Chapter One

Tonelli Lagrangians and Hamiltonians on Compact Manifolds

1.1 LAGRANGIAN POINT OF VIEW

In this section we want to introduce the basic setting that we will be considering hereafter. Let M be a compact and connected smooth manifold without boundary. Denote by $\mathrm{T}M$ its tangent bundle and by T^*M the cotangent one. A point of $\mathrm{T}M$ will be denoted by (x, v), where $x \in M$ and $v \in \mathrm{T}_x M$, and a point of T^*M by (x, p), where $p \in \mathrm{T}_x^*M$ is a linear form on the vector space $\mathrm{T}_x M$. Let us fix a Riemannian metric g on it and denote by d the induced metric on M; let $\|\cdot\|_x$ be the norm induced by g on $\mathrm{T}_x M$; we will use the same notation for the norm induced on T_x^*M.

We will consider functions $L : \mathrm{T}M \longrightarrow \mathbb{R}$ of class C^2, which are called *Lagrangians*. Associated with each Lagrangian is a flow on $\mathrm{T}M$ called the *Euler-Lagrange flow*, defined as follows. Let us consider the action functional A_L from the space of continuous piecewise C^1 curves $\gamma : [a, b] \to M$, with $a \leq b$, defined by:

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

Curves that extremize this functional among all curves with the same endpoints are solutions of the *Euler-Lagrange equation*:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t)) \qquad \forall\, t \in [a, b]\,. \tag{1.1}$$

Observe that this equation is equivalent to

$$\frac{\partial^2 L}{\partial v^2}(\gamma(t), \dot{\gamma}(t))\ddot{\gamma}(t) = \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t)) - \frac{\partial^2 L}{\partial v \partial x}(\gamma(t), \dot{\gamma}(t))\dot{\gamma}(t)\,;$$

therefore, if the second partial vertical derivative $\partial^2 L/\partial v^2(x, v)$ is nondegenerate at all points of $\mathrm{T}M$, we can solve for $\ddot{\gamma}(t)$. The condition

$$\det \frac{\partial^2 L}{\partial v^2} \neq 0$$

is called the *Legendre condition* and allows one to define a vector field X_L on $\mathrm{T}M$ such that the solutions of $\ddot{\gamma}(t) = X_L(\gamma(t), \dot{\gamma}(t))$ are precisely the curves satisfying the Euler-Lagrange equation. This vector field X_L is called the *Euler-Lagrange vector field* and its flow Φ_t^L is the *Euler-Lagrange flow* associated with L. It turns out that Φ_t^L is C^1 even if L is only C^2 (see Remark 1.2.2).

DEFINITION 1.1.1 (Tonelli Lagrangian). *A function $L : \mathrm{T}M \longrightarrow \mathbb{R}$ is called a Tonelli Lagrangian if:*

i) $L \in C^2(\mathrm{T}M)$*;*

ii) *L is strictly convex in each fiber, in the C^2 sense, i.e., the second partial vertical derivative $\partial^2 L/\partial v^2(x,v)$ is positive definite, as a quadratic form, for all (x,v);*

iii) *L is superlinear in each fiber, i.e.,*

$$\lim_{\|v\|_x \to +\infty} \frac{L(x,v)}{\|v\|_x} = +\infty.$$

This condition is equivalent to asking whether for each $A \in \mathbb{R}$ there exists $B(A) \in \mathbb{R}$ such that

$$L(x,v) \geq A\|v\| - B(A) \qquad \forall\,(x,v) \in \mathrm{T}M\,.$$

Observe that since the manifold is compact, condition *iii)* is independent of the choice of the Riemannian metric g.

REMARK 1.1.2. More generally, one can consider the case of a *time-periodic Tonelli Lagrangian* $L : \mathrm{T}M \times \mathbb{T} \longrightarrow \mathbb{R}$ (also called *non-autonomous case*), as it was originally done by John Mather [62]. In fact, as it was pointed out by Jürgen Moser, this was the right setting for generalizing Aubry and Mather's results for twist maps to higher dimensions; in fact, every twist map can be seen as the time-one map associated with the flow of a periodic Tonelli Lagrangian on the one-dimensional torus (see for instance [71]). In this case, a further condition on the Lagrangian is needed:

iv) *The Euler-Lagrange flow is complete, i.e., every maximal integral curve of the vector field X_L has all $\mathbb{R}$ as its domain of definition.*

In the non-autonomous case, in fact, this condition is necessary in order to have that action-minimizing curves (or *Tonelli minimizers*; see section 4.1) satisfy the Euler-Lagrange equation. Without such an assumption Ball and Mizel [8] have constructed an example of Tonelli minimizers that are not C^1 and therefore are not solutions of the Euler-Lagrange flow. The role of the completeness hypothesis can be explained as follows. It is possible to prove, under the above conditions, that action-minimizing curves not only exist and are absolutely continuous, but also are C^1 on an open and dense full measure subset of the interval in which they are defined. It is possible to check that they satisfy the Euler-Lagrange equation on this set, while their velocity goes to infinity on the exceptional set on which they are not C^1. A complete flow, therefore, implies that Tonelli minimizers are C^1 everywhere and that they are actual solutions of the Euler-Lagrange equation.

A sufficient condition for the completeness of the Euler-Lagrange flow, for example, can be expressed in terms of a growth condition for $\partial L/\partial t$:

$$-\frac{\partial L}{\partial t}(x,v,t) \le C\left(1 + \frac{\partial L}{\partial v}(x,v,t)\cdot v - L(x,v,t)\right) \quad \forall\,(x,v,t) \in \mathrm{T}M \times \mathbb{T}.$$

EXAMPLES OF TONELLI LAGRANGIANS.

- RIEMANNIAN LAGRANGIANS. Given a Riemannian metric g on M, the *Riemannian Lagrangian* on $(\mathrm{T}M, g)$ is given by the *Kinetic energy*:

  $$L(x,v) = \frac{1}{2}\|v\|_x^2 .$$

  Its Euler-Lagrange equation is the equation of the geodesics of g:

  $$\nabla_{\dot{x}}\dot{x} \equiv 0$$

  (where $\nabla_{\dot{x}}\dot{x}$ denotes the covariant derivative), and its Euler-Lagrange flow coincides with the geodesic flow.

- MECHANICAL LAGRANGIANS. These Lagrangians play a key role in the study of classical mechanics. They are given by the sum of the kinetic energy and a *potential* $U : M \longrightarrow \mathbb{R}$:

  $$L(x,v) = \frac{1}{2}\|v\|_x^2 + U(x) .$$

  The associated Euler-Lagrange equation is given by:

  $$\nabla_{\dot{x}}\dot{x} = \nabla U(x) ,$$

  where ∇U is the gradient of U with respect to the Riemannian metric g, i.e.,

  $$d_x U \cdot v = \langle \nabla U(x), v\rangle_x \quad \forall\,(x,v) \in \mathrm{T}M .$$

- MAÑÉ'S LAGRANGIANS. This is a particular class of Tonelli Lagrangians, introduced[1] by Ricardo Mañé in [51] (see also [39]). If X is a C^k vector field on M, with $k \ge 2$, one can embed its flow φ_t^X into the Euler-Lagrange flow associated with a certain Lagrangian, namely

  $$L_X(x,v) = \frac{1}{2}\,\|v - X(x)\|_x^2 .$$

[1] It is interesting to notice that similar ideas had already appeared in embryonic stages in the literature; see, for example, Freidlin and Wentzell's study of the exit time from a domain for a random perturbation of a vector field [44].

It is quite easy to check that the integral curves of the vector field X are solutions to the Euler-Lagrange equation. In particular, the Euler-Lagrange flow $\Phi_t^{L_X}$ restricted to $\mathrm{Graph}(X) = \{(x, X(x)),\ x \in M\}$ (which is clearly invariant) is conjugated to the flow of X on M, and the conjugation is given by $\pi|\mathrm{Graph}(X)$, where $\pi : \mathrm{T}M \to M$ is the canonical projection. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Graph}(X) & \xrightarrow{\ \Phi_t^{L_X}\ } & \mathrm{Graph}(X) \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow[\ \varphi_t^X\ ]{} & M \end{array}$$

that is, for every $x \in M$ and every $t \in \mathbb{R}$, $\Phi_t^{L_X}(x, X(x)) = (\gamma_x^X(t), \dot{\gamma}_x^X(t))$, where $\gamma_x^X(t) = \varphi_t^X(x)$.

1.2 HAMILTONIAN POINT OF VIEW

In the study of classical dynamics, it turns often very useful to consider the associated *Hamiltonian system*, which is defined on the cotangent bundle T^*M. Let us describe how to define this new system and what its relation is with the Lagrangian one.

A standard tool in the study of convex functions is the so-called *Fenchel transform*, which allows one to transform functions on a vector space into functions on the dual space (see for instance [37, 73] for excellent introductions to the topic). Given a Lagrangian L, we can define the associated *Hamiltonian* as its Fenchel transform (or *Legendre-Fenchel transform*):

$$\begin{array}{rccl} H: & \mathrm{T}^*M & \longrightarrow & \mathbb{R} \\ & (x,p) & \longmapsto & \sup\limits_{v \in \mathrm{T}_x M} \{\langle p,\, v\rangle_x - L(x,v)\} \end{array}$$

where $\langle \cdot, \cdot \rangle_x$ denotes the canonical pairing between the tangent and cotangent bundles.

If L is a Tonelli Lagrangian, one can easily prove that H is finite everywhere (as a consequence of the superlinearity of L) and C^2, superlinear and strictly convex in each fiber (in the C^2 sense). Such a Hamiltonian is called a *Tonelli* (or *optical*) *Hamiltonian.*

DEFINITION 1.2.1 (Tonelli Hamiltonian). *A function* $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ *is called a Tonelli (or optical) Hamiltonian if:*

i) H *is of class* C^2*;*

ii) *H is strictly convex in each fiber in the C^2 sense, i.e., the second partial vertical derivative $\partial^2 H/\partial p^2(x,p)$ is positive definite, as a quadratic form, for any $(x,p) \in \mathrm{T}^*M$;*

iii) *H is superlinear in each fiber, i.e.,*

$$\lim_{\|p\|_x \to +\infty} \frac{H(x,p)}{\|p\|_x} = +\infty \,.$$

EXAMPLES OF TONELLI HAMILTONIANS

Let us see what the Hamiltonians associated with the Tonelli Lagrangians that we have introduced in the previous examples are.

- RIEMANNIAN HAMILTONIANS. If $L(x,v) = \frac{1}{2}\|v\|_x^2$ is the Riemannian Lagrangian associated with a Riemannian metric g on M, the corresponding Hamiltonian will be
$$H(x,p) = \frac{1}{2}\|p\|_x^2,$$
where $\|\cdot\|$ represents—in this last expression—the induced norm on the cotangent bundle T^*M.

- MECHANICAL HAMILTONIANS. If $L(x,v) = \frac{1}{2}\|v\|_x^2 + U(x)$ is a mechanical Lagrangian, the associated Hamiltonian is:
$$H(x,p) = \frac{1}{2}\|p\|_x^2 - U(x),$$
sometimes referred to as *mechanical energy*.

- MAÑÉ'S HAMILTONIANS. If X is a C^k vector field on M, with $k \geq 2$, and $L_X(x,v) = \|v - X(x)\|_x^2$ is the associated Mañé Lagrangian, one can check that the corresponding Hamiltonian is given by:
$$H(x,p) = \frac{1}{2}\|p\|_x^2 + \langle p, X(x) \rangle \,.$$

Given a Hamiltonian, one can consider the associated *Hamiltonian flow* Φ_t^H on T^*M. In local coordinates, this flow can be expressed in terms of the so-called *Hamilton equations*:

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(x(t),p(t)) \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t),p(t)) \,. \end{cases} \tag{1.2}$$

We will denote by $X_H(x,p) := \left(\frac{\partial H}{\partial p}(x,p), -\frac{\partial H}{\partial x}(x,p)\right)$ the *Hamiltonian vector field* associated with H. This has a more intrinsic (geometric) definition in terms of the canonical symplectic structure ω on T^*M (see appendix A). In fact, X_H is the unique vector field that satisfies

$$\omega\left(X_H(x,p),\cdot\right) = d_x H(\cdot) \qquad \forall (x,p) \in \mathrm{T}^*M.$$

For this reason, it is sometime called the *symplectic gradient of* H. It is easy to check from both definitions that—only in the autonomous case—the Hamiltonian is a *prime integral of the motion*, i.e., it is constant along the solutions of these equations.

Now, we would like to explain what the relation is between the Euler-Lagrange flow and the Hamiltonian one. It follows easily from the definition of Hamiltonian (and Fenchel transform) that for each $(x,v) \in \mathrm{T}M$ and $(x,p) \in \mathrm{T}^*M$ the following inequality holds:

$$\langle p,\, v\rangle_x \leq L(x,v) + H(x,p)\,. \tag{1.3}$$

This is called *Fenchel inequality* (or *Legendre-Fenchel inequality*) and plays a crucial role in the study of Lagrangian and Hamiltonian dynamics and in the variational methods that we are going to describe. In particular, equality holds if and only if $p = \partial L/\partial v(x,v)$. One can therefore introduce the following diffeomorphism between $\mathrm{T}M$ and T^*M, known as the *Legendre transform*:

$$\begin{array}{rccc} \mathcal{L}: & \mathrm{T}M & \longrightarrow & \mathrm{T}^*M \\ & (x,v) & \longmapsto & \left(x,\, \dfrac{\partial L}{\partial v}(x,v)\right). \end{array} \tag{1.4}$$

Moreover the following relation with the Hamiltonian holds:

$$H \circ \mathcal{L}(x,v) = \left\langle \frac{\partial L}{\partial v}(x,v),\, v \right\rangle_x - L(x,v)\,.$$

A crucial observation is that this diffeomorphism $\mathcal{L}$ represents a conjugation between the two flows, namely the Euler-Lagrange flow on $\mathrm{T}M$ and the Hamiltonian flow on T^*M; in other words, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\;\Phi_t^L\;} & \mathrm{T}M \\ {\scriptstyle \mathcal{L}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}} \\ \mathrm{T}^*M & \xrightarrow[\;\Phi_t^H\;]{} & \mathrm{T}^*M \end{array}$$

REMARK 1.2.2. Since $\mathcal{L}$ and the Hamiltonian flow Φ^H are both C^1, then it follows from the commutative diagram above that the Euler-Lagrange flow is also C^1.

Therefore one can equivalently study the Euler-Lagrange flow or the Hamiltonian flow, obtaining in both cases information on the dynamics of the system.

Each of these equivalent approaches will provide different tools and advantages, which may be very useful for understanding the dynamical properties of the system. For instance, the tangent bundle is the natural setting for the classical calculus of variations and for Mather's and Mañé's approaches (chapters 3 and 4); on the other hand, the cotangent bundle is equipped with a canonical symplectic structure (see appendix A), which allows one to use several symplectic topological tools coming from the study of Lagrangian graphs, Hofer's theory, Floer homology, among other subjects. Moreover, a particular fruitful approach in T^*M is the so-called *Hamilton-Jacobi method* (or *weak KAM theory*), which is concerned with the study of *solutions* and *subsolutions* of Hamilton-Jacobi equations. In a certain sense, this approach represents the functional analytic counterpart of the above-mentioned variational approach (chapter 5). In the following chapters we will provide a complete description of these methods and their implications on the study of the dynamics of the system.