * \widetilde{\phi}_2^{(R_2)}.$$

Either way, we have

$$\psi^{(R_1)} = \left(\frac{R_1 \wedge R_2}{R_1 \vee R_2} \right)^{2M} \gamma_1^{(R_1)} * \gamma_2^{(R_2)},$$

where γ_1, γ_2 range over a bounded subset of $\mathcal{S}_0\left(\mathbb{R}^n\right)$. By replacing M with $M+N$, it suffices to prove the result with $N = 0$. For any fixed M, and using the above argument, we may write ψ in the form

$$\psi^{(R_1)} = \left(\frac{R_1 \wedge R_2}{R_1 \vee R_2} \right)^{2M} \gamma_1^{(R_1)} * \gamma_2^{(R_2)}, \tag{1.8}$$

where γ_1 and γ_2 range over a bounded subset of $\mathcal{S}_0\left(\mathbb{R}^n\right)$. Fix a semi-norm $\|\cdot\|_{\alpha,\beta}$. We will take M large in terms of α, β. We wish to show

$$\sup_{\psi} \left\| \widehat{\psi} \right\|_{\alpha,\beta} < \infty,$$

where ψ is defined by (1.8) and the supremum is taken as γ_1, γ_2 range over a bounded subset of $\mathcal{S}_0\left(\mathbb{R}^n\right)$ and R_1, R_2 range over $(0, \infty)$. But,

$$\widehat{\psi}\left(\xi\right) = \left(\frac{R_1 \wedge R_2}{R_1 \vee R_2} \right)^{2M} \widehat{\gamma}_1\left(\xi\right) \widehat{\gamma}_2\left(R_1 \xi / R_2\right).$$

Taking M sufficiently large in terms of α, β, the result follows. □

LEMMA 1.1.20. *Let* $\left\{\varsigma_j \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{S}_0(\mathbb{R}^n)$ *be a bounded set. The sum*

$$\sum_{j\in\mathbb{Z}} \varsigma_j^{\left(2^j\right)}$$

converges in the sense of distributions. Furthermore,

$$\sum_{j\in\mathbb{Z}} \mathrm{Op}\left(\varsigma_j^{\left(2^j\right)}\right) = \mathrm{Op}\left(\sum_{j\in\mathbb{Z}} \varsigma_j^{\left(2^j\right)}\right), \tag{1.9}$$

thought of as operators on $\mathcal{S}_0(\mathbb{R}^n)$*. Here, the sum on the left is taken in the topology of bounded convergence as operators* $\mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ *(see Definition A.1.28 for the definition of this topology), while the sum on the right is taken in the sense of distributions.*

PROOF. By Corollary 1.1.13, $\varsigma_j = \triangle\widetilde{\varsigma}_j$, where $\{\widetilde{\varsigma}_j \mid j \in \mathbb{Z}\} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set, and so, for $f \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\left|\int \varsigma_j^{\left(2^j\right)}(x)\, f(x)\; dx\right| &= \left|\int \widetilde{\varsigma}_j^{\left(2^j\right)}(x)\, 2^{-2j} \triangle f(t)\; dt\right| \\ &\leq 2^{-2j}\left(\int |\widetilde{\varsigma}_j|\right)\left(\int |\triangle f|\right) \\ &\lesssim 2^{-2j}\left(\int |\triangle f|\right).\end{aligned}$$

It follows immediately that the sum $\sum_{j\geq 0} \varsigma_j^{\left(2^j\right)}$ converges in the sense of distributions. For $j < 0$, we have

$$\left|\int \varsigma_j^{\left(2^j\right)}(x)\, f(x)\; dx\right| \lesssim 2^{jn} \int |f(x)|\; dx,$$

and so $\sum_{j<0} \varsigma_j^{\left(2^j\right)}$ also converges in the sense of distributions. Combining these, we see that $\sum_{j\in\mathbb{Z}} \varsigma_j^{\left(2^j\right)}$ converges in the sense of distributions. The same proof shows that the sum, in fact, converges in the sense of tempered distributions.

Let $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^n)$ be a bounded set. For $g \in \mathcal{B}$, define $g_j = 2^{|j|}\varsigma_j^{\left(2^j\right)} * g$. Lemma 1.1.19 shows $\left\{g_j \,\middle|\, j \in \mathbb{Z}, g \in \mathcal{B}\right\} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set, and we have

$$\sum_{j\in\mathbb{Z}} \mathrm{Op}\left(\varsigma_j^{\left(2^j\right)}\right) g = \sum_{j\in\mathbb{Z}} 2^{-|j|} g_j,$$

and it follows that $\sum_{j\in\mathbb{Z}} \mathrm{Op}\left(\varsigma_j^{\left(2^j\right)}\right)$ converges in the topology of bounded convergence as operators $\mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$. The equality (1.9) follows easily from the above proofs. □

DEFINITION 1.1.21. *For $x \in \mathbb{R}^n$, we define a tempered distribution δ_x by*

$$\int \delta_x (y) f (y) \, dy = f (x), \quad f \in \mathcal{S} (\mathbb{R}^n).$$

δ_x is called the Dirac δ function at x.

LEMMA 1.1.22. *There is a function $\varsigma \in \mathcal{S}_0 (\mathbb{R}^n)$ such that*

$$\delta_0 = \sum_{j \in \mathbb{Z}} \varsigma^{(2^j)},$$

where δ_0 denotes the Dirac δ function at 0.

PROOF. Let $\psi \in C_0^\infty (\mathbb{R}^n)$ equal 1 on a neighborhood of 0. Define

$$\widehat{\varsigma} (\xi) = \psi (\xi) - \psi (2\xi).$$

Notice,

$$\sum_{j \in \mathbb{Z}} \widehat{\varsigma} \left(2^{-j} \xi\right) = 1,$$

in the sense of tempered distributions. Taking the inverse Fourier transform of both sides yields the result. □

We have the following characterization of Calderón-Zygmund kernels.

THEOREM 1.1.23. *Let $K \in \mathcal{S}_0 (\mathbb{R}^n)'$. The following are equivalent:*

(i) *K is a Calderón-Zygmund kernel.*

(ii) $\mathrm{Op} (K) : \mathcal{S}_0 (\mathbb{R}^n) \to \mathcal{S}_0 (\mathbb{R}^n)$ *and for any bounded set $\mathcal{B} \subset \mathcal{S}_0 (\mathbb{R}^n)$, the set*

$$\left\{ g \in \mathcal{S}_0 (\mathbb{R}^n) \;\middle|\; \exists R > 0, f \in \mathcal{B}, g^{(R)} = \mathrm{Op} (K) f^{(R)} \right\} \subset \mathcal{S}_0 (\mathbb{R}^n)$$

is a bounded set.

(iii) *For each $j \in \mathbb{Z}$ there is a function $\varsigma_j \in \mathcal{S}_0 (\mathbb{R}^n)$ with $\left\{ \varsigma_j \;\middle|\; j \in \mathbb{Z} \right\} \subset \mathcal{S}_0 (\mathbb{R}^n)$ a bounded set and such that*

$$K = \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}.$$

See Lemma 1.1.20 for a discussion of the convergence of this sum.

PROOF. We begin with (i)⇒(ii), and we use Theorem 1.1.5. Let K be a Calderón-Zygmund kernel and let $\mathcal{B} \subset \mathcal{S}_0 (\mathbb{R}^n)$ be bounded. Let

$$\mathcal{T} := \left\{ g \;\middle|\; \exists R > 0, f \in \mathcal{B}, g^{(R)} = \mathrm{Op} (K) f^{(R)} \right\}.$$

We wish to show $\mathcal{T}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$ and moreover $\mathcal{T} \subset \mathcal{S}_0(\mathbb{R}^n)$. Notice that if $g^{(R)} = \mathrm{Op}(K) f^{(R)}$, then

$$\hat{g}(\xi) = \widehat{K}(R\xi) \hat{f}(\xi).$$

Thus, to show $\mathcal{T}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$, and using the fact that the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^n)$, it suffices to show

$$\widehat{\mathcal{T}} := \left\{ \widehat{K}(R\xi) \hat{f}(\xi) \;\middle|\; R > 0, f \in \mathcal{B} \right\}$$

is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$. But, we have for $f \in \mathcal{S}_0(\mathbb{R}^n)$, using Theorem 1.1.5,

$$\begin{aligned} \left\| \widehat{K}(R\xi) \hat{f}(\xi) \right\|_{\alpha,\beta} &= \left\| \left(|\xi|^{2|\alpha|} \widehat{K}(R\xi) \right) \left(|\xi|^{-2|\alpha|} \hat{f}(\xi) \right) \right\|_{\alpha,\beta} \\ &\lesssim \sum_{\substack{|\alpha'| \leq 3|\alpha| \\ |\beta'| \leq |\beta|}} \left\| \triangle^{-2|\alpha|} f \right\|_{\alpha',\beta'}. \end{aligned}$$

The right-hand side is a continuous semi-norm on $\mathcal{S}_0(\mathbb{R}^n)$ (by Corollary 1.1.13). Taking the supremum over $f \in \mathcal{B}$ and $R > 0$ shows that $\sup_{\hat{g} \in \widehat{\mathcal{T}}} \|\hat{g}\|_{\alpha,\beta} < \infty$, and it follows that $\mathcal{T}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$. We wish to show that $\mathcal{T} \subset \mathcal{S}_0(\mathbb{R}^n)$. Indeed, let $\hat{g} \in \widehat{\mathcal{T}}$, so that $\hat{g}(\xi) = \widehat{K}(R\xi) \hat{f}(\xi)$. By Lemma 1.1.12, we wish to show $\partial_\xi^\alpha \widehat{K}(R\xi) \hat{f}(\xi) \big|_{\xi=0} = 0, \forall \alpha$. But this follows immediately from Lemma 1.1.12 and Theorem 1.1.5.

We turn to (ii)⇒(iii); let K be as in (ii). We apply Lemma 1.1.22 to decompose $\delta_0 = \sum_{j \in \mathbb{Z}} \varsigma^{(2^j)}$. Lemma 1.1.20 shows that $I = \sum_{j \in \mathbb{Z}} \mathrm{Op}\left(\varsigma^{(2^j)}\right)$, where the sum is taken in the topology of bounded convergence as operators $\mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$, and I denotes the identity operator $\mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$. Let $\varsigma_j^{(2^j)} = \mathrm{Op}(K) \varsigma^{(2^j)}$, so that our assumption implies $\left\{ \varsigma_j \;\middle|\; j \in \mathbb{Z} \right\} \subset \mathcal{S}_0(\mathbb{R}^n)$ is bounded. We have

$$\mathrm{Op}(K) = \mathrm{Op}(K) I = \sum_{j \in \mathbb{Z}} \mathrm{Op}(K) \mathrm{Op}\left(\varsigma^{(2^j)}\right) = \sum_{j \in \mathbb{Z}} \mathrm{Op}\left(\varsigma_j^{(2^j)}\right).$$

Lemma 1.1.20 then shows that $K = \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}$.

Finally, we prove (iii)⇒(i); let $K = \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}$ as in (iii), where $\left\{ \varsigma_j \;\middle|\; j \in \mathbb{Z} \right\} \subset \mathcal{S}(\mathbb{R}^n)$ is bounded. We need to show $\sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}$ converges in distribution to a Calderón-Zygmund kernel. We have for $x \neq 0$,

$$\begin{aligned} \left| \partial_x^\alpha \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}(x) \right| &\leq \sum_{j \in \mathbb{Z}} 2^{nj+|\alpha|j} \left| (\partial_x^\alpha \varsigma_j) (2^j x) \right| \\ &\lesssim \sum_{j \in \mathbb{Z}} 2^{nj+|\alpha|j} \left(1 + |2^j x|\right)^{-n-|\alpha|-1} \\ &\lesssim |x|^{-n-|\alpha|}, \end{aligned}$$

which establishes the growth condition.

We now verify the cancellation condition. Let $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$ be a bounded set. Write $\varsigma_j = \triangle\widetilde{\varsigma}_j$ where, by Corollary 1.1.13, $\left\{\widetilde{\varsigma}_j \mid j \in \mathbb{Z}\right\} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set. For $\phi \in \mathcal{B}$ and $R > 0$, we have

$$\begin{aligned}
&\left|\int \sum_{j\in\mathbb{Z}} \varsigma_j^{(2^j)}(x)\,\phi(Rx)\,dx\right| \\
&\quad\lesssim \sum_{2^j \geq R} 2^{-2j} R^2 \int \left|\widetilde{\varsigma}_j^{(2^j)}(x)\,(\triangle\phi)(Rx)\right| dx + \sum_{2^j<R} \int \left|\phi\left(2^{-j}Rx\right)\right| dx \\
&\quad\lesssim \sum_{2^j \geq R} 2^{-2j} R^2 + \sum_{2^j<R} \left(2^j R^{-1}\right)^n \\
&\quad\lesssim 1,
\end{aligned}$$

which completes the proof. □

The proof of Theorem 1.1.23 used the Fourier transform many times. However, it is the statement of Theorem 1.1.23 (which does not mention the Fourier transform) which is of the most interest to us. Indeed, when we leave the translation invariant Euclidean setting, the Fourier transform will be much less applicable. However, we will be able to prove an analog of Theorem 1.1.23 using other methods.

It is not hard to see that many of the conclusions above concerning Calderón-Zygmund kernels, whose proofs used the Fourier transform, can be reproved using the conclusions of Theorem 1.1.23. For instance, that operators of the form $\mathrm{Op}(K)$ form an algebra follows immediately from (ii). Also, that $\mathrm{Op}(K)$ is bounded on L^2 follows from (iii) and the Cotlar-Stein Lemma (Lemma 1.2.26). Indeed, it follows easily from Lemma 1.1.19 that

$$\left\|\mathrm{Op}\left(\varsigma_j^{(2^j)}\right)^* \mathrm{Op}\left(\varsigma_k^{(2^k)}\right)\right\|_{L^2\to L^2} \lesssim 2^{-|j-k|},$$

and the L^2 boundedness of $\mathrm{Op}(K)$ follows from the Cotlar-Stein Lemma. Of course, these ideas are somewhat circular: the proof of Theorem 1.1.23 uses the Fourier transform. As we will see, it is possible to prove Theorem 1.1.23, and more general analogs, without direct use of the Fourier transform, whereby the above ideas become much more useful.

The operators discussed in this section are operators of "order 0." When we move to a more general setting, we will work with operators of other orders. We state here the basic definitions and results for operators of orders other than 0. We will prove more general analogs of these results in later chapters.

DEFINITION 1.1.24. *We say $K \in C_0^\infty(\mathbb{R}^n)'$ is a Calderón-Zygmund kernel of order $t \in (-n,\infty)$ if:*

(i) (Growth Condition) For every multi-index α, $\left|\partial_x^\alpha K(x)\right| \leq C_\alpha |x|^{-n-t-|\alpha|}$. In particular, we assume that $K(x)$ is a C^∞ function for $x \neq 0$.

(ii) (Cancellation Condition) For every bounded set $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$, we assume

$$\sup_{\substack{\phi \in \mathcal{B} \\ R>0}} R^{-t} \left| \int K(x)\, \phi(Rx)\; dx \right| \leq C_{\mathcal{B}}.$$

Remark 1.1.25 When $-n < t < 0$, the Cancellation Condition follows from the Growth Condition and the weaker assumption that K has no part supported at 0. I.e., suppose K is a distribution satisfying the Growth Condition where $t \in (-n, 0)$. Define the function

$$\widetilde{K}(x) = \begin{cases} K(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If $K - \widetilde{K} = 0$ as a distribution, then K is a Calderón-Zygmund kernel of order t. Conversely, if K is a Calderón-Zygmund kernel of order t, then $K - \widetilde{K} = 0$. In this sense, the Cancellation Condition is superfluous for $-n < t < 0$.

As before, given a Calderón-Zygmund kernel, we define an operator $\mathrm{Op}(K) : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $\mathrm{Op}(K) f = K * f$. Similar to the case of operators of order 0, a density argument shows that to uniquely determine a Calderón-Zygmund kernel K of any order $t \in (-n, \infty)$, it suffices to consider $K\big|_{\mathcal{S}_0(\mathbb{R}^n)}$. Just as in Theorem 1.1.23, we may use this to characterize Calderón-Zygmund kernels in other ways, as the next theorem shows. We state this theorem without proof.

THEOREM 1.1.26. *Fix $t \in (-n, \infty)$, and let $K \in \mathcal{S}_0(\mathbb{R}^n)'$. The following are equivalent:*

(i) K is a Calderón-Zygmund kernel of order t.

(ii) $\mathrm{Op}(K) : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ and for any bounded set $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^n)$, the set

$$\left\{ g \in \mathcal{S}_0(\mathbb{R}^n) \;\middle|\; \exists R > 0, f \in \mathcal{B}, g^{(R)} = R^{-t}\mathrm{Op}(K)\, f^{(R)} \right\} \subset \mathcal{S}_0(\mathbb{R}^n)$$

is a bounded set.

(iii) For each $j \in \mathbb{Z}$ there is a function $\varsigma_j \in \mathcal{S}_0(\mathbb{R}^n)$ with $\left\{ \varsigma_j \;\middle|\; j \in \mathbb{Z} \right\} \subset \mathcal{S}_0(\mathbb{R}^n)$ a bounded set and such that

$$K = \sum_{j \in \mathbb{Z}} 2^{jt} \varsigma_j^{(2^j)}.$$

The above sum converges in distribution, though the equality is taken in the sense of elements of $\mathcal{S}_0(\mathbb{R}^n)'$.

Furthermore, (ii) and (iii) are equivalent for any $t \in \mathbb{R}$.

For $t \leq -n$, we define Calderón-Zygmund kernels in the following way.

DEFINITION 1.1.27. *Let $K \in \mathcal{S}_0(\mathbb{R}^n)'$ and $t \in \mathbb{R}$. We say K is a Calderón-Zygmund kernel of order t if either of the two equivalent conditions (ii) or (iii) of Theorem 1.1.26 holds.*

Remark 1.1.28 Restricting a tempered distribution to $\mathcal{S}_0(\mathbb{R}^n)$ does not uniquely determine the distribution (polynomials are all 0, when thought of in the dual to $\mathcal{S}_0(\mathbb{R}^n)$). For Calderón-Zygmund operators of order $t > -n$, we used a density argument to uniquely pick out a distribution, given its values on $\mathcal{S}_0(\mathbb{R}^n)$. For $t \leq -n$, this procedure does not always work. For now, we satisfy ourselves by working only with operators defined on $\mathcal{S}_0(\mathbb{R}^n)$. Later in the monograph, (in Chapters 2 and 5) we restrict attention to non-translation invariant operators whose Schwartz kernels have compact support, and use this to avoid this non-uniqueness problem.

Remark 1.1.29 With the above definitions, $\triangle^s$ is an isomorphism of Calderón-Zygmund kernels of order t to Calderón-Zygmund kernels of order $t+2s$.[10] This gives another (equivalent) way to extend Definition 1.1.24 to kernels of order $t \leq -n$. Indeed, we say $K \in \mathcal{S}_0(\mathbb{R}^n)'$ is a Calderón-Zygmund kernel of order t if $\triangle^{-t/2}K$ is a Calderón-Zygmund kernel of order 0. As pointed out in the previous remark, though, this only uniquely specifies the kernel as an element of $\mathcal{S}_0(\mathbb{R}^n)'$, and not as a distribution.

Remark 1.1.30 Notice the scale invariance of conditions in Theorem 1.1.26. For instance, consider (ii). If $\mathrm{Op}(K) : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$,[11] then it follows by continuity that for any bounded set $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^n)$, $\mathrm{Op}(K)\mathcal{B}$ is also bounded. (ii) takes this automatic fact, and instead assumes a *scale invariant* version of it.

This leads us directly to the first main property of Calderón-Zygmund kernels.

THEOREM 1.1.31. *Suppose* $K_1, K_2 \in \mathcal{S}_0(\mathbb{R}^n)'$ *are Calderón-Zygmund kernels of order* $s, t \in \mathbb{R}$, *respectively. Then* $\mathrm{Op}(K_1)\,\mathrm{Op}(K_2) = \mathrm{Op}(K_3)$, *where* $K_3 \in \mathcal{S}_0(\mathbb{R}^n)'$ *is a Calderón-Zygmund kernel of order* $s+t$.

PROOF. This is an immediate consequence of (ii) of Theorem 1.1.26. □

To discuss the L^p boundedness of these operators, we need appropriate L^p Sobolev spaces. For $1 < p < \infty$ and $s \in \mathbb{R}$, we define $\mathring{L}^p_s(\mathbb{R}^n)$ to be the completion of $\mathcal{S}_0(\mathbb{R}^n)$ under the following norm:

$$\|f\|_{\mathring{L}^p_s(\mathbb{R}^n)} := \left\|\triangle^{s/2} f\right\|_{L^p(\mathbb{R}^n)}.$$

As mentioned before, $\mathring{L}^p_0(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ $(1 < p < \infty)$.

Remark 1.1.32 Unlike the usual Sobolev spaces, elements of $\mathring{L}^p_s$ cannot be identified with distributions if $s << 0$; instead they can be thought of as elements of $\mathcal{S}_0(\mathbb{R}^n)'$.

THEOREM 1.1.33. *Let* K *be a Calderón-Zygmund kernel of order* t. *Then,* $\mathrm{Op}(K) : \mathring{L}^p_s(\mathbb{R}^n) \to \mathring{L}^p_{s-t}(\mathbb{R}^n)$.

To prove Theorem 1.1.33, we need a lemma.

[10] This follows by combining Lemma 1.1.34 with Theorem 1.1.31.

[11] Here, and in the rest of the monograph, when we are given a linear operator $T : V \to W$, where V and W are topological vector spaces, we always assume that T is continuous.

LEMMA 1.1.34. *For $s \in \mathbb{R}$, the operator $\triangle^s : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ is of the form $\mathrm{Op}(K)$ for a Calderón-Zygmund kernel K of order $2s$.*

PROOF. This follows easily by using (iii) of Theorem 1.1.26. Indeed, let $\widehat{\phi} \in C_0^\infty(\mathbb{R}^n)$ equal 1 on a neighborhood of 0 and set $\widehat{\psi}(\xi) = \widehat{\phi}(\xi) - \widehat{\phi}(2\xi)$ so that $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ in the sense of tempered distributions. Let $\widehat{\varsigma} = |2\pi\xi|^{2s}\widehat{\psi}$, so that $|2\pi\xi|^{2s} = \sum_{j\in\mathbb{Z}} 2^{2js}\widehat{\varsigma}(2^{-j}\xi)$. Define $\varsigma = (\widehat{\varsigma})^\vee$–the inverse Fourier transform of $\widehat{\varsigma}$. Note that $\varsigma \in \mathcal{S}_0(\mathbb{R}^n)$. We have $\triangle^s = \sum_{j\in\mathbb{Z}} 2^{2js}\mathrm{Op}\left(\varsigma^{(2^j)}\right) = \sum_{j\in\mathbb{Z}} 2^{2js}\mathrm{Op}(\varsigma)^{(2^j)}$. Thus (iii) of Theorem 1.1.26 holds and $\triangle^s$ is a Calderón-Zygmund operator of order $2s$. □

PROOF OF THEOREM 1.1.33. Let K be a Calderón-Zygmund kernel of order t. For $f \in \mathcal{S}_0(\mathbb{R}^n)$, we wish to show

$$\left\|\triangle^{(s-t)/2}\mathrm{Op}(K) f\right\|_{L^p} \lesssim \left\|\triangle^{s/2} f\right\|_{L^p}.$$

By Corollary 1.1.13 we may write $f = \triangle^{-s/2}g$, $g \in \mathcal{S}_0(\mathbb{R}^n)$, and we therefore wish to show

$$\left\|\triangle^{(s-t)/2}\mathrm{Op}(K)\,\triangle^{-s/2}\,g\right\|_{L^p} \lesssim \|g\|_{L^p}.$$

This follows from the fact that $\triangle^{(s-t)/2}\mathrm{Op}(K)\,\triangle^{-s/2}$ is bounded on L^p, by Theorem 1.1.5, as it is an operator of order 0 (by Theorem 1.1.31 and Lemma 1.1.34). □

1.2 NON-TRANSLATION INVARIANT OPERATORS

Above, we discussed translation invariant operators: operators of the form $\mathrm{Op}(K) f = f * K$. To understand these operators, we made significant use of the Fourier transform. Even though the Fourier transform is less applicable, many of the ideas have generalizations to the non-translation invariant setting. First, a bit of notation. For an operator $T : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)'$, we identify T with its Schwartz kernel, $T(x,y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)'$ (see Theorem A.1.30 and the remarks surrounding it for this identification). We use the identification of functions in L^1_{loc} with certain distributions, as in the previous discussion. Thus, given an operator, we often treat it as a function, and when we do, we are assuming that its Schwartz kernel is given by integration against an $L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$ function. Conversely, given a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$ we will treat it as an operator $C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)'$, by this same identification. See Appendix A.1.1 for more details on this notation.

DEFINITION 1.2.1. *We say $T : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a Calderón-Zygmund operator of order $t \in (-n, \infty)$ if:*

(i) (Growth Condition) For every multi-indices α and β,

$$\left|\partial_x^\alpha \partial_y^\beta T(x,y)\right| \leq C_{\alpha,\beta}\,|x-y|^{-n-t-|\alpha|-|\beta|}.$$

In particular, we assume $T(x,y)$ is a C^∞ function for $x \neq y$.

(ii) (Cancellation Condition) For all bounded sets $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$ and for all $\phi \in \mathcal{B}$, $R > 0$, and $z \in \mathbb{R}^n$, define $\phi_{R,z}(x) = \phi(R(x-z))$. We assume, for every multi-index α,

$$\sup_{\phi\in\mathcal{B}} \sup_{R>0} \sup_{x,z\in\mathbb{R}^n} R^{-t-|\alpha|} \left|\partial_x^\alpha T\phi_{R,z}(x)\right| \leq C_{\mathcal{B},\alpha},$$

with the same estimates for T replaced by T^*, the formal L^2 adjoint of T.

Remark 1.2.2 Above we used the formal L^2 adjoint of T. To define this, we first define the transpose, T^t. The Schwartz kernel of T^t is defined by $T^t(x,y) = T(y,x)$; more precisely,

$$\int T^t(x,y)\,\phi(x,y)\;dx\,dy = \int T(x,y)\,\phi(y,x)\;dx\,dy,$$

for $\phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, where, as usual, we have written the pairing between distributions and test functions as integration. We define the Schwartz kernel of T^* by $T^* = \overline{T^t}$, where $\overline{z}$ denotes the complex conjugate of z. Here, for a distribution λ, we are defining the distribution $\overline{\lambda}$ by $\overline{\lambda}(f) = \overline{\lambda\left(\overline{f}\right)}$.

A key tool for studying these operators is a characterization similar to Theorem 1.1.26. For this, we need a generalization of operators of the form $\mathrm{Op}(\varsigma)$, where $\varsigma \in \mathcal{S}_0(\mathbb{R}^n)$, to the non-translation invariant setting. We begin with a generalization of $\mathrm{Op}(f)$, where $f \in \mathcal{S}(\mathbb{R}^n)$.

DEFINITION 1.2.3. *We define $\mathcal{P} \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ to be the Fréchet space of functions $E(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying for every $m \in \mathbb{N}$, and every multi-indices α, β,*

$$\left|\partial_x^\alpha \partial_y^\beta E(x,y)\right| \leq C_{\alpha,\beta,m}\,(1+|x-y|)^{-m}.$$

We give $\mathcal{P}$ the coarsest topology such that the least possible $C_{\alpha,\beta,m}$ defines a continuous semi-norm on $\mathcal{P}$, for each choice of α, β, and m.

LEMMA 1.2.4. *Let $\mathcal{B} \subset \mathcal{P}$. For each $E \in \mathcal{B}$ define two new functions $E_1(x,z) = E(x,x-z)$, $E_2(z,y) = E(y-z,y)$. Fix α and define $\mathcal{B}_\alpha \subset C^\infty(\mathbb{R}^n)$ by*

$$\mathcal{B}_\alpha := \left\{\partial_x^\alpha E_1(x,\cdot), \partial_y^\alpha E_2(\cdot,y) \;\middle|\; x,y \in \mathbb{R}^n, E \in \mathcal{B}\right\}.$$

Then, for every α, $\mathcal{B}_\alpha \subset \mathcal{S}(\mathbb{R}^n)$. Furthermore, $\mathcal{B}$ is a bounded subset of $\mathcal{P}$ if and only if $\mathcal{B}_\alpha$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$, for every α.

PROOF. This follows immediately from the definitions. □

Remark 1.2.5 We think of the elements of $\mathcal{P}$ as *operators*, by identifying the function $E \in \mathcal{P}$ with the operator whose Schwartz kernel is given by integration against E,

$$Ef(x) = \int E(x,y)\,f(y)\;dy.$$

DEFINITION 1.2.6. *We define $\mathcal{P}_0 \subset \mathcal{P}$ to consist of those $E \in \mathcal{P}$ such that $\forall x \in \mathbb{R}^n$, $E(x, \cdot) \in \mathcal{S}_0(\mathbb{R}^n)$ and $\forall y \in \mathbb{R}^n$, $E(\cdot, y) \in \mathcal{S}_0(\mathbb{R}^n)$.*

Remark 1.2.7 $\mathcal{P}_0$ is a closed subspace of $\mathcal{P}$ and we give it the induced Fréchet topology.

LEMMA 1.2.8. *For $s \in \mathbb{R}$, the maps $\triangle_x^s, \triangle_y^s : \mathcal{P}_0 \to \mathcal{P}_0$ given by $(\triangle_x^s E)(\cdot, y) = \triangle^s E(\cdot, y)$ and $\left(\triangle_y^s E\right)(x, \cdot) = \triangle^s E(x, \cdot)$ are automorphisms of $\mathcal{P}_0$.*

PROOF. It is easy to verify that $\triangle_x^s$ and $\triangle_y^s$ map $\mathcal{P}_0 \to \mathcal{P}_0$. The closed graph theorem (Theorem A.1.14) then shows that both maps are continuous. Their respective continuous inverses are $\triangle_x^{-s}$ and $\triangle_y^{-s}$. □

Example 1.2.9 *There is a continuous inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{P}$ given by*

$$f \mapsto f(x - y);$$

or if we think of elements of $\mathcal{P}$ as operators,

$$f \mapsto \mathrm{Op}(f).$$

Under this map, $\mathcal{S}_0(\mathbb{R}^n)$ maps in to $\mathcal{P}_0$. Thus, operators in $\mathcal{P}_0$ should be thought of as a non-translation invariant generalization of $\mathrm{Op}(f)$, where $f \in \mathcal{S}_0(\mathbb{R}^n)$.

We define dilations of $\mathcal{P}$ by, for $R > 0$,

$$E^{(R)}(x, y) = R^n E(Rx, Ry).$$

Using this, for $f \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathrm{Op}\left(f^{(R)}\right) = \mathrm{Op}(f)^{(R)}$. We have the following result, which is a generalization of Theorem 1.1.26.

THEOREM 1.2.10. *Fix $t \in (-n, \infty)$, and let $T : \mathcal{S}_0(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. The following are equivalent:*

(i) T is a Calderón-Zygmund operator of order t.[12]

(ii) $T : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$, and for any bounded set $\mathcal{B} \subset \mathcal{P}_0$, the set

$$\left\{E \,\middle|\, \exists R > 0, F \in \mathcal{B}, E^{(R)} = R^{-t} T F^{(R)}\right\} \subset \mathcal{P}_0$$

is a bounded set.

[12] As in Theorem 1.1.26, when we say $T : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ is a Calderón-Zygmund operator of order $t \in (-n, \infty)$ we mean that there is a Calderón-Zygmund operator $\widetilde{T} : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ such that $\widetilde{T}\big|_{\mathcal{S}_0(\mathbb{R}^n)} = T$. We will see in the proof that follows that if $\widetilde{T} : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a Calderón-Zygmund operator of order $t \in (-n, \infty)$, then $\widetilde{T}\big|_{\mathcal{S}_0(\mathbb{R}^n)} : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$.

(iii) For each $j \in \mathbb{Z}$ *there is* $E_j \in \mathcal{P}_0$ *with* $\left\{E_j \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ *a bounded set and such that*

$$T = \sum_{j\in\mathbb{Z}} 2^{jt} E_j^{\left(2^j\right)},$$

where above sum converges in the topology of bounded convergence as operators $\mathcal{S}_0\left(\mathbb{R}^n\right) \to \mathcal{S}_0\left(\mathbb{R}^n\right)$.

Furthermore, (ii) and (iii) are equivalent for any $t \in \mathbb{R}$.

Remark 1.2.11 When we move to more general situations, we will call operators analogous to operators like $E^{(R)}$, where $E \in \mathcal{P}_0$, *elementary operators*.

To prove Theorem 1.2.10, we need several lemmas.

LEMMA 1.2.12. *Let* $\mathcal{E} \subset \mathcal{P}$ *be a bounded set. For each* $E_1, E_2 \in \mathcal{E}$ *and* $j_1, j_2 \in \mathbb{R}$ *define an operator* F *by*

$$F^{\left(2^{j_1\wedge j_2}\right)} = E_1^{\left(2^{j_1}\right)} E_2^{\left(2^{j_2}\right)}.$$

Then, $\forall \alpha$, β, $\exists N = N\left(\alpha, \beta\right)$, $\forall m$, $\exists C$, $\forall E_1, E_2 \in \mathcal{E}$, $\forall j_1, j_2 \in \mathbb{R}$,

$$2^{-N|j_1-j_2|} \left|\partial_x^\alpha \partial_z^\beta F\left(x, z\right)\right| \leq C\left(1 + |x-y|\right)^{-m}.$$

PROOF. The conclusion of the lemma is equivalent to

$$2^{-N|j_1-j_2|} \left|\left(2^{-j_1\wedge j_2}\partial_x\right)^\alpha \left(2^{-j_1\wedge j_2}\partial_z\right)^\beta F^{\left(2^{j_1\wedge j_2}\right)}\left(x, z\right)\right| \leq C 2^{n j_1\wedge j_2}\left(1 + 2^{j_1\wedge j_2}|x-y|\right)^{-m}.$$

By taking N large in terms of α and β, the previous equation follows from

$$\left|\left(2^{-j_1}\partial_x\right)^\alpha \left(2^{-j_2}\partial_z\right)^\beta F^{\left(2^{j_1\wedge j_2}\right)}\left(x, z\right)\right| \leq C 2^{n j_1\wedge j_2}\left(1 + 2^{j_1\wedge j_2}|x-y|\right)^{-m}.$$

$\partial_x^\alpha E_1\left(x, y\right)$ and $\partial_z^\beta E_2\left(y, z\right)$ are of the same forms as E_1 and E_2 (i.e., as E_1 and E_2 range over $\mathcal{E}$, $\partial_x^\alpha E_1\left(x, y\right)$ and $\partial_z^\beta E_2\left(y, z\right)$ range over a bounded set in $\mathcal{P}$). Thus, we may replace $\left(2^{-j_1}\partial_x^\alpha\right) E_1^{\left(2^{j_1}\right)}\left(x, y\right)$ and $\left(2^{-j_2}\partial_z^\alpha\right) E_2^{\left(2^{j_2}\right)}\left(y, z\right)$ with $E_1^{\left(2^{j_1}\right)}$ and $E_2^{\left(2^{j_2}\right)}$, respectively; i.e., it suffices to prove the case when $\alpha = \beta = 0$ and $N = 0$. From here, the result follows by a straightforward estimate, and is left to the reader. □

LEMMA 1.2.13. *Let* $\mathcal{E} \subset \mathcal{P}$ *be a bounded set. For each* $E_1, E_2 \in \mathcal{E}$ *and* $j_1, j_2 \in \mathbb{R}$ *define two operators* F_1 *and* F_2 *by*

$$F_1^{\left(2^{j_1}\right)} = E_1^{\left(2^{j_1}\right)} E_2^{\left(2^{j_2}\right)}, \quad F_2^{\left(2^{j_2}\right)} = E_1^{\left(2^{j_1}\right)} E_2^{\left(2^{j_2}\right)}.$$

Then, $\forall \alpha$, β, m, $\exists N = N\left(\alpha, \beta, m\right)$, $\exists C$, $\forall E_1, E_2 \in \mathcal{E}$, $\forall j_1, j_2 \in \mathbb{R}$, *for* $k = 1, 2$,

$$2^{-N|j_1-j_2|} \left|\partial_x^\alpha \partial_z^\beta F_k\left(x, z\right)\right| \leq C\left(1 + |x-y|\right)^{-m}.$$

PROOF. We prove the result for $k = 1$; the proof for $k = 2$ is similar. The conclusion of the lemma, in this case, is equivalent to

$$2^{-N|j_1-j_2|} \left| \left(2^{-j_1}\partial_x\right)^\alpha \left(2^{-j_1}\partial_z\right)^\beta F_1^{\left(2^{j_1}\right)}(x,z) \right| \leq C 2^{nj_1} \left(1 + 2^{j_1}|x-y|\right)^{-m}.$$

By taking $N = N_1 + N_2$ where $N_1 = N_1(\alpha, \beta, m, n)$ is large and N_2 is to be chosen later, this follows from

$$\begin{aligned} 2^{-N_2|j_1-j_2|} & \left| \left(2^{-j_1\wedge j_2}\partial_x\right)^\alpha \left(2^{-j_1\wedge j_2}\partial_z\right)^\beta F_1^{\left(2^{j_1}\right)}(x,z) \right| \\ & \leq C 2^{nj_1\wedge j_2} \left(1 + 2^{j_1\wedge j_2}|x-y|\right)^{-m}. \end{aligned}$$

This is the conclusion of Lemma 1.2.12. □

LEMMA 1.2.14. *Suppose $\mathcal{B} \subset \mathcal{P}_0$ is a bounded set. For $E_1, E_2 \in \mathcal{B}$ and $j_1, j_2 \in \mathbb{R}$, define two operators $F_1 = F_1(E_1, E_2, j_1, j_2)$ and $F_2 = F_2(E_1, E_2, j_1, j_2)$ by*

$$F_1^{\left(2^{j_1}\right)} = E_1^{\left(2^{j_1}\right)} E_2^{\left(2^{j_2}\right)}, \quad F_2^{\left(2^{j_2}\right)} = E_1^{\left(2^{j_1}\right)} E_2^{\left(2^{j_2}\right)}.$$

Then, for every N, the set

$$\left\{ 2^{N|j_1-j_2|} F_1, 2^{N|j_1-j_2|} F_2 \;\middle|\; E_1, E_2 \in \mathcal{B}, j_1, j_2 \in \mathbb{R} \right\} \subset \mathcal{P}_0$$

is a bounded set.

PROOF. We prove that

$$\left\{ 2^{N|j_1-j_2|} F_1, 2^{N|j_1-j_2|} F_2 \;\middle|\; E_1, E_2 \in \mathcal{B}, j_1, j_2 \in \mathbb{R} \right\} \subset \mathcal{P} \tag{1.10}$$

is a bounded set. Once this is done, the result will follow, since it is immediate to verify that if $F_1, F_2 \in \mathcal{P}$, they are in fact in $\mathcal{P}_0$, by using that $E_1, E_2 \in \mathcal{P}_0$. We show

$$\left\{ 2^{N|j_1-j_2|} F_1 \;\middle|\; E_1, E_2 \in \mathcal{B}, j_1, j_2 \in \mathbb{R} \right\} \subset \mathcal{P}$$

is a bounded set. The proof where F_1 is replaced by F_2 is similar.

Fix N, α, β, and m. Take $M = M(N, \alpha, \beta, m)$ large, to be chosen later. We separate into two cases. If $j_1 \geq j_2$, we define $\widetilde{E}_1(x,y) = \triangle_y^M E_1(x,y)$ and $\widetilde{E}_2(x,y) = \triangle_x^{-M} E_2(x,y)$. If $j_2 > j_1$ we define $\widetilde{E}_1(x,y) = \triangle_y^{-M} E_1$ and $\widetilde{E}_2(x,y) = \triangle_x^M E_2(x,y)$. Notice that $\left\{ \widetilde{E}_1, \widetilde{E}_2 \;\middle|\; E_1, E_2 \in \mathcal{B}, j_1, j_2 \in \mathbb{R} \right\} \subset \mathcal{P}_0$ is a bounded set and we have

$$F_1^{\left(2^{j_1}\right)} = 2^{-2M|j_1-j_2|} \widetilde{E}_1^{\left(2^{j_1}\right)} \widetilde{E}_2^{\left(2^{j_2}\right)}.$$

Applying Lemma 1.2.13, and by taking $M = M(N, \alpha, \beta, m)$ sufficiently large, we have

$$\left| \partial_x^\alpha \partial_y^\beta 2^{N|j_1-j_2|} F_1(x,y) \right| \lesssim (1 + |x-y|)^{-m}.$$

This completes the proof. □

LEMMA 1.2.15. *Suppose $\mathcal{B} \subset \mathcal{P}$ and $\mathcal{B}_0 \subset \mathcal{P}_0$ are bounded sets. For $E_1 \in \mathcal{B}_0$, $E_2 \in \mathcal{B}$ and $j_1, j_2 \in \mathbb{R}$ with $j_1 \geq j_2$. Define two operators*

$$F_1^{\left(2^{j_2}\right)} = E_1^{\left(2^{j_1}\right)} E_2^{\left(2^{j_2}\right)}, \quad F_2^{\left(2^{j_2}\right)} = E_2^{\left(2^{j_2}\right)} E_1^{\left(2^{j_1}\right)}.$$

Then, for every N, $\left\{2^{N|j_1-j_2|}F_1, 2^{N|j_1-j_2|}F_2 \,\middle|\, E_1 \in \mathcal{B}_0, E_2 \in \mathcal{B}, j_1 \geq j_2\right\} \subset \mathcal{P}$ is a bounded set. Note F_1 and F_2 are defined in a slightly different way than the operators of the same name in Lemma 1.2.14.

PROOF. The proof follows in the same way as Lemma 1.2.14 and we leave the details to the interested reader. □

LEMMA 1.2.16. *Let T be a Calderón-Zygmund operator of order $t \in (-n, \infty)$, and let $\mathcal{B} \subset \mathcal{P}_0$ be a bounded set. For $E \in \mathcal{B}$, $j \in \mathbb{R}$, define $F^{\left(2^j\right)} = 2^{-jt} T E^{\left(2^j\right)}$. Then,*

$$\left\{F \,\middle|\, E \in \mathcal{B}, j \in \mathbb{R}\right\} \subset \mathcal{P}$$

is a bounded set. The same result holds for $F^{\left(2^j\right)} = 2^{-jt} E^{\left(2^j\right)} T$.

PROOF. We prove the result for $2^{-jt} T E^{\left(2^j\right)}$; the result for $2^{-jt} E^{\left(2^j\right)} T$ follows by taking adjoints. Fix multi-indices α, β and fix $m \in \mathbb{N}$. We wish to show

$$\left|\left(2^{-j}\partial_x\right)^{\alpha}\left(2^{-j}\partial_z\right)^{\beta} F^{\left(2^j\right)}(x,z)\right| \lesssim 2^{nj}\left(1 + 2^j|x-z|\right)^{-m}.$$

As E ranges over $\mathcal{B}$, $\partial_z^{\beta} E$ ranges over a bounded subset of $\mathcal{P}_0$. Thus we may, without loss of generality, assume that $\beta = 0$.

Fix $\phi \in C_0^{\infty}\left(B^n(2)\right)$, with $\phi \equiv 1$ on $B^n(1)$. Take $M = M(\alpha, \beta, m)$ and $m' = m'(m)$ large to be chosen later. Set $\widetilde{E}(x,z) = \triangle_x^{-M} E(x,z)$, so that $\widetilde{E}$ ranges over a bounded subset of $\mathcal{P}_0$ as E ranges over $\mathcal{B}$. We have,

$$\begin{aligned}
\left(2^{-j}\partial_x\right)^{\alpha} F^{\left(2^j\right)}(x,z) &= 2^{-2Mj-tj}\left[\left(2^{-j}\partial_x\right)^{\alpha} T \triangle^M \widetilde{E}^{\left(2^j\right)}(\cdot, z)\right](x) \\
&= 2^{-2Mj-tj}\left[\left(2^{-j}\partial_x\right)^{\alpha} T \triangle^M \phi\left(2^j(\cdot - x)\right)\widetilde{E}^{\left(2^j\right)}(\cdot, z)\right](x) \\
&+ 2^{-2Mj-tj}\left[\left(2^{-j}\partial_x\right)^{\alpha} T \triangle^M \left(1 - \phi\left(2^j(\cdot - x)\right)\right)\widetilde{E}^{\left(2^j\right)}(\cdot, z)\right](x).
\end{aligned}$$

We bound these two terms separately.

Using the cancellation condition applied with $\phi_{R,z}$ replaced by

$$2^{-nj-2Mj} \triangle^M \phi\left(2^j(\cdot - x)\right)\widetilde{E}^{\left(2^j\right)}(\cdot, z) =: \psi$$

we see

$$\begin{aligned}
&\left|2^{-2Mj-tj}\left[\left(2^{-j}\partial_x\right)^{\alpha} T \triangle^M \phi\left(2^j(\cdot - x)\right)\widetilde{E}^{\left(2^j\right)}(\cdot, z)\right](x)\right| \\
&\quad\lesssim 2^{nj-tj}\left|\left(2^{-j}\partial_x\right)^{\alpha} T\psi\right| \\
&\quad\lesssim 2^{nj}\left(1 + 2^j|x-z|\right)^{-m};
\end{aligned}$$

here, we have used the rapid decrease of $\widetilde{E}$ to obtain the factor $\left(1+2^j\left|x-z\right|\right)^{-m}$.

Using, now, the growth condition

$$\begin{aligned}
&\left|2^{-2Mj-tj}\left[\left(2^{-j}\partial_x\right)^{\alpha}T\,\triangle^M\left(1-\phi\left(2^j\left(\cdot-x\right)\right)\right)\widetilde{E}^{\left(2^j\right)}\left(\cdot,z\right)\right]\left(x\right)\right| \\
&\quad\lesssim 2^{-2Mj-tj}\int_{|y-x|>2^{-j}}\left|\triangle_y^M T\left(x,y\right)\left(1-\phi\left(2^j\left(y-x\right)\right)\right)\widetilde{E}^{\left(2^j\right)}\left(y,z\right)\right|\,dy \\
&\quad\lesssim 2^{-2Mj-tj}\int_{|y-x|>2^{-j}}\left|x-y\right|^{-n-t-2M}2^{nj}\left(1+2^j\left|y-z\right|\right)^{-m'}\,dy \\
&\quad\lesssim 2^{nj}\int\left(1+\left|2^jx-y\right|\right)^{-n-t-2M}\left(1+\left|y-2^jz\right|\right)^{-m'}\,dy \\
&\quad\lesssim 2^{nj}\left(1+2^j\left|x-z\right|\right)^{-m},
\end{aligned}$$

provided M and m' are sufficiently large. This completes the proof. □

LEMMA 1.2.17. *Let T be a Calderón-Zygmund operator of order $t\in(-n,\infty)$, and let $\mathcal{B}\subset\mathcal{P}_0$ be a bounded set. For $E_1,E_2\in\mathcal{B}$, $j_1,j_2\in\mathbb{R}$, define $F^{\left(2^{j_1\wedge j_2}\right)}=2^{-j_1\wedge j_2 t}E_1^{\left(2^{j_1}\right)}TE_2^{\left(2^{j_2}\right)}$. Then, for every N, $\left\{2^{N|j_1-j_2|}F\,\middle|\,E_1,E_2\in\mathcal{B},j_1,j_2\in\mathbb{R}\right\}\subset\mathcal{P}_0$ is a bounded set.*

PROOF. We prove that $\left\{2^{N|j_1-j_2|}F\middle|E_1,E_2\in\mathcal{B},j_1,j_2\in R\right\}\subset\mathcal{P}$ is a bounded set. The result follows, as it is easy to then show that $F\in\mathcal{P}_0$, given that $E_1,E_2\in\mathcal{P}_0$.

Suppose that $j_2\le j_1$. We apply Lemma 1.2.16 to $G^{\left(2^{j_2}\right)}:=2^{-tj_2}TE_2^{\left(2^{j_2}\right)}$ to see that $\left\{G\,\middle|\,E_2\in\mathcal{B},j_2\in\mathbb{R}\right\}\subset\mathcal{P}$ is a bounded set. The lemma then follows from Lemma 1.2.15.

If, instead, $j_1\le j_2$, we instead apply Lemma 1.2.16 to $2^{-tj_1}E_1^{\left(2^{j_1}\right)}T$ and then apply Lemma 1.2.15 to complete the proof. □

LEMMA 1.2.18. *For each $j\in\mathbb{Z}$, let $E_j\in\mathcal{P}_0$. Suppose that $\left\{E_j\,\middle|\,j\in\mathbb{Z}\right\}\subset\mathcal{P}_0$ is a bounded set. Fix $t\in\mathbb{R}$. Then,*

$$\sum_{j\in\mathbb{Z}}2^{jt}E_j^{\left(2^j\right)}$$

converges in the topology of bounded convergence as operators $\mathcal{S}_0\left(\mathbb{R}^n\right)\to\mathcal{S}_0\left(\mathbb{R}^n\right)$.

PROOF. This follows just as in Lemma 1.1.20. □

LEMMA 1.2.19. *There is an operator $E\in\mathcal{P}_0$ with*

$$I=\sum_{j\in\mathbb{Z}}E^{\left(2^j\right)},$$

with the convergence of the sum taken in the topology of bounded convergence as operators $\mathcal{S}_0\left(\mathbb{R}^n\right)\to\mathcal{S}_0\left(\mathbb{R}^n\right)$.

PROOF. Take ς as in Lemma 1.1.22, and set $E = \mathrm{Op}(\varsigma)$. The result follows. □

PROOF OF THEOREM 1.2.10. (i)⇒(iii): Let T be a Calderón-Zygmund operator of order $t \in (-n, \infty)$. Let E be as in Lemma 1.2.19. We have,

$$T = ITI = \sum_{j,k\in\mathbb{Z}} E^{(2^j)} T E^{(2^k)}.$$

Setting $F_{j,k}^{(2^{j\wedge k})} = 2^{-(j\wedge k)t} 2^{|j-k|} E^{(2^j)} T E^{(2^k)}$, Lemma 1.2.17 shows that

$$\left\{ F_{j,k} \,\middle|\, j,k \in \mathbb{Z} \right\} \subset \mathcal{P}_0$$

is a bounded set. We define

$$F_l = \sum_{j\wedge k = l} 2^{-|j-k|} F_{j,k},$$

so that $\left\{ F_l \,\middle|\, l \in \mathbb{Z} \right\} \subset \mathcal{P}_0$ is a bounded set and

$$T = \sum_{l\in\mathbb{Z}} 2^{lt} F_l^{(2^l)}.$$

This competes the proof of (i)⇒(iii).

(iii)⇒(i): Let $\left\{ E_j \,\middle|\, j \in \mathbb{Z} \right\} \subset \mathcal{P}_0$ be a bounded set, and let $t \in (-n, \infty)$. We wish to show that $T = \sum_{j\in\mathbb{Z}} 2^{jt} E_j^{(2^j)}$ is a Calderón-Zygmund operator of order t. First we verify the growth condition. Fix multi-indices α and β and let $m = m(\alpha, \beta, t)$ be large.

$$\begin{aligned} \left| \partial_x^\alpha \partial_z^\beta T(x,z) \right| &\leq \sum_{j\in\mathbb{Z}} 2^{j(|\alpha|+|\beta|+t)} \left| \left(2^{-j}\partial_x\right)^\alpha \left(2^{-j}\partial_z\right)^\beta 2^{jn} E_j\left(2^j x, 2^j z\right) \right| \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(|\alpha|+|\beta|+t+n)} \left(1 + 2^j |x-z|\right)^{-m}. \end{aligned}$$

We separate the above sum into two terms: when $|x-z| \geq 2^{-j}$ and when $|x-z| < 2^{-j}$. For the first, we have

$$\begin{aligned} &\sum_{|x-z|\geq 2^{-j}} 2^{j(|\alpha|+|\beta|+t+n)} \left(1 + 2^j |x-z|\right)^{-m} \\ &\lesssim \sum_{|x-z|\geq 2^{-j}} 2^{j(|\alpha|+|\beta|+t+n)} 2^{-mj} |x-z|^{-m} \\ &\lesssim |x-z|^{-|\alpha|-|\beta|-t-n}. \end{aligned}$$

For the second, we have

$$\begin{aligned}
&\sum_{|x-z|<2^{-j}} 2^{j(|\alpha|+|\beta|+t+n)} \left(1+2^j |x-z|\right)^{-m} \\
&\lesssim \sum_{|x-z|<2^{-j}} 2^{j(|\alpha|+|\beta|+t+n)} \\
&\lesssim |x-z|^{-|\alpha|-|\beta|-t-n},
\end{aligned}$$

where the last line uses $t > -n$. We, therefore, have

$$\left|\partial_x^\alpha \partial_z^\beta T(x,z)\right| \lesssim |x-z|^{-n-|\alpha|-|\beta|-t},$$

thereby establishing the growth condition.

We now turn to the cancellation condition. Let $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$ be a bounded set. For $\phi \in \mathcal{B}$, define $F \in \mathcal{P}$ by $F(x,y) = \phi(x-y)$. Note that $\left\{F \,\middle|\, \phi \in \mathcal{B}\right\} \subset \mathcal{P}$ is a bounded set, and that $\phi_{R,z}(x) = R^{-n} F^{(R)}(x,z)$. Fix $R > 0$ and let $R = 2^k$ for $k \in \mathbb{R}$. In light of the above remarks, to prove the cancellation condition, we need to show

$$\left|\partial_x^\alpha \left[TF^{(2^k)}\right](x,z)\right| \lesssim 2^{k(n+t+|\alpha|)},$$

with implicit constant depending on α and $\mathcal{B}$, but independent of $x, z \in \mathbb{R}^n$ and $k \in \mathbb{R}$. Fix N large to be chosen later. For $j \geq k$, define $G_j^{(2^k)} = 2^{N|j-k|} E_j^{(2^j)} F^{(2^k)}$. In light of Lemma 1.2.15, we have $\left\{G_j \,\middle|\, j \geq k, \phi \in \mathcal{B}\right\} \subset \mathcal{P}$ is a bounded set. For $j < k$ define $G_j^{(2^j)} = E_j^{(2^j)} F^{(2^k)}$. A simple estimate shows $|\partial_x^\alpha G_j(x,z)| \lesssim 1$. We have, for N sufficiently large,

$$\sum_{j\geq k} 2^{jt} 2^{-N|j-k|} \left|\partial_x^\alpha G_j^{(2^k)}(x,z)\right| \lesssim \sum_{j\geq k} 2^{jt} 2^{-N|j-k|} 2^{k(n+|\alpha|)} \lesssim 2^{k(t+n+|\alpha|)},$$

$$\sum_{j<k} 2^{jt} \left|\partial_x^\alpha G_j^{(2^j)}(x,z)\right| \lesssim \sum_{j<k} 2^{j(t+n+|\alpha|)} \lesssim 2^{k(t+n+|\alpha|)},$$

where in the later sum we have used $t > -n$. Since

$$\partial_x^\alpha TF^{(2^k)} = \sum_{j<k} 2^{jt} \partial_x^\alpha G_j^{(2^j)} + \sum_{j\geq k} 2^{jt} 2^{-N|j-k|} \partial_x^\alpha G_j^{(2^k)},$$

combining the above two estimates completes the proof of the cancellation condition.

(ii)⇒(iii): Let T be as in (ii). Take $E \in \mathcal{P}_0$ as in Lemma 1.2.19 so that $I = \sum_{j\in\mathbb{Z}} E^{(2^j)}$. Define $E_j^{(2^j)} = 2^{-jt} TE^{(2^j)}$. By our assumption, $\left\{E_j \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ is a bounded set. We have

$$T = TI = \sum_{j\in\mathbb{Z}} TE^{(2^j)} = \sum_{j\in\mathbb{Z}} 2^{jt} E_j,$$

completing the proof of (iii).

(iii)⇒(ii): Let $T = \sum_{j\in\mathbb{Z}} 2^{jt} E_j^{(2^j)}$, where $\left\{E_j \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ is a bounded set, as in (iii). Let $\mathcal{B} \subset \mathcal{P}_0$ be a bounded set, and fix N large to be chosen later. For $E \in \mathcal{B}$, $k \in \mathbb{R}$, define $F_j = F_j(E,k)$ by $F_j^{(2^k)} = 2^{N|j-k|} E_j^{(2^j)} E^{(2^k)}$. By Lemma 1.2.14, $\left\{F_j \,\middle|\, j \in \mathbb{Z}, E \in \mathcal{B}, k \in \mathbb{R}\right\} \subset \mathcal{P}_0$ is a bounded set. Define $\widetilde{E} = \widetilde{E}(E,k)$ by $2^{kt}\widetilde{E}^{(2^k)} = \sum_{j\in\mathbb{Z}} 2^{jt-N|j-k|} F_j^{(2^k)}$. We take $N > t$ and have that

$$\left\{\widetilde{E} \,\middle|\, E \in \mathcal{B}, k \in \mathbb{R}\right\} \subset \mathcal{P}_0$$

is a bounded set. Consider,

$$TE^{(2^k)} = \sum_{j\in\mathbb{Z}} 2^{jt-N|j-k|} F_j^{(2^k)} = 2^{kt}\widetilde{E}^{(2^k)},$$

completing the proof. □

As in the translation invariant setting, we may extend the definition of Calderón-Zygmund operators to any order in the following way.

DEFINITION 1.2.20. *We say $T : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ is a Calderón-Zygmund operator of order $t \in \mathbb{R}$ if either of the equivalent conditions (ii) or (iii), of Theorem 1.2.10, holds.*

Remark 1.2.21 Suppose $K \in \mathcal{S}_0(\mathbb{R}^n)'$ is a Calderón-Zygmund kernel of order s. Then $\mathrm{Op}(K)$ is a Calderón-Zygmund operator of order s. Indeed, (iii) of Theorem 1.1.26 shows that $\mathrm{Op}(K)$ may be written as $\mathrm{Op}(K) = \sum_{j\in\mathbb{Z}} \mathrm{Op}(\varsigma_j)^{(2^j)}$, where $\{\varsigma_j \mid j \in \mathbb{Z}\} \subset \mathcal{S}_0(\mathbb{R}^n)$ is a bounded set. We have $\left\{\mathrm{Op}(\varsigma_j) \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ is a bounded set and (iii) of Theorem 1.2.10 shows that $\mathrm{Op}(K)$ is a Calderón-Zygmund operator of order s.

PROPOSITION 1.2.22. *If $T, S : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ are Calderón-Zygmund operators of orders $t, s \in \mathbb{R}$, respectively, then $TS : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ is a Calderón-Zygmund operator of order $t + s$.*

PROOF. This follows immediately from (ii) of Theorem 1.2.10. □

LEMMA 1.2.23. *For $s \in \mathbb{R}$, $\triangle^s : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{R}^n)$ is a Calderón-Zygmund operator of order $s/2$.*

PROOF. This follows by combining Lemma 1.1.34 and Remark 1.2.21. □

Remark 1.2.24 Just as in Remark 1.1.29, Lemma 1.2.23 when combined with Proposition 1.2.22 shows that $\triangle^s$ is an isomorphism between Calderón-Zygmund operators of order t and Calderón-Zygmund operators of order $t + 2s$. Thus, T is a Calderón-Zygmund operator of order $t \in \mathbb{R}$ if and only if $\triangle^{-t/2}T$ is a Calderón-Zygmund operator of order 0.

THEOREM 1.2.25. *Let T be a Calderón-Zygmund operator of order t. Then, $T : \mathring{L}^p_s \to \mathring{L}^p_{s-t}$.*

We prove Theorem 1.2.25 in three steps, and each step involves one of the equivalent characterizations from Theorem 1.2.10. The first is the L^2 boundedness of operators of order 0. This uses (iii) and the Cotlar-Stein Lemma, which we state without proof; see [Ste93] for details.

LEMMA 1.2.26 (Cotlar-Stein Lemma). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. For each $j \in \mathbb{N}$, let $T_j : \mathcal{H}_1 \to \mathcal{H}_2$ be such that*

$$\sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \left\|T_j^* T_k\right\|_{\mathcal{H}_1\to\mathcal{H}_1}^{1/2}, \sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \|T_j T_k^*\|_{\mathcal{H}_2\to\mathcal{H}_2}^{1/2} \leq C.$$

Then, the sum

$$\sum_{j\in\mathbb{N}} T_j$$

converges in the strong operator topology to a bounded operator $\mathcal{H}_1 \to \mathcal{H}_2$ and

$$\left\|\sum_{j\in\mathbb{N}} T_j\right\|_{\mathcal{H}_1\to\mathcal{H}_2} \leq C.$$

LEMMA 1.2.27. *Let $\mathcal{E} \subset \mathcal{P}$ be a bounded set. Then,*

$$\sup_{E\in\mathcal{E}} \sup_{R>0} \sup_{1\leq p\leq\infty} \left\|E^{(R)}\right\|_{L^p\to L^p} < \infty.$$

PROOF. That $\left\|E^{(R)}\right\|_{L^1\to L^1} \lesssim 1$ and $\left\|E^{(R)}\right\|_{L^\infty\to L^\infty} \lesssim 1$ follows immediately from the definitions. The result follows by interpolation. □

PROPOSITION 1.2.28. *Suppose T is a Calderón-Zygmund operator of order 0. Then $T : L^2 \to L^2$.*

PROOF. We use (iii) and write $T = \sum_{j\in\mathbb{Z}} E_j^{(2^j)}$, where $\left\{E_j \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ is a bounded set. Note that $\left\{E_j^* \,\middle|\, j \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ is also a bounded set, and that $\left(E_j^*\right)^{(2^j)} = \left(E_j^{(2^j)}\right)^*$. We define $F_{j,k}^{(2^j)} = E_j^{(2^j)} \left(E_k^{(2^k)}\right)^*$ and $G_{j,k}^{(2^j)} = \left(E_j^{(2^j)}\right)^* E_k^{(2^k)}$. Lemma 1.2.14 shows that $\left\{2^{|j-k|}F_{j,k}, 2^{|j-k|}G_{j,k} \,\middle|\, j,k \in \mathbb{Z}\right\} \subset \mathcal{P}_0$ is a bounded set. Lemma 1.2.27 then shows

$$\left\|E_j^{(2^j)} \left(E_k^{(2^k)}\right)^*\right\|_{L^2\to L^2}, \left\|\left(E_j^{(2^j)}\right)^* E_k^{(2^k)}\right\|_{L^2\to L^2} \lesssim 2^{-|j-k|}.$$

The result follows from the Cotlar-Stein Lemma. □

Next, we extend the above result to L^p, $1 < p < \infty$. This uses (i).

PROPOSITION 1.2.29. *Let T be a Calderón-Zygmund operator of order* 0. *Then* $T : L^p \to L^p$, $1 < p < \infty$.

PROOF SKETCH. Because $T : L^2 \to L^2$ by Proposition 1.2.28, to show $T : L^p \to L^p$, $1 < p \leq 2$, it suffices to show T is weak-type $(1,1)$. This follows just as in Proposition 1.1.7. Applying this to T^*, which is also a Calderón-Zygmund operator of order 0, we see that $T^* : L^p \to L^p$, $1 < p \leq 2$, and therefore we see $T : L^p \to L^p$, $2 \leq p < \infty$, completing the proof. □

Finally we prove Theorem 1.2.25. Here we use Proposition 1.2.22, and thus implicitly use (ii).

PROOF OF THEOREM 1.2.25. Let T be a Calderón-Zygmund operator of order t. For $f \in \mathcal{S}_0(\mathbb{R}^n)$, we wish to show

$$\left\| \triangle^{(s-t)/2} T f \right\|_{L^p} \lesssim \left\| \triangle^{s/2} f \right\|_{L^p}.$$

By Corollary 1.1.13 we may write $f = \triangle^{-s/2} g$, $g \in \mathcal{S}_0(\mathbb{R}^n)$, and we therefore wish to show

$$\left\| \triangle^{(s-t)/2} T \, \triangle^{-s/2} \, g \right\|_{L^p} \lesssim \|g\|_{L^p}.$$

This follows from the fact that $\triangle^{(s-t)/2} \mathrm{Op}(K) \, \triangle^{-s/2}$ is bounded on L^p, by Proposition 1.2.29, as it is an operator of order 0 (by Proposition 1.2.22 and Lemma 1.2.23). □

Remark 1.2.30 Let T be a Calderón-Zygmund operator of order t, with $-n < t < 0$. Let $T_0(x,y)$ be the function which equals $T(x,y)$ for $x \neq y$ and which equals 0 for $x = y$. From the Growth Condition it follows that $T_0(x,y) \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$, and we may therefore identify T_0 with an operator $T_0 : C_0^\infty(\Omega) \to C_0^\infty(\Omega)'$. In fact, it is easy to see that $T_0 = T$; i.e., T is itself given by integration against an L^1_{loc} function. To see this, merely use (iii) of Theorem 1.2.10. Indeed, if $\left\{ E_j \mid j \in \mathbb{Z} \right\} \subset \mathcal{P}_0$ is a bounded set and $-n < t < 0$, then it is easy to see $\sum_{j \in \mathbb{Z}} 2^{jt} E_j^{(2^j)}$ converges in distribution to an L^1_{loc} function.

1.3 PSEUDODIFFERENTIAL OPERATORS

We now wish to introduce another, closely related, class of singular integral operators: the standard pseudodifferential operators on $\mathbb{R}^n$. To a distribution $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)'$, we may associate an operator, $a(x,D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)'$, by

$$\int g(x) \left(a(x,D) f \right)(x) \, dx := \int g(x) \, a(x,\xi) \, \hat{f}(\xi) \, e^{2\pi i x \cdot \xi} \, dx \, d\xi, \text{ for } f, g \in \mathcal{S}(\mathbb{R}^n)$$

where we have, as usual, written the pairing between distributions and test functions as integration. An analog of the Schwartz kernel theorem (see Theorem A.1.30) states that the map $a \mapsto a(x, D)$ is a *bijection* between distributions in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)'$ and operators $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)'$. a is called the "symbol" of the operator, and for appropriate choices of a, operators written in this way are referred to as "pseudodifferential operators."

The first example to consider, and the one that justifies the name, is that of linear partial differential operators with smooth coefficients, to which we now turn.

DEFINITION 1.3.1. *We let $C_b^\infty(\mathbb{R}^n)$ denote the Fréchet space consisting of those $f \in C^\infty(\mathbb{R}^n)$ such that, for every multi-index α,*

$$\sup_x |\partial_x^\alpha f(x)| < \infty.$$

We give $C_b^\infty(\mathbb{R}^n)$ the coarsest topology such that the left-hand side of the above equation defines a continuous semi-norm, for each α.

Example 1.3.2 *Let $D = \frac{1}{2\pi i}\frac{\partial}{\partial x}$, so that $(D^\alpha f)^\wedge(\xi) = \xi^\alpha \hat{f}(\xi)$. Consider a linear partial differential operator of order M, $p(x, D)$, defined by*

$$p(x, D) f(x) = \sum_{|\alpha| \leq M} a_\alpha(x) D^\alpha f(x),$$

where $a_\alpha \in C_b^\infty(\mathbb{R}^n)$. $p(x, D)$ is a pseudodifferential operator corresponding to the tempered distribution $p(x, \xi) = \sum_{|\alpha| \leq M} a_\alpha(x) \xi^\alpha$ (where we are identifying the function $p(x, \xi)$ with a distribution in the usual way).

We now define a class of symbols which we study. The class that follows is the most commonly used class of symbols, and we refer to them as the "standard symbols."

DEFINITION 1.3.3. *Fix $m \in \mathbb{R}$. The space of standard symbols of order m, S^m, is the Fréchet space of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfy, for all multi-indices α, β,*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m - |\beta|}.$$

We give S^m the coarsest topology such that the least possible $C_{\alpha,\beta}$ defines a continuous semi-norm on S^m. Operators of the form $a(x, D)$, with $a \in S^m$, are called "pseudodifferential operators of order m." When we wish to be explicit that we are referring to standard symbols, we call $a(x, D)$ a "standard pseudodifferential operator of order m."

Remark 1.3.4 We defined Calderón-Zygmund operators via estimates on the Schwartz kernel of the operator (the growth condition), along with an extra condition (the cancellation condition). In a similar vein, pseudodifferential operators are defined via estimates on the symbol. This highlights a major convenience of pseudodifferential operators: the estimates on the symbols play the role of both the growth condition *and* the cancellation condition from Calderón-Zygmund operators.

Remark 1.3.5 As we shall see, pseudodifferential operators of order m are closely related to Calderón-Zygmund operators of order m. However, neither class is contained in the other. For instance, if $\phi \in C_b^\infty(\mathbb{R}^n)$, the operator given by $f \mapsto \phi f$ is a pseudodifferential operator of order 0, however it is not a Calderón-Zygmund operator of order 0 except in very special cases (it does not satisfy the cancellation condition for $R \ll 1$). Conversely, we shall see that the Schwartz kernels of pseudodifferential operators satisfy estimates which are somewhat better than those satisfied by Calderón-Zygmund operators.

As with Calderón-Zygmund operators, the two main theorems concerning pseudodifferential operators are that they form an algebra, and that they are bounded on appropriate Sobolev spaces. We now turn to discussing these results in more detail. We do not include proofs of all results in the section. We refer the reader to [Ste93, Chapter VI] for more details and proofs.

THEOREM 1.3.6. *Suppose $a \in S^{m_1}$ and $b \in S^{m_2}$. We consider the operator $a(x,D)\,b(x,D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. There is $c \in S^{m_1+m_2}$ such that $c(x,D) = a(x,D)\,b(x,D)$. Furthermore, for every N,*

$$c - \sum_{|\alpha|<N} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha a\right)\left(\partial_x^\alpha b\right) \in S^{m_1+m_2-N}.$$

Remark 1.3.7 Taking $N = 1$ in Theorem 1.3.6 we see

$$c(x,\xi) \equiv a(x,\xi)\,b(x,\xi) \mod S^{m_1+m_2-1};$$

and we think of the remainder term in $S^{m_1+m_2-1}$ as a "lower order" term.

Remark 1.3.8 Theorem 1.3.6 has a much better conclusion than Proposition 1.2.22: not only do we have $c \in S^{m_1+m_2}$, but we know exactly what c is, modulo elements of $\bigcap_{m\in\mathbb{R}} S^m$. This makes pseudodifferential operators far easier to work with, but is closely tied to the fact that we may use the Fourier transform extensively.

Remark 1.3.9 Theorem 1.3.6 is often referred to as the "calculus of pseudodifferential operators."

For $s \in \mathbb{R}$, let Λ^s denote the pseudodifferential operator of order s with symbol $\left(1 + |\xi|^2\right)^{s/2}$. It is easy to verify that $\Lambda^s\Lambda^t = \Lambda^{s+t}$. As such, $\Lambda^s : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an automorphism with inverse Λ^{-s}.

DEFINITION 1.3.10. *For $1 \le p \le \infty$, we define $L_s^p(\mathbb{R}^n)$, the standard L^p Sobolev space of order $s \in \mathbb{R}$, to be the completion of $\mathcal{S}(\mathbb{R}^n)$ in the norm*

$$\|f\|_{L_s^p} := \|\Lambda^s f\|_{L^p}.$$

THEOREM 1.3.11. *Let $a \in S^m$. Then $a(x,D) : L_s^p \to L_{s-m}^p$, $1 < p < \infty$.*

As in Theorem 1.2.25, the proof of Theorem 1.3.11 separates into three main parts, each of which is proved in a manner closely analogous to the corresponding part of the proof of Theorem 1.2.25. Details of the proofs may be found in [Ste93]. The three parts are:

(I) If $a \in S^0$, then $a(x,D) : L^2 \to L^2$.

(II) If $a \in S^0$, then $a(x,D) : L^p \to L^p$, $1 < p < \infty$.

(III) Using the previous part and Theorem 1.3.6, Theorem 1.3.11 follows.

The proof of (I) relies on a decomposition of $a(x,D)$ analogous to the one used in Proposition 1.2.28. Indeed, we decompose

$$a(x,D) = a(x,D)\, I = \sum_{j\in\mathbb{Z}} a(x,D)\, \mathrm{Op}\left(\varsigma^{(2^j)}\right) =: \sum_{j\in\mathbb{Z}} A_j,$$

where $\varsigma \in \mathcal{S}_0(\mathbb{R}^n)$ is as in Lemma 1.1.22. The operators A_j can be shown to satisfy

$$\left\|A_j^* A_k\right\|_{L^2\to L^2}, \left\|A_j A_k^*\right\|_{L^2\to L^2} \lesssim 2^{-|j-k|},$$

and the L^2 boundedness of $a(x,D)$ follows from the Cotlar-Stein Lemma (Lemma 1.2.26).

For (II), the key is to show that $a(x,D)$ is weak-type $(1,1)$. This follows by taking the inverse Fourier transform in the ξ variable. To a distribution $a(x,\xi) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)'$ we associate a distribution $\breve{a}(x,z) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)'$ by, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\int \breve{a}(x,z)\, f(x)\, g(x-z)\; dx\, dz := \int f(x)\, (a(x,D)\, g)(x)\; dx.$$

I.e., $\breve{a}(x,z)$ is the inverse Fourier transform of $a(x,\xi)$ in the ξ variable. It is not hard to see that $a \mapsto \breve{a}$ is an automorphism of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)'$. To characterize symbols in S^m, we define a class of kernels.

DEFINITION 1.3.12. *We say $K \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)'$ is a kernel of order m if the following conditions hold.*

- *(Growth Condition) For every multi-indices α, β and every N,*

$$\left|\partial_x^\beta \partial_z^\alpha K(x,z)\right| \le C_{\alpha,\beta,N}\, |z|^{-n-m-|\alpha|-N},$$

  in particular, $K(x,z)$ is C^∞ for $z \ne 0$.

- *(Cancellation Condition) For $m \ge 0$, we assume for every bounded set $\mathcal{B} \subset C_0^\infty(\mathbb{R}^n)$, and for every multi-index α,*

$$\sup_{x\in\mathbb{R}^n} \sup_{R\ge 1} \sup_{\phi\in\mathcal{B}} R^{-m} \left|\int \partial_x^\alpha K(x,z)\, \phi(Rz)\; dz\right| < \infty.$$

- *If $m < 0$, the growth condition implies that for $\phi \in C_0^\infty(\mathbb{R}^n)$,*

$$\lim_{\epsilon\to 0} \int_{|z|>\epsilon} K(x,z)\, \phi(z)\; dz$$

  converges to a C^∞ function. We assume that $\int K(x,z)\, \phi(z)\, dz$ agrees with this limit.

PROPOSITION 1.3.13. *$a \mapsto \check{a}$ is a bijection between S^m and kernels of order m.*

PROOF. See [Ste93, Chapter VI]. □

Remark 1.3.14 In light of Proposition 1.3.13, we may write pseudodifferential operators in another way. Namely, for $a \in S^m$,

$$a(x, D) f(x) = \int \check{a}(x, z) f(x - z)\ dz,$$

where $\check{a}$ is a kernel of order m. In the sequel, we use this idea to generalize some aspects of pseudodifferential operators to settings where we have no immediate analog of the Fourier transform.

PROPOSITION 1.3.15. *If $a \in S^0$, then $a(x, D) : L^p \to L^p$, $1 < p \leq 2$.*

MAIN IDEA OF PROOF. Writing $a(x, D)$ as in Remark 1.3.14, the proof is nearly identical to the proof in Proposition 1.1.7. □

The proof of (II) is then completed by duality using the next result, which we state without proof.

PROPOSITION 1.3.16. *Let $a \in S^m$, then $a(x, D)^*$ is a pseudodifferential operator of order m.*

(III) now follows just as in the proof of Theorem 1.1.33, using (II) and the fact that if $a(x, D)$ is a pseudodifferential operator of order t, then $\Lambda^{s-t} a(x, D) \Lambda^{-s}$ is a pseudodifferential operator of order 0, by Theorem 1.3.6.

1.4 ELLIPTIC EQUATIONS

Theorem 1.3.6 allows us to approximately invert certain pseudodifferential operators while staying in the class of pseudodifferential operators. In what follows, we write $\mathrm{Op}(S^m)$ to denote the class of pseudodifferential operators of order m.

DEFINITION 1.4.1. *We say $a \in S^m$ is elliptic if there exists $R > 0$ such that $|a(x, \xi)| \geq C |\xi|^m$, for some C and all $|\xi| > R$.*

PROPOSITION 1.4.2. *Suppose $a \in S^m$ is elliptic. Then there is $b \in S^{-m}$ such that $a(x, D) b(x, D) \equiv I \mod \mathrm{Op}\left(S^{-1}\right)$.*

PROOF. Let $a \in S^m$ be elliptic, and take R as in Definition 1.4.1. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on a neighborhood of $B^n(R)$. Define

$$b(x, \xi) = (1 - \eta(\xi))\, a(x, \xi)^{-1}.$$

It is simple to verify that $b \in S^{-m}$. Furthermore, by Theorem 1.3.6, $b(x, D) a(x, D) \equiv I \mod \mathrm{Op}\left(S^{-1}\right)$. □

COROLLARY 1.4.3. *Suppose $a \in S^m$ is elliptic. Then, for every N, there is $b_N \in S^{-m}$ such that $b_N(x,D)\, a(x,D) \equiv I \mod \mathrm{Op}\left(S^{-N}\right)$.*

PROOF. Let $b \in S^{-m}$ be as in Proposition 1.4.2. Define $R \in \mathrm{Op}\left(S^{-1}\right)$ by $R = b(x,D)\, a(x,D) - I$. Then let

$$b_N(x,D) = \sum_{j=0}^{N-1} (-1)^j R^j b(x,D).$$

Theorem 1.3.6 shows that $b_N(x,\xi) \in S^{-m}$ satisfies the conclusions of the corollary. □

Remark 1.4.4 Actually, one may improve Corollary 1.4.3 by showing that there is $b_\infty \in S^{-m}$ such that $b_\infty(x,D)\, a(x,D) \equiv I \mod \mathrm{Op}(S^{-\infty})$, where $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m$. See [Ste93]. We do not pursue this here.

THEOREM 1.4.5. *Let $a \in S^m$ be elliptic, and suppose $u \in C^\infty(\mathbb{R}^n)'$ satisfies $a(x,D)\, u \in L^p_s$ for some $s \in \mathbb{R}$ and some $p \in (1,\infty)$. Then, $u \in L^p_{s+m}$.*

Remark 1.4.6 The space $C^\infty(\mathbb{R}^n)'$ is the space of "distributions with compact support." The reader wishing for more details on this space is referred to Appendix A.1.1.

PROOF OF THEOREM 1.4.5. It is classical that u, being a distribution with compact support, is in L^p_{-M} for some M. Take N large, and let b_N be as in Corollary 1.4.3. Define $R_N = b_N(x,D)\, a(x,D) - I \in \mathrm{Op}\left(S^{-N}\right)$. We have

$$u = b_N(x,D)\, a(x,D)\, u - R_N u.$$

By Theorem 1.3.11, we have $b_N(x,D)\, a(x,D)\, u \in L^p_{s+m}$ and $R_N u \in L^p_{-M+N}$. Taking $N \geq M + s + m$, the result follows. □

The most interesting examples of elliptic psuedodifferential operators come from elliptic differential operators.[13] See Example 1.3.2. In this case, Theorem 1.4.5 gives the optimal L^p regularity of elliptic differential operators.

In the sequel, we use similar ideas in a non-Euclidean setting, where we do not have a convenient analog of the Fourier transform. To motivate our later definitions, we present an equivalent characterization of elliptic pseudodifferential operators, which we state without proof.

THEOREM 1.4.7. *Let $a \in S^m$. The following are equivalent:*

(i) a is elliptic.

[13] An elliptic differential operator is a differential operator which is an elliptic pseudodifferential operator.

(ii) For every N there exists C_N such that $\forall f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|f\|_{L^2_m} \leq C_N \left(\|a(x,D) f\|_{L^2} + \|f\|_{L^2_{-N}} \right). \tag{1.11}$$

(iii) For some N with $-N < m$ there exists C_N so that (1.11) *holds.*

Suppose $a(x,D) \in \mathrm{Op}(S^m)$ is an elliptic pseudodifferential operator of order m. Fix p, $1 < p < \infty$, and suppose $u \in C^\infty(\mathbb{R}^n)'$ satisfies $a(x,D)u \in L^p_s$, for some $s \in \mathbb{R}$. Theorem 1.4.5 shows that $u \in L^p_{s+m}$.

More is true, the above holds "locally." For $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^n)$, we write $\phi_1 \prec \phi_2$ to mean that $\phi_2 \equiv 1$ on a neighborhood of the support of ϕ_1. Suppose $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^n)$ with $\phi_1 \prec \phi_2$ and suppose $u \in C^\infty(\mathbb{R}^n)'$, with $\phi_2 a(x,D)u \in L^p_s$ (we are still taking $a(x,D) \in \mathrm{Op}(S^m)$ to be elliptic, and fixing $p \in (1,\infty)$). Then, we have $\phi_1 u \in L^p_{s+m}$. To formally discuss this, we state a definition and a theorem.

DEFINITION 1.4.8. *Let $T : C^\infty(\mathbb{R}^n)' \to C_0^\infty(\mathbb{R}^n)'$ be an operator. We say T is L^p-subelliptic of order $\epsilon > 0$ on an open set $V \subseteq \mathbb{R}^n$, if for every $\phi_1 \prec \phi_2 \in C_0^\infty(V)$, we have the following estimate for all $u \in C_0^\infty(V)'$, and all s, N,*

$$\|\phi_1 u\|_{L^p_{s+\epsilon}} \leq C_{\phi_1,\phi_2,s,N} \left(\|\phi_2 T u\|_{L^p_s} + \|\phi_2 u\|_{L^p_s} \right),$$

where if the left-hand side is infinite, the right-hand side is infinite, and if the right-hand side is finite, the left-hand side is finite. We say T is L^p-subelliptic on V, if T is L^p-subelliptic of order $\epsilon > 0$ on V for some ϵ. We say T is subelliptic on V, if T is L^2-subelliptic on V.

We state the following theorem without proof.

THEOREM 1.4.9. *Let $a(x,D) \in \mathrm{Op}(S^m)$, $m > 0$, be elliptic. Then $a(x,D)$ is L^p-subelliptic of order m on $\mathbb{R}^n$, for every $1 < p < \infty$.*

Theorem 1.4.9 shows that ellipticity implies subellipticity. There is a third, even weaker, condition which will also be of interest to us.

DEFINITION 1.4.10. *We say $T : C^\infty(\mathbb{R}^n)' \to C_0^\infty(\mathbb{R}^n)'$ is hypoelliptic on an open set $V \subseteq \mathbb{R}^n$ if the following holds. For every distribution $u \in C^\infty(\mathbb{R}^n)'$ with $Tu\big|_V \in C^\infty(V)$, we have $u\big|_V \in C^\infty(V)$.*

THEOREM 1.4.11. *Suppose $T : C^\infty(\mathbb{R}^n)' \to C_0^\infty(\mathbb{R}^n)'$ is L^p-subelliptic on V for some $p \in [1,\infty]$. Then, T is hypoelliptic on V.*

PROOF. Take $u \in C^\infty(\mathbb{R}^n)'$ with $Tu\big|_V \in C^\infty(V)$. Fix $x_0 \in V$, we wish to show u is C^∞ near x_0. Take $\phi_1 \in C_0^\infty(V)$ with $\phi_1 \equiv 1$ on a neighborhood of x_0 and take $\phi_2 \in C_0^\infty(V)$ with $\phi_1 \prec \phi_2$. Since $u \in C^\infty(\mathbb{R}^n)' \subset \bigcup_N L^p_{-N}$, we have $\phi_2 u \in L^p_{-N}$ for some N. Using that $\phi_2 T u \in C_0^\infty(V) \subset L^p_s$, $\forall s$, subellipticity shows $\phi_1 u \in L^p_{s+\epsilon}$, $\forall s$. Since $\bigcap_s L^p_s \subset C^\infty$ (by the Sobolev embedding theorem), we have $\phi_1 u \in C^\infty$, and the result follows. □

One may succinctly restate Theorems 1.4.9 and 1.4.11 as:

$$\text{Ellipticity} \Rightarrow \text{Subellipticity} \Rightarrow \text{Hypoellipticity.}$$

In general, none of the reverse implications hold, even for partial differential operators– see Sections 2.6 and 4.3.1.

1.5 FURTHER READING AND REFERENCES

Our prototypical example of a singular integer operator, the Hilbert transform, first arose in Hilbert's work on what is now known as the Riemann-Hilbert problem where he studied a similar operator on the unit circle (instead of on $\mathbb{R}$). Hilbert's proof was published by Weyl [Wey08]. Schur improved these results and introduced the form of the Hilbert transform mentioned in the introduction [Sch11]. All of these results were restricted to L^2. It was Marcel Riesz who extended these results to L^p $(1 < p < \infty)$ [Rie28]. The above citations were focused on "complex analysis methods," and did not generalize to higher dimensions. Besicovitch [Bes26], Titchmarsh [Tit29], and Marcinkiewicz [Mar36] offered a "real-variable" analysis of the Hilbert transform.

This real variable analysis of the Hilbert transform was a main motivating example for Calderón and Zygmund when they introduced the homogeneous Calderón-Zygmund kernels as discussed in the introduction to this chapter [CZ52]. The proof of (a) of Theorem 1.1.2 uses their methods. The interpolation theorem used in that proof (the "Marcinkiewicz Interpolation theorem") was proved by Marcinkiewicz [Mar39].

The concept of a pseudodifferential operator is rooted in the work of Marcinkiewicz [Mar39], later work by Calderón and Zygmund, and the work of Seeley in his thesis [See59]. This was followed by further work of Seeley [See65] and Unterberger and Bokobza [UB64]. This culminated in the work of Kohn and Nirenberg [KN65] and Hörmander [Hör65]. It was these last two references that first exhibited the theory of pseudodifferential operators as covered in Section 1.3. Our presentation more closely follows the one from Chapter VI of [Ste93].

Non-translation invariant, non-homogeneous Calderón-Zygmund operators (as discussed in Section 1.2) was the work of many authors. A systematic approach for operators of order 0, working in more general "spaces of homogeneous type," and working with much less regular kernels was developed by Coifman and Weiss [CW71]. Definition 1.2.1 was taken from much more recent work of Nagel, Rosay, Stein, and Wainger [NRSW89] and Koenig [Koe02] who worked in the more general setting discussed in the next chapter (see Section 2.16 for further comments on their work). The concept of defining the cancellation condition as in Definition 1.2.1 is closely related to the hypothesis of the $T(1)$ theorem of David and Journé [DJ84]; see also the presentation in [Ste93, pages 293-294]. This idea was further championed by E. M. Stein; see, e.g., [Ste93, page 248]. The idea to extend operators to order $\leq -n$ by considering them acting on $\mathcal{S}_0(\mathbb{R}^n)$ appears in the work by Christ, Geller, Głowacki, and Polin [CGGP92] though it has been used in many situations.

The decomposition of a Calderón-Zygmund kernel as a sum of dilates of functions in $\mathcal{S}_0(\mathbb{R}^n)$ (see, e.g, (iii) of Theorem 1.1.23, (iii) of Theorem 1.1.26, and more generally, (iii) of Theorem 1.2.10) is called a Littlewood-Paley decomposition of the operator, named so because of the first place similar decompositions appeared: in the work of Littlewood and Paley on Fourier series [LP31, LP37, LP38]. These ideas were later worked on by Zygmund and Marcinkiewicz, but were moved to higher dimensions and used in greater generality by E. M. Stein (see, e.g., [Ste70b]). A decomposition, which is very similar to the ones we use, appears in a "multi-parameter" situation in the work of Nagel, Ricci, Stein, and Wainger on flag kernels [NRS01, NRSW12] (see Section 4.2 for a discussion of their work). Since our main goal is to generalize such concepts to a multi-parameter setting, these works were of the greatest inspiration to us. It is worth noting that in these papers the authors used bounded subsets of $C_0^\infty\left(B^n\left(1\right)\right)$ with one moment vanishing (in place of bounded subsets of $\mathcal{S}_0\left(\mathbb{R}^n\right)$) to study operators of order 0. For our purposes (here, to study operators of all orders, and later to study a more complicated multi-parameter setting in Chapter 5) we need to have many moments vanish and this is why we moved to the space $\mathcal{S}_0\left(\mathbb{R}^n\right)$. In the non-translation invariant setting, results similar to Theorem 1.2.10 are known to experts but we could not find this exact statement in the literature. It is probably most closely related to the translation invariant settings described above.

Finally, a key idea in this monograph is that one may characterize singular integral operators in terms of their actions on certain special functions or operators. Our simplest example of this is (ii) of Theorem 1.1.23–which leads to a short proof that such operators form an algebra. The author first heard of this characterization in a graduate class given by E. M. Stein at Princeton University in 2007.

Further history on some of the above topics and more references can be found in the expository articles by Stein [Ste99, Ste82a] along with Stein's book [Ste93].