CHAPTER ONE

Introduction

Partial differential equations (PDE) describe physical systems, such as solid and fluid mechanics, the evolution of populations and disease, and mathematical physics. The many different kinds of PDE each can exhibit different properties. For example, the heat equation describes the spreading of heat in a conducting medium, smoothing the spatial distribution of temperature as it evolves in time; it also models the molecular diffusion of a solute in its solvent as the concentration varies in both space and time. The wave equation is at the heart of the description of time-dependent displacements in an elastic material, with wave solutions that propagate disturbances. It describes the propagation of p-waves and s-waves from the epicenter of an earthquake, the ripples on the surface of a pond from the drop of a stone, the vibrations of a guitar string, and the resulting sound waves. Laplace's equation lies at the heart of potential theory, with applications to electrostatics and fluid flow as well as other areas of mathematics, such as geometry and the theory of harmonic functions. The mathematics of PDE includes the formulation of techniques to find solutions, together with the development of theoretical tools and results that address the properties of solutions, such as existence and uniqueness.

This text provides an introduction to a fascinating, intricate, and useful branch of mathematics. In addition to covering specific solution techniques that provide an insight into how PDE work, the text is a gateway to theoretical studies of PDE, involving the full power of real, complex and functional analysis. Often we will refer to applications to provide further intuition into specific equations and their solutions, as well as to show the modeling of real problems by PDE.

The study of PDE takes many forms. Very broadly, we take two approaches in this book. One approach is to describe methods of constructing solutions, leading to formulas. The second approach is more theoretical, involving aspects of analysis, such as the theory of distributions and the theory of function spaces.

1.1. Linear PDE

To introduce PDE, we begin with four linear equations. These equations are basic to the study of PDE, and are prototypes of classes of equations, each with different properties. The primary elementary methods of solution are related to the techniques we develop for these four equations.

For each of the four equations, we consider an unknown (real-valued) function u on an open set $U \subset \mathbb{R}^n$. We refer to u as the *dependent variable,* and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in U$ as the vector of *independent variables.* A *partial differential equation* is an equation that involves $\mathbf{x}$, u, and partial derivatives of u. Quite often, $\mathbf{x}$ represents only spatial variables. However, many equations are *evolutionary,* meaning that $u = u(\mathbf{x}, t)$ depends also on time t and the PDE has time derivatives. The *order* of a PDE is defined as the order of the highest derivative that appears in the equation.

The Linear Transport Equation:

$$u_t + cu_x = 0. \tag{1.1}$$

This simple first-order linear PDE describes the motion at constant speed c of a quantity u depending on a single spatial variable x and time t. Each solution is a *traveling wave* that moves with the speed c. If $c > 0$, the wave moves to the right; if $c < 0$, the wave moves left. The solutions are all given by a formula $u(x, t) = f(x - ct)$. The function $f = f(\xi)$, depending on a single variable $\xi = x - ct$, is determined from side conditions, such as boundary or initial conditions.

The next three equations are prototypical second-order linear PDE.

The Heat Equation:

$$u_t = k\Delta u. \tag{1.2}$$

In this equation, $u(\mathbf{x}, t)$ is the temperature in a homogeneous heat-conducting material, the parameter $k > 0$ is constant, and the Laplacian Δ is defined by

$$\Delta f(\mathbf{x}) = \nabla \cdot \nabla f(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2}$$

in Cartesian coordinates. The heat equation, also known as the diffusion equation, models diffusion in other contexts, such as the diffusion of a dye in a clear liquid. In such cases, u represents the concentration of the diffusing quantity.

The Wave Equation:

$$u_{tt} = c^2\Delta u. \tag{1.3}$$

As the name suggests, the wave equation models wave propagation. The parameter c is the wave speed. The dependent variable $u = u(\mathbf{x}, t)$ is a displacement, such as the displacement at each point of a guitar string as the string vibrates, if

$\mathbf{x} \in \mathbb{R}$, or of a drum membrane, in which case $\mathbf{x} \in \mathbb{R}^2$. The acceleration u_{tt}, being a second time derivative, gives the wave equation quite different properties from those of the heat equation.

Laplace's Equation:

$$\Delta u = 0. \tag{1.4}$$

Laplace's equation models equilibria or steady states in diffusion processes, in which $u(\mathbf{x}, t)$ is independent of time t,[1] and appears in many other contexts, such as the motion of fluids, and the equilibrium distribution of heat.

These three second-order equations arise often in applications, so it is very useful to understand their properties. Moreover, their study turns out to be useful theoretically as well, since the three equations are prototypes of second-order linear equations, namely, elliptic, parabolic, and hyperbolic PDE.

1.2. Solutions; Initial and Boundary Conditions

A *solution* of a PDE such as any of (1.1)–(1.4) is a real-valued function u satisfying the equation. Often this means that u is as differentiable as the PDE requires, and the PDE is satisfied at each point of the domain of u. However, it can be appropriate or even necessary to consider a more general notion of solution, in which u is not required to have all the derivatives appearing in the equation, at least not in the usual sense of calculus. We will consider this kind of *weak solution* later (see Chapter 11).

As with ordinary differential equations (ODE), solutions of PDE are not unique; identifying a unique solution relies on side conditions, such as initial and boundary conditions. For example, the heat equation typically comes with an *initial condition* of the form

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in U, \tag{1.5}$$

in which $u_0 : U \to \mathbb{R}$ is a given function.

Example 1. (Simple initial condition) The functions $u(x, t) = ae^{-t} \sin x + be^{-4t} \sin(2x)$ are solutions of the heat equation $u_t = u_{xx}$ for any real numbers a, b. However, $a = 3, b = -7$ would be uniquely determined by the initial condition $u(x, 0) = 3 \sin x - 7 \sin(2x)$. Then $u(x, t) = 3e^{-t} \sin x - 7e^{-4t} \sin(2x)$.

Boundary conditions are specified on the boundary ∂U of the (spatial) domain. *Dirichlet boundary conditions* take the following form, for a given function $f : \partial U \to \mathbb{R}$:

$$u(\mathbf{x}, t) = f(\mathbf{x}), \quad \mathbf{x} \in \partial U, \quad t > 0.$$

1. However, there are time-dependent solutions, for example $u(x, t)$ linear in x or independent of x.

Neumann boundary conditions specify the normal derivative of u on the boundary:

$$\frac{\partial u}{\partial \nu}(\mathbf{x}, t) = f(\mathbf{x}), \quad \mathbf{x} \in \partial U, \;\; t > 0,$$

where $\nu(\mathbf{x})$ is the unit outward normal to the boundary at $\mathbf{x}$. These boundary conditions are called *homogeneous* if $f \equiv 0$. Similarly, a linear PDE is called *homogeneous* if $u = 0$ is a solution. If it is not homogeneous, then the equation or boundary condition is called *inhomogeneous*.

Equations and boundary conditions that are linear and homogeneous have the property that any linear combination $u = av + bw$ of solutions v, w, with $a, b \in \mathbb{R}$, is also a solution. This special property, sometimes called the principle of superposition, is crucial to constructive methods of solution for linear equations.

1.3. Nonlinear PDE

We introduce a selection of nonlinear PDE that are significant by virtue of specific properties, special solutions, or their importance in applications.

The Inviscid Burgers Equation:

$$u_t + uu_x = 0 \tag{1.6}$$

is an example of a nonlinear first-order equation. Notice that this equation is *nonlinear* due to the uu_x term. It is related to the linear transport equation (1.1), but the wave speed c is now u and depends on the solution. We shall see in Chapter 3 that this equation and other first-order equations can be solved systematically using a procedure called the *method of characteristics*. However, the method of characteristics only gets you so far; solutions typically develop a singularity, in which the graph of u as a function of x steepens in places until at some finite time the slope becomes infinite at some x. The solution then continues with a shock wave. The solution is not even continuous at the shock, but the solution still makes sense, because the PDE expresses a *conservation law* and the shock preserves conservation.

For higher-order nonlinear equations, there are no methods of solution that work in as much generality as the method of characteristics for first-order equations. Here is a sample of higher-order nonlinear equations with interesting and accessible solutions.

Fisher's Equation:

$$u_t = \Delta u + f(u),$$

with $f(u) = u(1-u)$. This equation is a model for population dynamics when the spatial distribution of the population is taken into account. Notice the resemblance to the heat equation; also note that the ODE $u'(t) = f(u(t))$ is the *logistic equation,* describing population growth limited by a maximum population normalized to $u = 1$. In Chapter 12, we shall construct *traveling waves,* special solutions in which the population distribution moves with a constant speed in one direction. Recall that all solutions of the linear transport equation (1.1) are traveling waves, but they all have the same speed c. For Fisher's equation, we have to determine the speeds of traveling waves as part of the problem, and the traveling waves are special solutions, not the general solution.

The Porous Medium Equation:

$$u_t = \Delta(u^m). \tag{1.7}$$

In this equation, $m > 0$ is constant. The porous medium equation models flow in porous rock or compacted soil. The variable $u(\mathbf{x}, t) \geq 0$ measures the density of a compressible gas in a given location $\mathbf{x}$ at time t. The value of m depends on the equation of state relating pressure in the gas to its density. For $m = 1$, we recover the heat equation, but for $m \neq 1$, the equation is nonlinear. In fact, $m \geq 2$ for gas flow.

The Korteweg-deVries (KdV) Equation:

$$u_t + uu_x + u_{xxx} = 0.$$

This third-order equation is a model for water waves in which the height of the wave is $u(x, t)$. The KdV equation has particularly interesting traveling wave solutions called *solitary waves,* in which the height is symmetric about a single crest. The equation is a model in the sense that it relies on an approximation of the equations of fluid mechanics in which the length of the wave is large compared to the depth of the water.

Burgers' Equation:

$$u_t + uu_x = \nu u_{xx}.$$

The parameter $\nu > 0$ represents viscosity, hence the name *inviscid* Burgers equation for the first-order equation (1.6) having $\nu = 0$. Burgers' equation is a combination of the heat equation with a nonlinear term that convects the solution in a way typical of fluid flow. (See the Navier-Stokes system later in this list.) This important equation can be reduced to the heat equation with a clever change of dependent variable, called the *Cole-Hopf transformation* (see Chapter 13, Section 13.5).

Finally, we mention two *systems* of nonlinear PDE.

The Shallow Water Equations:

$$h_t + (hv)_x = 0,$$
$$v_t + vv_x - gh_x = 0,$$

in which $g > 0$ is the gravitational acceleration. The dependent variables h, v represent the height and velocity, respectively, of a shallow layer of water. The variable x is the horizontal spatial variable, along a flat bottom, and it is assumed that there is no dependence or motion in the orthogonal horizontal direction. Moreover, the velocity v is taken to be independent of depth.

The Navier-Stokes Equations:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u},$$
$$\nabla \cdot \mathbf{u} = 0.$$

describe the velocity $\mathbf{u} \in \mathbb{R}^3$ and pressure p in the flow of an incompressible viscous fluid. In this system of four equations, the parameter $\nu > 0$ is the viscosity, the first three equations (for $\mathbf{u}$) represent conservation of momentum, and the final equation is a constraint that expresses the incompressibility of the fluid. In an incompressible fluid, local volumes are unchanged in time as they follow the flow. Apart from special types of flow (such as in a stratified fluid), incompressibility also means that the density is constant (and is incorporated into ν, the kinematic viscosity).

Interestingly, the momentum equation, regarded as an evolution equation for $\mathbf{u}$, resembles Burgers' equation in structure. The pressure p does not have its own evolution equation; it serves merely to maintain incompressibility. In the limit $\nu \to 0$, we recover the incompressible Euler equations for an inviscid fluid. This is a singular limit in the sense that the order of the momentum equation is reduced. It is also a singular limit for Burgers' equation.

1.4. Beginning Examples with Explicit Wave-like Solutions

The linear and nonlinear first order equations described in Sections 1.1 and 1.3 nicely illustrate mathematical properties and representation of wave-like solutions. We discuss these equations and their solutions as a starting point for more general considerations.

1.4.1. The Linear Transport Equation

Solutions of the linear transport equation,

$$u_t + cu_x = 0, \tag{1.8}$$

where $c \in \mathbb{R}$ is a constant (the wave speed), are traveling waves $u(x,t) = f(x - ct)$. We can determine a unique solution by specifying the function

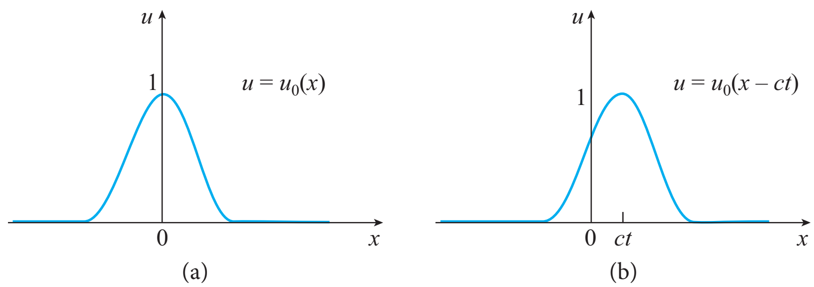

Figure 1.1. Linear transport equation: traveling wave solution. (a) $t = 0$; (b) $t > 0$.

$f : \mathbb{R} \to \mathbb{R}$ from an initial condition

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty, \tag{1.9}$$

in which $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function. Then the unique solution of the *initial value problem* (1.8), (1.9) is the traveling wave $u(x, t) = u_0(x - ct)$. A typical traveling wave is shown in Figure 1.1.

Instead of initial conditions, we can also specify a *boundary condition* for this PDE. Here is an example of how this would look, for functions ϕ, ψ given on the interval $[0, \infty)$:

$$\begin{aligned} \text{(initial condition)} \quad u(x, 0) &= \phi(x), \quad \text{if } x \geq 0, \\ \text{(boundary condition)} \quad u(0, t) &= \psi(t), \quad \text{if } t \geq 0. \end{aligned} \tag{1.10}$$

The solution u of (1.8), (1.10) will be a function defined on the first quadrant $Q_1 = \{(x, t) : x \geq 0, t \geq 0\}$ in the x-t plane. The general solution of the PDE is $u(x, t) = f(x - ct)$; the initial condition specifies $f(y)$ for $y > 0$, and the boundary condition gives $f(y)$ for $y < 0$. Both are needed to determine the solution $u(x, t)$ on Q_1.

1.4.2. The Inviscid Burgers Equation

This equation,

$$u_t + uu_x = 0, \tag{1.11}$$

has wave speed u that depends on the solution, in contrast to the linear transport equation (1.8) in which the wave speed c is constant. If we use the wave speed to track the solution, we can sketch its evolution. In Figure 1.2 we show how an initial condition (1.9) evolves for small $t > 0$. Points nearer the crest travel faster, since u is larger there, so the front of the wave tends to steepen, while the back spreads out. Notice how Figure 1.2 differs from Figure 1.1. The solution $u = u(x, t)$ can be specified implicitly in an equation without derivatives:

$$u = u_0(x - ut). \tag{1.12}$$

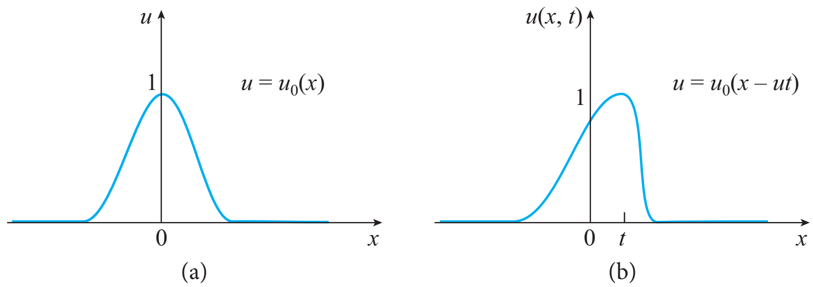

Figure 1.2. Inviscid Burgers equation: nonlinear wave propagation. (a) $t = 0$; (b) $t > 0$.

Eventually, the graph becomes infinitely steep, and the implicit solution in (1.12) is no longer valid. The solution is continued to larger time by including a shock wave, defined in Chapter 13.

PROBLEMS

1. Show that the traveling wave $u(x, t) = f(x - 3t)$ satisfies the linear transport equation $u_t + 3u_x = 0$ for any differentiable function $f : \mathbb{R} \to \mathbb{R}$.

2. Find an equation relating the parameters k, m, n so that the function $u(x, t) = e^{mt} \sin(nx)$ satisfies the heat equation $u_t = ku_{xx}$.

3. Find an equation relating the parameters c, m, n so that the function $u(x, t) = \sin(mt) \sin(nx)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$.

4. Find all functions $a, b, c : \mathbb{R} \to \mathbb{R}$ such that $u(x, t) = a(t)e^{2x} + b(t)e^x + c(t)$ satisfies the heat equation $u_t = u_{xx}$ for all x, t.

5. For $m > 1$, define the conductivity $k = k(u)$ so that the porous medium equation (1.7) can be written as the (quasilinear) heat equation

$$u_t = \nabla \cdot (k(u)\nabla u).$$

6. Solve the initial value problem

$$\begin{aligned} u_t + 4u_x &= 0, \quad -\infty < x < \infty, \quad t > 0, \\ u(x, 0) &= (1 + x^2)^{-1}, \quad -\infty < x < \infty. \end{aligned}$$

7. Solve the initial boundary value problem

$$\begin{aligned} u_t + 4u_x &= 0, \quad 0 < x < \infty, \quad t > 0, \\ u(x, 0) &= 0, \quad 0 < x < \infty, \\ u(0, t) &= te^{-t}, \quad t > 0. \end{aligned}$$

Explain why there is no solution if the PDE is changed to $u_t - 4u_x = 0$.

8. Consider the linear transport equation (1.8) with initial and boundary conditions (1.10).

(a) Suppose the data ϕ, ψ are differentiable functions. Show that the function $u : Q_1 \to \mathbb{R}$ given by

$$u(x,t) = \begin{cases} \phi(x-ct), & \text{if } x \geq ct, \\ \psi(t - x/c), & \text{if } x \leq ct \end{cases} \tag{1.13}$$

satisfies the PDE away from the line $x = ct$, the boundary condition, and initial condition. To see where (1.13) comes from, start from the general solution $u(x,t) = f(x-ct)$ of the PDE and substitute into the side conditions (1.10).

(b) In solution (1.13), the line $x = ct$, which emerges from the origin $x = t = 0$, separates the quadrant Q_1 into two regions. On the line, the solution has one-sided limits given by ϕ, ψ. Consequently, the solution will in general have singularities on the line.

(i) Find conditions on the data ϕ, ψ so that the solution is continuous across the line $x = ct$.

(ii) Find conditions on the data ϕ, ψ so that the solution is differentiable across the line $x = ct$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Verify that if $u(x,t)$ is differentiable and satisfies (1.12), that is, $u = f(x - ut)$, then $u(x,t)$ is a solution of the initial value problem

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad -\infty < x < \infty.$$

10. Let $u_0(x) = 1 - x^2$ if $-1 \leq x \leq 1$, and $u_0(x) = 0$ otherwise.

(a) Use (1.12) to find a formula for the solution $u = u(x,t)$ of the inviscid Burgers equation (1.11), (1.9) with $-1 < x < 1$, $0 < t < \frac{1}{2}$.

(b) Verify that $u(1,t) = 0, \quad 0 < t < \frac{1}{2}$.

(c) Differentiate your formula to find $u_x(1^-, t)$, and deduce that $u_x(1^-, t) \to -\infty$ as $t \to \frac{1}{2}^-$.

Note: $u_x(x,t)$ is discontinuous at $x = \pm 1$; the notation $u_x(1^-, t)$ means the one-sided limit: $u_x(1^-, t) = \lim\limits_{x<1, x\to 1} u_x(x,t)$. Similarly, $t \to \frac{1}{2}^-$ means $t \to \frac{1}{2}$, with $t < \frac{1}{2}$.