Chapter One

Layer Potential Techniques

The asymptotic theory for elasticity imaging described in this book relies on layer potential techniques. In this chapter we prepare the way by reviewing a number of basic facts and preliminary results regarding the layer potentials associated with both the static and time-harmonic elasticity systems. The most important results in this chapter are on one hand the decomposition formulas for the solutions to transmission problems in elasticity and characterization of eigenvalues of the elasticity system as characteristic values of layer potentials and on the other hand, the Helmholtz-Kirchhoff identities. As will be shown later, the Helmholtz-Kirchhoff identities play a key role in the analysis of resolution in elastic wave imaging. We also note that when dealing with exterior problems for harmonic elasticity, one should introduce a radiation condition, known as the Sommerfeld radiation condition, in order to select the physical solution to the problem.

This chapter is organized as follows. In Section 1.1 we first review commonly used function spaces. Then we introduce in Section 1.2 equations of linear elasticity and use the Helmholtz decomposition theorem to decompose the displacement field into the sum of an irrotational (curl-free) and a solenoidal (divergence-free) field. Section 1.3 is devoted to the radiation condition for the time-harmonic elastic waves, which is used to select the physical solution to exterior problems. In Section 1.4 we introduce the layer potentials associated with the operators of static and time-harmonic elasticity, study their mapping properties, and prove decomposition formulas for the displacement fields. In Section 1.5 we derive the Helmholtz-Kirchhoff identities, which play a key role in the resolution analysis in Chapters 7 and 8. In Section 1.6 we characterize the eigenvalues of the elasticity operator on a bounded domain with Neumann or Dirichlet boundary conditions as the characteristic values of certain layer potentials which are meromorphic operator-valued functions. We also introduce Neumann and Dirichlet functions and write their spectral decompositions. These results will be used in Chapter 11. Finally, in Section 1.7 we state a generalization of Meyer's theorem concerning the regularity of solutions to the equations of linear elasticity, which will be needed in Chapter 11 in order to establish an asymptotic theory of eigenvalue elastic problems.

1.1 SOBOLEV SPACES

Throughout the book, symbols of scalar quantities are printed in italic type, symbols of vectors are printed in bold italic type, symbols of matrices or 2-tensors are printed in bold type, and symbols of 4-tensors are printed in blackboard bold type.

The following Sobolev spaces are needed for the study of mapping properties of layer potentials for elasticity equations.

Let ∂_i denote $\partial/\partial x_i$. We use $\nabla = (\partial_i)_{i=1}^d$ and $\partial^2 = (\partial^2_{ij})_{i,j=1}^d$ to denote the gradient and the Hessian, respectively.

Let Ω be a smooth domain in $\mathbb{R}^d$, with $d = 2$ or 3. We define the Hilbert space $H^1(\Omega)$ by

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega) \right\},$$

where ∇u is interpreted as a distribution and $L^2(\Omega)$ is defined in the usual way, with

$$||u||_{L^2(\Omega)} = \left(\int_\Omega |u|^2 \, d\boldsymbol{x} \right)^{1/2}.$$

The space $H^1(\Omega)$ is equipped with the norm

$$||u||_{H^1(\Omega)} = \left(\int_\Omega |u|^2 \, d\boldsymbol{x} + \int_\Omega |\nabla u|^2 \, d\boldsymbol{x} \right)^{1/2}.$$

If Ω is bounded, another Banach space $H_0^1(\Omega)$ arises by taking the closure of $\mathcal{C}_0^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in Ω, in $H^1(\Omega)$. We will also need the space $H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{\Omega})$ of functions $u \in L^2_{\text{loc}}(\mathbb{R}^d \setminus \overline{\Omega})$, the set of locally square summable functions in $\mathbb{R}^d \setminus \overline{\Omega}$, such that

$$h\, u \in H^1(\mathbb{R}^d \setminus \overline{\Omega}) \quad \forall\, h \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus \overline{\Omega}).$$

Furthermore, we define $H^2(\Omega)$ as the space of functions $u \in H^1(\Omega)$ such that $\partial^2_{ij} u \in L^2(\Omega)$, for $i, j = 1, \ldots, d$, and the space $H^{3/2}(\Omega)$ as the interpolation space $[H^1(\Omega), H^2(\Omega)]_{1/2}$ (see, for example, the book by Bergh and Löfström [49]).

It is known that the trace operator $u \mapsto u|_{\partial\Omega}$ is a bounded linear surjective operator from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$, where $H^{1/2}(\partial\Omega)$ is the collection of functions $f \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^d} \, d\sigma(\boldsymbol{x}) \, d\sigma(\boldsymbol{y}) < +\infty.$$

We set $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$ and let $\langle\ ,\ \rangle_{1/2,-1/2}$ denote the duality pair between these dual spaces.

We introduce a weighted norm, $\|u\|_{H^1_w(\mathbb{R}^d \setminus \overline{\Omega})}$, in two dimensions. Let

$$\|u\|^2_{H^1_w(\mathbb{R}^2 \setminus \overline{\Omega})} := \int_{\mathbb{R}^2 \setminus \overline{\Omega}} \frac{|u(\boldsymbol{x})|^2}{\sqrt{1 + |\boldsymbol{x}|^2}} d\boldsymbol{x} + \int_{\mathbb{R}^2 \setminus \overline{\Omega}} |\nabla u(\boldsymbol{x})|^2 d\boldsymbol{x}. \tag{1.1}$$

This weighted norm is introduced because, as will be shown later, the solutions of the static elasticity equation behave like $O(|\boldsymbol{x}|^{-1})$ in two dimensions as $|\boldsymbol{x}| \to \infty$. For convenience, we set

$$W(\mathbb{R}^d \setminus \overline{\Omega}) := \begin{cases} H^1_w(\mathbb{R}^2 \setminus \overline{\Omega}) & \text{for } d = 2, \\ H^1(\mathbb{R}^3 \setminus \overline{\Omega}) & \text{for } d = 3. \end{cases} \tag{1.2}$$

In three dimensions, $W(\mathbb{R}^d \setminus \overline{\Omega})$ is the usual Sobolev space.

We also define the Banach space $W^{1,\infty}(\Omega)$ by

$$W^{1,\infty}(\Omega) = \left\{ u \in L^\infty(\Omega) : \nabla u \in L^\infty(\Omega) \right\}, \tag{1.3}$$

where ∇u is interpreted as a distribution and $L^\infty(\Omega)$ is defined in the usual way, with

$$||u||_{L^\infty(\Omega)} = \inf \Big\{ C \geq 0 : |u(\boldsymbol{x})| \leq C \quad \text{a.e. } \boldsymbol{x} \in \Omega \Big\}.$$

We will need the following Hilbert spaces for deriving the Helmholtz decomposition theorem

$$H_{\text{curl}}(\Omega) := \{\boldsymbol{u} \in L^2(\Omega)^d, \nabla \times \boldsymbol{u} \in L^2(\Omega)^d\},$$

equipped with the norm

$$||\boldsymbol{u}||_{\text{curl}}(\Omega) = \Big(\int_\Omega |\boldsymbol{u}|^2 \, d\boldsymbol{x} + \int_\Omega |\nabla \times \boldsymbol{u}|^2 \, d\boldsymbol{x} \Big)^{1/2},$$

and

$$H_{\text{div}}(\Omega) := \{\boldsymbol{u} \in L^2(\Omega)^d, \nabla \cdot \boldsymbol{u} \in L^2(\Omega)\},$$

equipped with the norm

$$||\boldsymbol{u}||_{\text{div}}(\Omega) = \Big(\int_\Omega |\boldsymbol{u}|^2 \, d\boldsymbol{x} + \int_\Omega |\nabla \cdot \boldsymbol{u}|^2 \, d\boldsymbol{x} \Big)^{1/2}.$$

Finally, let $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_{d-1}$ be an orthonormal basis for the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$ and let

$$\partial/\partial \boldsymbol{T} = \sum_{p=1}^{d-1} (\partial/\partial T_p) \, \boldsymbol{T}_p \tag{1.4}$$

denote the tangential derivative on $\partial\Omega$. We say that $f \in H^1(\partial\Omega)$ if $f \in L^2(\partial\Omega)$ and $\partial f/\partial \boldsymbol{T} \in L^2(\partial\Omega)^{d-1}$. Furthermore, we define $H^{-1}(\partial\Omega)$ as the dual of $H^1(\partial\Omega)$ and the space $H^s(\partial\Omega)$, for $0 \leq s \leq 1$, as the interpolation space $[L^2(\partial\Omega), H^1(\partial\Omega)]_s$ or, equivalently, as the set of functions $f \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{d-1+2s}} \, d\sigma(\boldsymbol{x}) \, d\sigma(\boldsymbol{y}) < +\infty;$$

see again [49].

1.2 ELASTICITY EQUATIONS

Let Ω be a domain in $\mathbb{R}^d$, $d = 2, 3$. Let λ and μ be the Lamé constants for Ω satisfying the strong convexity condition

$$\mu > 0 \quad \text{and} \quad d\lambda + 2\mu > 0. \tag{1.5}$$

The constants λ and μ are respectively referred to as the compression modulus and the shear modulus. The compression modulus measures the resistance of the material to compression and the shear modulus measures the resistance to shearing. We also introduce the bulk modulus $\beta := \lambda + 2\mu/d$. We refer the reader to [122, p.11] for an explanation of the physical significance of (1.5).

In a homogeneous isotropic elastic medium, the elastostatic operator corresponding to the Lamé constants λ, μ is given by

$$\mathcal{L}^{\lambda,\mu}\boldsymbol{u} := \mu\triangle\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}, \quad \boldsymbol{u} : \Omega \to \mathbb{R}^d. \tag{1.6}$$

If Ω is bounded with a connected Lipschitz boundary, then we define the conormal derivative $\partial\boldsymbol{u}/\partial\boldsymbol{\nu}$ by

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}} = \lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{n} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t)\boldsymbol{n}, \tag{1.7}$$

where $\nabla\boldsymbol{u}$ is the matrix $(\partial_j u_i)_{i,j=1}^d$ with u_i being the i-th component of $\boldsymbol{u}$, the superscript t denotes the transpose, and $\boldsymbol{n}$ is the outward unit normal to the boundary $\partial\Omega$.

Note that the conormal derivative has a direct physical meaning:

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}} = \text{traction on } \partial\Omega.$$

The vector $\boldsymbol{u}$ is the displacement field of the elastic medium having the Lamé coefficients λ and μ, and the symmetric gradient

$$\nabla^s\boldsymbol{u} := (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t)/2$$

is the strain tensor.

In $\mathbb{R}^d$, $d = 2, 3$, let

$$\mathbf{I} := \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j,$$
$$\mathbb{I} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l,$$

with $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$ being the canonical basis of $\mathbb{R}^d$ and $\otimes$ denoting the tensor product between vectors in $\mathbb{R}^d$. Here, $\mathbf{I}$ is the $d \times d$ identity matrix or 2-tensor while $\mathbb{I}$ is the identity 4-tensor.

Define the elasticity tensor $\mathbb{C} = (C_{ijkl})_{i,j,k,l=1}^d$ for $\mathbb{R}^d$ by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.8}$$

which can be written as

$$\mathbb{C} := \lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathbb{I}.$$

With this notation, we have

$$\mathcal{L}^{\lambda,\mu}\boldsymbol{u} = \nabla\cdot\mathbb{C}\nabla^s\boldsymbol{u},$$

and

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}} = (\mathbb{C}\nabla^s\boldsymbol{u})\boldsymbol{n} = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n},$$

where $\boldsymbol{\sigma}(\boldsymbol{u})$ is the stress tensor given by

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{C}\nabla^s\boldsymbol{u}.$$

Now, we consider the elastic wave equation

$$\rho \partial_t^2 \boldsymbol{U} - \mathcal{L}^{\lambda,\mu} \boldsymbol{U} = 0,$$

where the positive constant ρ is the density of the medium. Then, we obtain a time-harmonic solution $\boldsymbol{U}(\boldsymbol{x}, t) = \Re e[e^{-i\omega t} \boldsymbol{u}(\boldsymbol{x})]$ if the space-dependent part $\boldsymbol{u}$ satisfies the time-harmonic elasticity equation for the displacement field,

$$(\mathcal{L}^{\lambda,\mu} + \omega^2 \rho) \boldsymbol{u} = 0, \tag{1.9}$$

with ω being the angular frequency.

The time-harmonic elasticity equation (1.9) has a special family of solutions called p- and s-plane waves:

$$\boldsymbol{U}^p(\boldsymbol{x}) = e^{i\omega\sqrt{\rho/(\lambda+2\mu)}\boldsymbol{x}\cdot\boldsymbol{\theta}} \boldsymbol{\theta} \quad \text{and} \quad \boldsymbol{U}^s(\boldsymbol{x}) = e^{i\omega\sqrt{\rho/\mu}\boldsymbol{x}\cdot\boldsymbol{\theta}} \boldsymbol{\theta}^\perp \tag{1.10}$$

for $\boldsymbol{\theta} \in \mathbb{S}^{d-1} := \{\boldsymbol{\theta} \in \mathbb{R}^d : |\boldsymbol{\theta}| = 1\}$ the direction of the wavevector and $\boldsymbol{\theta}^\perp$ is such that $|\boldsymbol{\theta}^\perp| = 1$ and $\boldsymbol{\theta}^\perp \cdot \boldsymbol{\theta} = 0$. Note that $\boldsymbol{U}^p$ is irrotational while $\boldsymbol{U}^s$ is solenoidal.

Taking the limit $\omega \to 0$ in (1.9) yields the static elasticity equation

$$\mathcal{L}^{\lambda,\mu} \boldsymbol{u} = 0. \tag{1.11}$$

In a bounded domain Ω, the equations (1.9) and (1.11) need to be supplemented with boundary conditions at $\partial\Omega$. If $\partial\Omega$ is a stress-free surface, the traction acting on $\partial\Omega$ vanishes:

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = 0.$$

This boundary condition is appropriate when the surface $\partial\Omega$ forms the outer boundary on the elastic body that is surrounded by empty space.

In a homogeneous, isotropic medium, using the Helmholtz decomposition theorem, the displacement field can be decomposed into the sum of an irrotational and a solenoidal field. Assume that Ω is simply connected and its boundary $\partial\Omega$ is connected. The Helmholtz decomposition states that for $\boldsymbol{w} \in L^2(\Omega)^d$ there exist $\phi_{\boldsymbol{w}} \in H^1(\Omega)$ and $\boldsymbol{\psi}_{\boldsymbol{w}} \in H_{\text{curl}}(\Omega) \cap H_{\text{div}}(\Omega)$ such that

$$\boldsymbol{w} = \nabla \phi_{\boldsymbol{w}} + \nabla \times \boldsymbol{\psi}_{\boldsymbol{w}}. \tag{1.12}$$

The Helmholtz decomposition (1.12) can be found by solving the following weak Neumann problem in Ω [38, 78]:

$$\int_\Omega \nabla \phi_{\boldsymbol{w}} \cdot \nabla p \, d\boldsymbol{x} = \int_\Omega \boldsymbol{w} \cdot \nabla p \, d\boldsymbol{x} \quad \forall\, p \in H^1(\Omega). \tag{1.13}$$

The function $\phi_{\boldsymbol{w}} \in H^1(\Omega)$ is uniquely defined up to an additive constant. In order to uniquely define the function $\boldsymbol{\psi}_{\boldsymbol{w}}$, we impose that it satisfies the following properties [53]:

$$\begin{cases} \nabla \cdot \boldsymbol{\psi}_{\boldsymbol{w}} = 0 & \text{in } \Omega, \\ \boldsymbol{\psi}_{\boldsymbol{w}} \cdot \boldsymbol{n} = (\nabla \times \boldsymbol{\psi}_{\boldsymbol{w}}) \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

The boundary condition $(\nabla \times \boldsymbol{\psi}_{\boldsymbol{w}}) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ shows that the gradient and curl parts in (1.12) are orthogonal.

We define, respectively, the Helmholtz decomposition operators $\mathcal{H}^p$ and $\mathcal{H}^s$ for $\boldsymbol{w} \in L^2(\Omega)^d$ by

$$\mathcal{H}^p[\boldsymbol{w}] := \nabla \phi_{\boldsymbol{w}} \quad \text{and} \quad \mathcal{H}^s[\boldsymbol{w}] := \nabla \times \boldsymbol{\psi}_{\boldsymbol{w}}, \tag{1.15}$$

where $\phi_{\boldsymbol{w}}$ is a solution to (1.13) and $\boldsymbol{\psi}_{\boldsymbol{w}}$ satisfy $\nabla \times \boldsymbol{\psi}_{\boldsymbol{w}} = \boldsymbol{w} - \nabla \phi_{\boldsymbol{w}}$ together with (1.14). The L^2-projectors $\mathcal{H}^p$ and $\mathcal{H}^s$ are pseudo-differential operators of symbols $\boldsymbol{\pi}_p(\boldsymbol{x}, \boldsymbol{\xi})$ equal to the orthogonal projector onto $\mathbb{R}\boldsymbol{\xi}$ and $\boldsymbol{\pi}_s = \mathbf{I} - \boldsymbol{\pi}_p$; see [161].

The following lemma holds.

LEMMA 1.1 (Properties of the Helmholtz decomposition operators). *Let the Lamé parameters (λ, μ) be constants satisfying (1.5). We have the orthogonality relations*

$$\mathcal{H}^s \mathcal{H}^p = \mathcal{H}^p \mathcal{H}^s = 0. \tag{1.16}$$

Moreover, $\mathcal{H}^s$ and $\mathcal{H}^p$ commute with $\mathcal{L}^{\lambda,\mu}$: For any smooth vector field $\boldsymbol{w}$ in Ω,

$$\mathcal{H}^\alpha[\mathcal{L}^{\lambda,\mu}\boldsymbol{w}] = \mathcal{L}^{\lambda,\mu}\mathcal{H}^\alpha[\boldsymbol{w}], \quad \alpha = p, s. \tag{1.17}$$

PROOF. We only prove (1.17). The orthogonality relations (1.16) are easy to see. Let $\mathcal{H}^s[\boldsymbol{w}] = \nabla \phi_{\boldsymbol{w}}$ and let $\mathcal{H}^p[\boldsymbol{w}] = \nabla \times \boldsymbol{\psi}_{\boldsymbol{w}}$. Then we have

$$\mathcal{L}^{\lambda,\mu}\boldsymbol{w} = (\lambda + 2\mu)\nabla\Delta\phi_{\boldsymbol{w}} + \mu\nabla \times \Delta\boldsymbol{\psi}_{\boldsymbol{w}},$$

and therefore,

$$\mathcal{H}^s[\mathcal{L}^{\lambda,\mu}\boldsymbol{w}] = (\lambda + 2\mu)\nabla\Delta\phi_{\boldsymbol{w}} = \mathcal{L}^{\lambda,\mu}\mathcal{H}^s[\boldsymbol{w}],$$

and

$$\mathcal{H}^p[\mathcal{L}^{\lambda,\mu}\boldsymbol{w}] = \mu\nabla \times \Delta\boldsymbol{\psi}_{\boldsymbol{w}} = \mathcal{L}^{\lambda,\mu}\mathcal{H}^p[\boldsymbol{w}]$$

as desired. □

It is worth emphasizing that in the exterior (unbounded) domain $\mathbb{R}^d \setminus \overline{\Omega}$ or in the free space $\mathbb{R}^d$, the Helmholtz decomposition (1.12) stays valid with $H^1(\Omega)$ replaced by $\{v \in L^2_{\text{loc}} : \nabla v \in L^2\}$; see, for instance, [93, 88].

In the time-harmonic regime, if the medium is infinite, then the irrotational and solenoidal fields solve two separate Helmholtz equations with different wavenumbers. As will be shown in the next section, radiation conditions should be imposed in order to select the physical solutions. The irrotational field is called compressional wave (p-wave) and the solenoidal field is called shear wave (s-wave). The displacement field associated with the p-wave is in the same direction as the wave propagates while the displacement field associated with the s-wave propagates orthogonally to the direction of propagation of the wave. Note that, in three dimensions, the s-wave has two directions of oscillations. Note also that if the medium is bounded, then the p- and s-waves are coupled by the boundary conditions at the boundary of the medium.

Let the wave numbers κ_s and κ_p be given by

$$\kappa_s = \frac{\omega}{c_s} \quad \text{and} \quad \kappa_p = \frac{\omega}{c_p}, \tag{1.18}$$

where c_s is the wave velocity for shear waves and c_p is the wave velocity for compressive waves:

$$c_s = \sqrt{\frac{\mu}{\rho}} \quad \text{and} \quad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{1.19}$$

The α-wave, $\alpha = p, s$, propagates with a wave number κ_α, through space and the corresponding wave velocity is given by c_α. Note that if $\lambda > 0$, then c_p is larger than c_s provided that (1.5) holds. This means that the p-wave arrives faster than the s-wave in the time domain.

Finally, it is worth mentioning that by antiplane elasticity equation we mean the conductivity equation $\nabla \cdot \mu \nabla u_3 = 0$, where u_3 is the x_3-component of the displacement field $\boldsymbol{u}$. When the elastic material is invariant under the transformation $x_3 \to -x_3$, the equations of linearized elasticity can be reduced to the antiplane elasticity equation.

1.3 RADIATION CONDITION

Let us formulate the *radiation condition* for the time-harmonic elastic waves when $\operatorname{Im} \omega \geq 0$ and $\omega \neq 0$.

Since $\mathcal{H}^s$ and $\mathcal{H}^p$ commute with $\mathcal{L}^{\lambda,\mu}$, as shown in Lemma 1.1, it follows from the Helmholtz decomposition (1.12) that any smooth solution $\boldsymbol{u}$ to the constant-coefficient equation $(\mathcal{L}^{\lambda,\mu} + \omega^2 \rho)\boldsymbol{u} = 0$ can be decomposed as follows [120, Theorem 2.5]:

$$\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_s, \tag{1.20}$$

where $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ satisfy the equations

$$\begin{cases} (\triangle + \kappa_p^2)\boldsymbol{u}_p = 0, & \nabla \times \boldsymbol{u}_p = 0, \\ (\triangle + \kappa_s^2)\boldsymbol{u}_s = 0, & \nabla \cdot \boldsymbol{u}_s = 0. \end{cases} \tag{1.21}$$

In fact, $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ are given by $\boldsymbol{u}_p = \mathcal{H}^p[\boldsymbol{u}]$ and $\boldsymbol{u}_s = \mathcal{H}^s[\boldsymbol{u}]$.

In order to select the physical solutions, we impose on $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ the radiation condition for solutions of the Helmholtz equation by requiring, as $r = |\boldsymbol{x}| \to +\infty$, that

$$\begin{cases} \partial_r \boldsymbol{u}_p(\boldsymbol{x}) - \mathrm{i}\kappa_p \boldsymbol{u}_p(\boldsymbol{x}) = O(r^{-(d+1)/2}), \\ \partial_r \boldsymbol{u}_s(\boldsymbol{x}) - \mathrm{i}\kappa_s \boldsymbol{u}_s(\boldsymbol{x}) = O(r^{-(d+1)/2}). \end{cases} \tag{1.22}$$

We say that $\boldsymbol{u}$ satisfies the Sommerfeld-Kupradze radiation condition if it can be decomposed in the form (1.20) with $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ satisfying (1.21) and (1.22).

We recall the following uniqueness results for the exterior problem [120].

LEMMA 1.2 (Uniqueness result). *Let $\boldsymbol{u}$ be a solution to $(\mathcal{L}^{\lambda,\mu} + \omega^2 \rho)\boldsymbol{u} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ satisfying the Sommerfeld-Kupradze radiation condition (1.22). If either $\boldsymbol{u} = 0$ or $\partial \boldsymbol{u}/\partial \boldsymbol{\nu} = 0$ on $\partial\Omega$, then $\boldsymbol{u}$ is identically zero in $\mathbb{R}^d \setminus \overline{\Omega}$.*

1.4 INTEGRAL REPRESENTATION OF SOLUTIONS TO THE LAMÉ SYSTEM

1.4.1 Fundamental Solutions

In dimension d, the Kupradze matrix $\mathbf{\Gamma}^\omega = (\Gamma^\omega_{ij})_{i,j=1}^d$ of the fundamental solution to the operator $\mathcal{L}^{\lambda,\mu} + \omega^2\rho$ satisfies

$$(\mathcal{L}^{\lambda,\mu} + \omega^2\rho)\mathbf{\Gamma}^\omega(\boldsymbol{x}-\boldsymbol{y}) = \delta_{\boldsymbol{y}}(\boldsymbol{x})\mathbf{I}, \qquad \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \neq \boldsymbol{y}, \tag{1.23}$$

where $\delta_{\boldsymbol{y}}$ is the Dirac mass at $\boldsymbol{y}$ and $\mathbf{I}$ is the $d \times d$ identity matrix. See [120, Chapter 2]. The function $\mathbf{\Gamma}^\omega$ can be decomposed into shear and pressure components [3]:

$$\mathbf{\Gamma}^\omega(\boldsymbol{x}) = \mathbf{\Gamma}^\omega_s(\boldsymbol{x}) + \mathbf{\Gamma}^\omega_p(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \boldsymbol{x} \neq 0, \tag{1.24}$$

where

$$\mathbf{\Gamma}^\omega_p(\boldsymbol{x}) = -\frac{1}{\mu\kappa_s^2}\mathbf{D}\Gamma^\omega_p(\boldsymbol{x}) \quad \text{and} \quad \mathbf{\Gamma}^\omega_s(\boldsymbol{x}) = \frac{1}{\mu\kappa_s^2}(\kappa_s^2\mathbf{I} + \mathbf{D})\Gamma^\omega_s(\boldsymbol{x}). \tag{1.25}$$

Here, the tensor $\mathbf{D}$ is defined by

$$\mathbf{D} = \nabla \otimes \nabla = (\partial^2_{ij})_{i,j=1}^d, \tag{1.26}$$

where the function Γ^ω_α is the fundamental solution to the Helmholtz operator, *i.e.*,

$$(\Delta + \kappa_\alpha^2)\Gamma^\omega_\alpha(\boldsymbol{x}) = \delta_{\mathbf{0}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \neq 0,$$

subject to the Sommerfeld radiation condition:

$$\partial_r\Gamma^\omega_\alpha(\boldsymbol{x}) - \mathrm{i}\kappa_\alpha\Gamma^\omega_\alpha(\boldsymbol{x}) = O(r^{-(d+1)/2}) \quad \text{as } r = |\boldsymbol{x}| \to +\infty.$$

Note that $\nabla \cdot \mathbf{\Gamma}^\omega_s = 0$ and $\nabla \times \mathbf{\Gamma}^\omega_p = 0$. Moreover, $\mathbf{\Gamma}^\omega$ satisfies the Sommerfeld-Kupradze radiation condition (1.22). See [2] and [120, Chapter 2]. Here, the vector field $\nabla \cdot \mathbf{\Gamma}^\omega_s$ and the matrix field $\nabla \times \mathbf{\Gamma}^\omega_p$ are defined by

$$\begin{cases} (\nabla \cdot \mathbf{\Gamma}^\omega_s)\boldsymbol{p} = \nabla \cdot (\mathbf{\Gamma}^\omega_s\boldsymbol{p}), \\ (\nabla \times \mathbf{\Gamma}^\omega_p)\boldsymbol{p} = \nabla \times (\mathbf{\Gamma}^\omega_p\boldsymbol{p}) \end{cases}$$

for all $\boldsymbol{p} \in \mathbb{R}^d$.

The function Γ^ω_α is given by

$$\Gamma^\omega_\alpha(\boldsymbol{x}) = \begin{cases} -\dfrac{\mathrm{i}}{4}H_0^{(1)}(\kappa_\alpha|\boldsymbol{x}|), & d = 2, \\ -\dfrac{e^{\mathrm{i}\kappa_\alpha|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}, & d = 3, \end{cases} \tag{1.27}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0. For definition and properties of the Hankel function we refer, for instance, to [125]. The only relevant fact we shall recall here is the following behavior of the Hankel function near 0:

$$-\frac{i}{4}H_0^{(1)}(\kappa_\alpha|\boldsymbol{x}|) = \frac{1}{2\pi}\ln(\kappa_\alpha|\boldsymbol{x}|) + \tau + \sum_{n=1}^{+\infty}(b_n \ln(\kappa_\alpha|\boldsymbol{x}|) + c_n)(\kappa_\alpha|\boldsymbol{x}|)^{2n}, \quad \alpha = p, s, \tag{1.28}$$

where

$$b_n = \frac{(-1)^n}{2\pi}\frac{1}{2^{2n}(n!)^2}, \quad c_n = -b_n\left(\gamma - \ln 2 - \frac{\pi \mathrm{i}}{2} - \sum_{j=1}^{n}\frac{1}{j}\right),$$

and the constant $\tau = (1/2\pi)(\gamma - \ln 2) - \mathrm{i}/4$, γ being the Euler constant. It is known (see, for example, [75, 125]) that, as $t \to +\infty$, we have

$$\begin{aligned} H_0^{(1)}(t) &= \sqrt{\frac{2}{\pi t}}e^{\mathrm{i}(t-\frac{\pi}{4})}\left[1 + O\left(\frac{1}{t}\right)\right], \\ \frac{d}{dt}H_0^{(1)}(t) &= \sqrt{\frac{2}{\pi t}}e^{\mathrm{i}(t+\frac{\pi}{4})}\left[1 + O\left(\frac{1}{t}\right)\right]. \end{aligned} \tag{1.29}$$

Using (1.29), one can see that in the two-dimensional case

$$\hat{\boldsymbol{x}} \cdot \nabla H_0^{(1)}(\kappa_\alpha|\boldsymbol{x}|) - \mathrm{i}\kappa_\alpha H_0^{(1)}(\kappa_\alpha|\boldsymbol{x}|) = O(|\boldsymbol{x}|^{-3/2}), \tag{1.30}$$

where $\hat{\boldsymbol{x}} := \boldsymbol{x}/|\boldsymbol{x}|$. This is exactly the two-dimensional Sommerfeld radiation condition one should impose in order to select the physical solution of the Helmholtz equation.

In the three-dimensional case, the Kupradze matrix $\boldsymbol{\Gamma}^\omega = (\Gamma_{ij}^\omega)_{i,j=1}^3$ is given by

$$\Gamma_{ij}^\omega(\boldsymbol{x}) = -\frac{\delta_{ij}}{4\pi\mu|\boldsymbol{x}|}e^{\mathrm{i}\kappa_s|\boldsymbol{x}|} + \frac{1}{4\pi\omega^2\rho}\partial_i\partial_j\frac{e^{\mathrm{i}\kappa_p|\boldsymbol{x}|} - e^{\mathrm{i}\kappa_s|\boldsymbol{x}|}}{|\boldsymbol{x}|}, \tag{1.31}$$

where κ_α, $\alpha = p, s$, is given by (1.18). One can easily show that Γ_{ij}^ω has the series representation:

$$\begin{aligned} \Gamma_{ij}^\omega(\boldsymbol{x}) = &-\frac{1}{4\pi}\sum_{n=0}^{+\infty}\frac{\mathrm{i}^n}{(n+2)n!}\left(\frac{n+1}{c_s^{n+2}} + \frac{1}{c_p^{n+2}}\right)\omega^n\delta_{ij}|\boldsymbol{x}|^{n-1} \\ &+ \frac{1}{4\pi\rho}\sum_{n=0}^{+\infty}\frac{\mathrm{i}^n(n-1)}{(n+2)n!}\left(\frac{1}{c_s^{n+2}} - \frac{1}{c_p^{n+2}}\right)\omega^n|\boldsymbol{x}|^{n-3}x_ix_j. \end{aligned} \tag{1.32}$$

If $\omega = 0$, then $\boldsymbol{\Gamma} := \boldsymbol{\Gamma}^0$ is the Kelvin matrix of the fundamental solution to the Lamé system; *i.e.*,

$$\Gamma_{ij}(\boldsymbol{x}) = -\frac{\gamma_1}{4\pi}\frac{\delta_{ij}}{|\boldsymbol{x}|} - \frac{\gamma_2}{4\pi}\frac{x_ix_j}{|\boldsymbol{x}|^3}, \tag{1.33}$$

where

$$\gamma_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad \gamma_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right). \tag{1.34}$$

In the two-dimensional case, the Kupradze matrix $\boldsymbol{\Gamma}^\omega = (\Gamma_{ij}^\omega)_{i,j=1}^2$ of the fundamental solution to the operator $\mathcal{L}^{\lambda,\mu} + \omega^2\rho$, $\omega \neq 0$, is given by

$$\Gamma_{ij}^{\omega}(\boldsymbol{x}) = -\frac{\mathrm{i}}{4\mu}\delta_{ij} H_0^{(1)}(\kappa_s|\boldsymbol{x}|) + \frac{\mathrm{i}}{4\omega^2\rho}\partial_i\partial_j \left(H_0^{(1)}(\kappa_p|\boldsymbol{x}|) - H_0^{(1)}(\kappa_s|\boldsymbol{x}|) \right). \quad (1.35)$$

For $\omega = 0$, we set $\boldsymbol{\Gamma}$ to be the Kelvin matrix of fundamental solutions to the Lamé system; *i.e.*,

$$\Gamma_{ij}(\boldsymbol{x}) = \frac{\gamma_1}{2\pi}\delta_{ij}\ln|\boldsymbol{x}| - \frac{\gamma_2}{2\pi}\frac{x_i x_j}{|\boldsymbol{x}|^2}. \quad (1.36)$$

1.4.2 Single- and Double-Layer Potentials

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with a connected Lipschitz boundary. The single- and double-layer potentials for the operator $\mathcal{L}^{\lambda,\mu} + \omega^2\rho$ are given by

$$\mathcal{S}_{\Omega}^{\omega}[\boldsymbol{\varphi}](\boldsymbol{x}) = \int_{\partial\Omega} \boldsymbol{\Gamma}^{\omega}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d, \quad (1.37)$$

$$\mathcal{D}_{\Omega}^{\omega}[\boldsymbol{\varphi}](\boldsymbol{x}) = \int_{\partial\Omega} \frac{\partial \boldsymbol{\Gamma}^{\omega}}{\partial \boldsymbol{\nu}(\boldsymbol{y})}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \partial\Omega, \quad (1.38)$$

for $\boldsymbol{\varphi} \in L^2(\partial\Omega)^d$, where $\partial/\partial\boldsymbol{\nu}$ denotes the conormal derivative defined in (1.7). Thus, for $i = 1, \ldots, d$,

$$(\mathcal{D}_{\Omega}^{\omega}[\boldsymbol{\varphi}](\boldsymbol{x}))_i = \int_{\partial\Omega} \lambda \frac{\partial \Gamma_{ij}^{\omega}}{\partial y_j}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y}) + \mu\Big(\frac{\partial \Gamma_{ij}^{\omega}}{\partial y_l} + \frac{\partial \Gamma_{il}^{\omega}}{\partial y_j}\Big)(\boldsymbol{x}-\boldsymbol{y}) n_j(\boldsymbol{y})\varphi_l(\boldsymbol{y})\, d\sigma(\boldsymbol{y}).$$

The following formulas give the jump relations satisfied by the conormal derivative of the single-layer potential and by the double-layer potential:

$$\frac{\partial(\mathcal{S}_{\Omega}^{\omega}[\boldsymbol{\varphi}])}{\partial\boldsymbol{\nu}}\Big|_{\pm}(\boldsymbol{x}) = \Big(\pm\frac{1}{2}\mathbf{I} + (\mathcal{K}_{\Omega}^{\omega})^*\Big)[\boldsymbol{\varphi}](\boldsymbol{x}) \ \text{ a.e. } \boldsymbol{x} \in \partial\Omega, \quad (1.39)$$

$$(\mathcal{D}_{\Omega}^{\omega}[\boldsymbol{\varphi}])\Big|_{\pm}(\boldsymbol{x}) = \Big(\mp\frac{1}{2}\mathbf{I} + \mathcal{K}_{\Omega}^{\omega}\Big)[\boldsymbol{\varphi}](\boldsymbol{x}) \ \text{ a.e. } \boldsymbol{x} \in \partial\Omega, \quad (1.40)$$

where $\mathcal{K}_{\Omega}^{\omega}$ is the operator defined by

$$\mathcal{K}_{\Omega}^{\omega}[\boldsymbol{\varphi}](\boldsymbol{x}) = \text{p.v.}\int_{\partial\Omega} \frac{\partial \boldsymbol{\Gamma}^{\omega}}{\partial \boldsymbol{\nu}(\boldsymbol{y})}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \quad (1.41)$$

and $(\mathcal{K}_{\Omega}^{\omega})^*$ is the L^2-adjoint of $\mathcal{K}_{\Omega}^{-\omega}$; that is,

$$(\mathcal{K}_{\Omega}^{\omega})^*[\boldsymbol{\varphi}](\boldsymbol{x}) = \text{p.v.}\int_{\partial\Omega} \frac{\partial \boldsymbol{\Gamma}^{\omega}}{\partial \boldsymbol{\nu}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}).$$

See [77, 120]. Here, p.v. stands for the Cauchy principal value and the subscripts + and − indicate the limits from outside and inside Ω, respectively. The operators $(\mathcal{K}_{\Omega}^{\omega})^*$ and $\mathcal{K}_{\Omega}^{\omega}$ are called the Neumann-Poincaré operators.

By a straightforward calculation, one can see that the single- and double-layer potentials, $\mathcal{S}_{\Omega}^{\omega}[\boldsymbol{\varphi}]$ and $\mathcal{D}_{\Omega}^{\omega}[\boldsymbol{\varphi}]$ for $\boldsymbol{\varphi} \in L^2(\partial\Omega)^d$, satisfy the time-harmonic elasticity equation in Ω and $\mathbb{R}^d \setminus \overline{\Omega}$ together with the Sommerfeld-Kupradze radiation condition (1.22). We refer to [1, 120] for details.

Let $\mathcal{S}_\Omega$, $\mathcal{D}_\Omega$, $(\mathcal{K}_\Omega)^*$, and $\mathcal{K}_\Omega$ be the layer potentials for the operator $\mathcal{L}^{\lambda,\mu}$. Analogously to (1.39) and (1.40), the following formulas give the jump relations obeyed by $\mathcal{D}_\Omega[\boldsymbol{\varphi}]$ and by $\partial(\mathcal{S}_\Omega[\boldsymbol{\varphi}])/\partial\boldsymbol{\nu}$ on general Lipschitz domains for $\boldsymbol{\varphi} \in L^2(\partial\Omega)^d$:

$$\frac{\partial(\mathcal{S}_\Omega[\boldsymbol{\varphi}])}{\partial\boldsymbol{\nu}}\Big|_\pm(\boldsymbol{x}) = \Big(\pm\frac{1}{2}\mathbf{I} + (\mathcal{K}_\Omega)^*\Big)[\boldsymbol{\varphi}](\boldsymbol{x}) \quad \text{a.e. } \boldsymbol{x} \in \partial\Omega, \tag{1.42}$$

$$(\mathcal{D}_\Omega[\boldsymbol{\varphi}])\Big|_\pm(\boldsymbol{x}) = \Big(\mp\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega\Big)[\boldsymbol{\varphi}](\boldsymbol{x}) \quad \text{a.e. } \boldsymbol{x} \in \partial\Omega. \tag{1.43}$$

Again, the layer potentials $\mathcal{S}_\Omega[\boldsymbol{\varphi}]$, $\mathcal{D}_\Omega[\boldsymbol{\varphi}]$ for $\boldsymbol{\varphi} \in L^2(\partial\Omega)^d$ satisfy

$$\mathcal{L}^{\lambda,\mu}\mathcal{S}_\Omega[\boldsymbol{\varphi}] = \mathcal{L}^{\lambda,\mu}\mathcal{D}_\Omega[\boldsymbol{\varphi}] = 0 \quad \text{in } \Omega \cup (\mathbb{R}^d \setminus \overline{\Omega}).$$

We emphasize that the singular integral operators $\mathcal{K}_\Omega$ and $\mathcal{K}_\Omega^\omega$ are not compact, even on smooth domains. This causes some difficulties in solving the elasticity system using layer potential techniques.

Let Ψ be the vector space of all linear solutions to the equation $\mathcal{L}^{\lambda,\mu}\boldsymbol{u} = 0$ satisfying $\partial\boldsymbol{u}/\partial\boldsymbol{\nu} = 0$ on $\partial\Omega$, or, equivalently,

$$\begin{aligned} \Psi &= \Big\{\boldsymbol{\psi} \in H^1(\Omega)^d : \partial_i\psi_j + \partial_j\psi_i = 0,\ 1 \le i, j \le d\Big\}, \\ &= \Big\{\boldsymbol{a} + \mathbf{B}\boldsymbol{x}, \boldsymbol{a} \in \mathbb{R}^d, \mathbf{B} \in M_d^A\Big\}, \end{aligned} \tag{1.44}$$

where M_d^A is the space of antisymmetric matrices. One has

$$\dim \Psi = d(d+1)/2.$$

Define a subspace of $L^2(\partial\Omega)^d$ by

$$L^2_\Psi(\partial\Omega) = \Big\{\boldsymbol{f} \in L^2(\partial\Omega)^d : \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{\psi}\, d\sigma = 0\ \forall\ \boldsymbol{\psi} \in \Psi\Big\}. \tag{1.45}$$

In particular, since Ψ contains constant functions, we get

$$\int_{\partial\Omega} \boldsymbol{f}\, d\sigma = 0$$

for any $\boldsymbol{f} \in L^2_\Psi(\partial\Omega)$.

Define

$$H_\Psi^{-1/2}(\partial\Omega) := \Big\{\ \boldsymbol{\varphi} \in H^{-1/2}(\partial\Omega)^d : \langle\boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{1/2,-1/2} = 0\ \forall\ \boldsymbol{\psi} \in \Psi\Big\}. \tag{1.46}$$

Then the following result holds.

Lemma 1.3 (Mapping properties of $\mathcal{K}_\Omega^*$). *The operator $\pm\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega^*$ is invertible on $H_\Psi^{-1/2}(\partial\Omega)$. Moreover, there exists a positive constant C such that*

$$\|\mathcal{S}_\Omega[\boldsymbol{\varphi}]\|_{W(\mathbb{R}^d)} \le C\|\boldsymbol{\varphi}\|_{H^{-1/2}(\partial\Omega)} \tag{1.47}$$

for all $\boldsymbol{\varphi} \in H^{-1/2}(\partial\Omega)^d$. *Furthermore, the null space of* $-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega$ *on* $H^{-1/2}(\partial\Omega)$ *is* Ψ.

The following invertibility results will be also needed.

LEMMA 1.4 (Mapping properties of $(\mathcal{K}_\Omega^\omega)^*$). *The operator* $\frac{1}{2}\mathbf{I}+(\mathcal{K}_\Omega^\omega)^*$ *is invertible on* $H^{-1/2}(\partial\Omega)^d$. *If* ω^2 *is not a Dirichlet eigenvalue for* $-\mathcal{L}^{\lambda,\mu}$ *on* Ω, *then* $-\frac{1}{2}\mathbf{I} + (\mathcal{K}_\Omega^\omega)^*$ *is invertible on* $H^{-1/2}(\partial\Omega)^d$ *as well.*

Next, we recall Green's formulas for the Lamé system, which can be obtained by integration by parts. The first formula is

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} \, d\sigma = \int_\Omega \boldsymbol{u} \cdot \mathcal{L}^{\lambda,\mu} \boldsymbol{v} \, d\boldsymbol{x} + Q(\boldsymbol{u}, \boldsymbol{v}), \tag{1.48}$$

where $\boldsymbol{u} \in H^1(\Omega)^d$, $\boldsymbol{v} \in H^{3/2}(\Omega)^d$, and

$$Q(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \Big(\lambda(\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v}) + 2\mu \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{v} \Big) \, d\boldsymbol{x}. \tag{1.49}$$

Here and throughout this book $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^d a_{ij} b_{ij}$ for matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$.

The strong convexity condition (1.5) shows that the quadratic form

$$\boldsymbol{u} \mapsto Q(\boldsymbol{u}, \boldsymbol{u})$$

is positive definite. Note that $H^1(\Omega)^d$ is the closure of this quadratic form since $\boldsymbol{u} \mapsto \nabla^s \boldsymbol{u}$ is elliptic of order 1.

Formula (1.48) yields Green's second formula

$$\int_{\partial\Omega} \Big(\boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} - \boldsymbol{v} \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} \Big) \, d\sigma(\boldsymbol{x}) = \int_\Omega \big(\boldsymbol{u} \cdot \mathcal{L}^{\lambda,\mu} \boldsymbol{v} - \boldsymbol{v} \cdot \mathcal{L}^{\lambda,\mu} \boldsymbol{u} \big) \, d\boldsymbol{x} \tag{1.50}$$

for $\boldsymbol{u}, \ \boldsymbol{v} \in H^{3/2}(\Omega)^d$.

Formula (1.50) shows that if $\boldsymbol{u} \in H^{3/2}(\Omega)^d$ satisfies $\mathcal{L}^{\lambda,\mu}\boldsymbol{u} = 0$ in Ω, then $\partial \boldsymbol{u}/\partial \boldsymbol{\nu}|_{\partial\Omega} \in L^2_\Psi(\partial\Omega)$.

The following formulation of Korn's inequality will be of interest to us. See [144] and [73, Theorem 6.3.4].

LEMMA 1.5 (Korn's inequality). *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$. *Let* $\boldsymbol{u} \in H^1(\Omega)^d$ *satisfy*

$$\int_\Omega \Big(\boldsymbol{u} \cdot \boldsymbol{\psi} + \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi} \Big) \, d\boldsymbol{x} = 0 \quad \textit{for all } \boldsymbol{\psi} \in \Psi. \tag{1.51}$$

Then there is a constant C *depending only on the Lipschitz character of* Ω *such that*

$$\int_\Omega \Big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2 \Big) \, d\boldsymbol{x} \le C \int_D |\nabla^s \boldsymbol{u}|^2 \, d\boldsymbol{x}. \tag{1.52}$$

Here, $|\nabla \boldsymbol{u}|^2 = \nabla \boldsymbol{u} : \nabla \boldsymbol{u}$ *and* $|\nabla^s \boldsymbol{u}|^2 = \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{u}$.

Finally, we prove using Green's formulas that $-\mathcal{S}_\Omega$ is positive.

LEMMA 1.6. *The operator* $-\mathcal{S}_\Omega : L^2(\partial\Omega)^d \to L^2(\partial\Omega)^d$ *is positive and self-adjoint.*

PROOF. It is clear that $\mathcal{S}_\Omega$ is self-adjoint. Let $\boldsymbol{\varphi} \in L^2(\partial\Omega)^d$. Since

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathcal{S}_\Omega[\boldsymbol{\varphi}] = 0 \quad \text{in } \Omega \cup (\mathbb{R}^d \setminus \overline{\Omega}), \\ \mathcal{S}_\Omega[\boldsymbol{\varphi}](\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d}) \quad \text{as } |\boldsymbol{x}| \to +\infty, \end{cases}$$

then we have

$$-\int_{\partial\Omega} \frac{\partial}{\partial\boldsymbol{\nu}}\mathcal{S}_\Omega[\boldsymbol{\varphi}]\Big|_+ \cdot \mathcal{S}_\Omega[\boldsymbol{\varphi}]\, d\sigma = \int_{\mathbb{R}^d\setminus\overline{\Omega}} \Big(2\mu|\nabla^s\mathcal{S}_\Omega[\boldsymbol{\varphi}]|^2 + \lambda|\nabla\cdot\mathcal{S}_\Omega[\boldsymbol{\varphi}]|^2\Big)\, d\boldsymbol{x},$$

and

$$\int_{\partial\Omega} \frac{\partial}{\partial\boldsymbol{\nu}}\mathcal{S}_\Omega[\boldsymbol{\varphi}]\Big|_- \cdot \mathcal{S}_\Omega[\boldsymbol{\varphi}]\, d\sigma = \int_{\Omega} \Big(2\mu|\nabla^s\mathcal{S}_\Omega[\boldsymbol{\varphi}]|^2 + \lambda|\nabla\cdot\mathcal{S}_\Omega[\boldsymbol{\varphi}]|^2\Big)\, d\boldsymbol{x}.$$

Summing up the above two identities we find by using the jump relation

$$\frac{\partial}{\partial\boldsymbol{\nu}}\mathcal{S}_\Omega[\boldsymbol{\varphi}]\Big|_+ - \frac{\partial}{\partial\boldsymbol{\nu}}\mathcal{S}_\Omega[\boldsymbol{\varphi}]\Big|_- = \boldsymbol{\varphi},$$

that

$$-\int_{\partial\Omega} \mathcal{S}_\Omega[\boldsymbol{\varphi}]\cdot\boldsymbol{\varphi}\, d\sigma = \int_{\mathbb{R}^d} \Big(2\mu|\nabla^s\mathcal{S}_\Omega[\boldsymbol{\varphi}]|^2 + \lambda|\nabla\cdot\mathcal{S}_\Omega[\boldsymbol{\varphi}]|^2\Big)\, d\boldsymbol{x}.$$

Thus $-\mathcal{S}_\Omega \geq 0$. Moreover, if $\mathcal{S}_\Omega[\boldsymbol{\varphi}] = 0$ on $\partial\Omega$, then by the uniqueness of a solution to both the interior and exterior Dirichlet boundary problems for $\mathcal{L}^{\lambda,\mu}$ it follows that $\mathcal{S}_\Omega[\boldsymbol{\varphi}] = 0$ in $\Omega\cup(\mathbb{R}^d\setminus\overline{\Omega})$ and therefore, by the jump relation (1.42), $\boldsymbol{\varphi} = 0$. □

1.4.3 Transmission Problem

In this subsection we consider a Lipschitz bounded inclusion D with Lamé parameters $\widetilde{\lambda}, \widetilde{\mu}$ different from those λ and μ of the background medium. We assume that the pair of Lamé parameters $\widetilde{\lambda}, \widetilde{\mu}$ satisfy the strong convexity condition (1.5) and is such that

$$(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0, \quad (\lambda - \widetilde{\lambda})^2 + (\mu - \widetilde{\mu})^2 \neq 0. \tag{1.53}$$

Let $\widetilde{\mathcal{S}}_D^\omega$ denote the single-layer potential defined by (1.37) with λ, μ replaced by $\widetilde{\lambda}, \widetilde{\mu}$. We also denote by $\partial\boldsymbol{u}/\partial\widetilde{\boldsymbol{\nu}}$ the conormal derivative associated with $\widetilde{\lambda}, \widetilde{\mu}$. We now have the following solvability result which can be viewed as a compact perturbation result of the case $\omega = 0$.

THEOREM 1.7. *Let D be a Lipschitz bounded domain in $\mathbb{R}^d$. Suppose that $(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0$ and $0 < \widetilde{\lambda}, \widetilde{\mu} < +\infty$. Suppose that $\Im m\, \omega \geq 0$ and $\omega^2\rho$ is not a Dirichlet eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ on D. For any given $(\boldsymbol{F}, \boldsymbol{G}) \in H^1(\partial D)^d \times L^2(\partial D)^d$, there exists a unique pair $(\boldsymbol{f}, \boldsymbol{g}) \in L^2(\partial D)^d \times L^2(\partial D)^d$ such that*

$$\begin{cases} \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{f}]|_- - \mathcal{S}_D^\omega[\boldsymbol{g}]|_+ = \boldsymbol{F}, \\ \dfrac{\partial}{\partial\widetilde{\boldsymbol{\nu}}}\widetilde{\mathcal{S}}_D^\omega[\boldsymbol{f}]\Big|_- - \dfrac{\partial}{\partial\boldsymbol{\nu}}\mathcal{S}_D^\omega[\boldsymbol{g}]\Big|_+ = \boldsymbol{G}. \end{cases}$$

A positive constant C exists such that

$$||\boldsymbol{f}||_{L^2(\partial D)^d} + ||\boldsymbol{g}||_{L^2(\partial D)^d} \leq C\Big(||\boldsymbol{F}||_{H^1(\partial D)^d} + ||\boldsymbol{G}||_{L^2(\partial D)^d}\Big). \tag{1.54}$$

Moreover, if $\omega = 0$ and $\boldsymbol{G} \in L^2_\Psi(\partial D)$, then $\boldsymbol{g} \in L^2_\Psi(\partial D)$.

PROOF. For $\omega = 0$, the theorem is proved in [84]. Here, we only consider the case $\omega \neq 0$, which can be treated as a compact perturbation of the case $\omega = 0$. In fact, let us define the operators

$$T, T_0 : L^2(\partial D)^d \times L^2(\partial D)^d \to H^1(\partial D)^d \times L^2(\partial D)^d$$

by

$$T(\boldsymbol{f}, \boldsymbol{g}) := \left(\widetilde{\mathcal{S}}_D^\omega[\boldsymbol{f}]|_- - \mathcal{S}_D^\omega[\boldsymbol{g}]|_+, \ \frac{\partial}{\partial \widetilde{\boldsymbol{\nu}}}\widetilde{\mathcal{S}}_D^\omega[\boldsymbol{f}]\Big|_- - \frac{\partial}{\partial \boldsymbol{\nu}}\mathcal{S}_D^\omega[\boldsymbol{g}]\Big|_+\right)$$

and

$$T_0(\boldsymbol{f}, \boldsymbol{g}) := \left(\widetilde{\mathcal{S}}_D[\boldsymbol{f}]|_- - \mathcal{S}_D[\boldsymbol{g}]|_+, \ \frac{\partial}{\partial \widetilde{\boldsymbol{\nu}}}\widetilde{\mathcal{S}}_D[\boldsymbol{f}]\Big|_- - \frac{\partial}{\partial \boldsymbol{\nu}}\mathcal{S}_D[\boldsymbol{g}]\Big|_+\right).$$

It is easily checked that $T - T_0$ is a compact operator. Since we know that T_0 is invertible, by the Fredholm alternative, it is enough to show that T is injective. Suppose that $T(\boldsymbol{f}, \boldsymbol{g}) = 0$. Then the function $\boldsymbol{u}$ given by

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{cases} \mathcal{S}_D^\omega[\boldsymbol{g}](\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{D}, \\ \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{f}](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases}$$

is a solution to the transmission problem

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{u} + \omega^2\boldsymbol{u} = 0 & \text{in } \ \mathbb{R}^d \setminus \overline{D}, \\ \mathcal{L}^{\widetilde{\lambda},\widetilde{\mu}}\boldsymbol{u} + \omega^2\boldsymbol{u} = 0 & \text{in } \ D, \\ \boldsymbol{u}\big|_+ - \boldsymbol{u}\big|_- = 0 & \text{on } \ \partial D, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\Big|_+ - \dfrac{\partial \boldsymbol{u}}{\partial \widetilde{\boldsymbol{\nu}}}\Big|_- = 0 & \text{on } \ \partial D, \end{cases}$$

satisfying the radiation condition. By the uniqueness of a solution to this transmission problem (see, for instance, [120, Chapter 3]), we have $\boldsymbol{u} = 0$. From the assumption on ω, we conclude by using Lemma 1.4 that $\boldsymbol{f} = \boldsymbol{g} = 0$. This completes the proof. □

Later in this book, we will consider the following transmission problem:

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{u} + \omega^2\rho\boldsymbol{u} = 0 & \text{in } \Omega\setminus\overline{D}, \\ \mathcal{L}^{\widetilde{\lambda},\widetilde{\mu}}\boldsymbol{u} + \omega^2\rho\boldsymbol{u} = 0 & \text{in } D, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{g} & \text{on } \partial\Omega, \\ \boldsymbol{u}\big|_+ - \boldsymbol{u}\big|_- = 0 & \text{on } \partial D, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\Big|_+ - \dfrac{\partial \boldsymbol{u}}{\partial \widetilde{\boldsymbol{\nu}}}\Big|_- = 0 & \text{on } \partial D, \end{cases} \tag{1.55}$$

where D and Ω are Lipschitz bounded domains in $\mathbb{R}^d$ with $\overline{D} \subset \Omega$. Note that the p- and s-waves cannot be decoupled because of the boundary and transmission conditions.

For problem (1.55) the following representation formula holds.

THEOREM 1.8 (Representation formula). *Let $\Im m\, \omega \geq 0$. Suppose that $\omega^2\rho$ is not a Dirichlet eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ on D. Let $\boldsymbol{u}$ be a solution of (1.55) and $\boldsymbol{f} := \boldsymbol{u}|_{\partial\Omega}$. Define*

$$\boldsymbol{H}(\boldsymbol{x}) := \mathcal{D}_\Omega^\omega[\boldsymbol{f}](\boldsymbol{x}) - \mathcal{S}_\Omega^\omega[\boldsymbol{g}](\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \partial\Omega. \tag{1.56}$$

Then $\boldsymbol{u}$ can be represented as

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{cases} \boldsymbol{H}(\boldsymbol{x}) + \mathcal{S}_D^\omega[\boldsymbol{\psi}](\boldsymbol{x}), & \boldsymbol{x} \in \Omega\setminus\overline{D}, \\ \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases} \tag{1.57}$$

where the pair $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in L^2(\partial D)^d \times L^2(\partial D)^d$ is the unique solution of

$$\begin{cases} \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}] - \mathcal{S}_D^\omega[\boldsymbol{\psi}] = \boldsymbol{H}|_{\partial D}, \\ \dfrac{\partial}{\partial \widetilde{\boldsymbol{\nu}}}\widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}] - \dfrac{\partial}{\partial \boldsymbol{\nu}}\mathcal{S}_D^\omega[\boldsymbol{\psi}] = \dfrac{\partial \boldsymbol{H}}{\partial \boldsymbol{\nu}}\Big|_{\partial D}. \end{cases} \tag{1.58}$$

Moreover, we have

$$\boldsymbol{H}(\boldsymbol{x}) + \mathcal{S}_D^\omega[\boldsymbol{\psi}](\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}. \tag{1.59}$$

PROOF. We consider the following two-phase transmission problem:

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{v} + \omega^2\rho\boldsymbol{v} = 0 & \text{in } (\Omega\setminus\overline{D})\bigcup(\mathbb{R}^d\setminus\overline{\Omega}), \\ \mathcal{L}^{\widetilde{\lambda},\widetilde{\mu}}\boldsymbol{v} + \omega^2\rho\boldsymbol{v} = 0 & \text{in } D, \\ \boldsymbol{v}\big|_- - \boldsymbol{v}\big|_+ = \boldsymbol{f},\ \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}\Big|_- - \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}\Big|_+ = \boldsymbol{g} & \text{on } \partial\Omega, \\ \boldsymbol{v}\big|_- - \boldsymbol{v}\big|_+ = 0,\ \dfrac{\partial \boldsymbol{v}}{\partial \widetilde{\boldsymbol{\nu}}}\Big|_- - \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}\Big|_+ = 0 & \text{on } \partial D, \end{cases} \tag{1.60}$$

with the radiation condition. This problem has a unique solution. See [120, Chapter

3]. It is easily checked that both $\boldsymbol{v}$ and $\widetilde{\boldsymbol{v}}$ defined by

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{cases} \boldsymbol{u}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ 0, & \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}, \end{cases}$$

and

$$\widetilde{\boldsymbol{v}}(\boldsymbol{x}) = \begin{cases} \boldsymbol{H}(\boldsymbol{x}) + \mathcal{S}_D^\omega[\boldsymbol{\psi}](\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^d \setminus \left(\overline{D} \cup \partial\Omega\right), \\ \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases}$$

are solutions to (1.60). Hence $\boldsymbol{v} = \widetilde{\boldsymbol{v}}$, which concludes the proof of the theorem. □

We now consider the static case. For a given $\boldsymbol{g} \in L^2_\Psi(\partial D)$, let $\boldsymbol{u}$ be the solution of the transmission problem

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{u} = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}^{\widetilde{\lambda},\widetilde{\mu}}\boldsymbol{u} = 0 \quad \text{in } D, \\ \boldsymbol{u}\big|_- = \boldsymbol{u}\big|_+ \quad \text{on } \partial D, \\ \dfrac{\partial \boldsymbol{u}}{\partial \widetilde{\boldsymbol{\nu}}}\bigg|_- = \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\bigg|_+ \quad \text{on } \partial D, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\bigg|_{\partial\Omega} = \boldsymbol{g}, \\ \boldsymbol{u}\big|_{\partial\Omega} \in L^2_\Psi(\partial\Omega). \end{cases} \tag{1.61}$$

The following representation theorem for the solution of the transmission problem (1.61) holds [28].

Theorem 1.9. *Let $\boldsymbol{H}$ be defined by*

$$\boldsymbol{H}(\boldsymbol{x}) = \mathcal{D}_\Omega[\boldsymbol{u}|_{\partial\Omega}](\boldsymbol{x}) - \mathcal{S}_\Omega[\boldsymbol{g}](\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \partial\Omega. \tag{1.62}$$

Then the solution $\boldsymbol{u}$ of (1.61) can represented by

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{cases} \boldsymbol{H}(\boldsymbol{x}) + \mathcal{S}_D[\boldsymbol{\psi}](\boldsymbol{x}), & \boldsymbol{x} \in \Omega \setminus \overline{D}, \\ \widetilde{\mathcal{S}}_D[\boldsymbol{\varphi}](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases} \tag{1.63}$$

where $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ is the unique solution in $L^2(\partial D)^d \times L^2_\Psi(\partial D)$ of

$$\begin{cases} \widetilde{\mathcal{S}}_D[\boldsymbol{\varphi}]\big|_- - \mathcal{S}_D[\boldsymbol{\psi}]\big|_+ = \boldsymbol{H}|_{\partial D} \quad \text{on } \partial D, \\ \dfrac{\partial}{\partial \widetilde{\boldsymbol{\nu}}}\widetilde{\mathcal{S}}_D[\boldsymbol{\varphi}]\bigg|_- - \dfrac{\partial}{\partial \boldsymbol{\nu}}\mathcal{S}_D[\boldsymbol{\psi}]\bigg|_+ = \dfrac{\partial \boldsymbol{H}}{\partial \boldsymbol{\nu}}\bigg|_{\partial D} \quad \text{on } \partial D. \end{cases} \tag{1.64}$$

There exists a positive constant C such that

$$\|\boldsymbol{\varphi}\|_{L^2(\partial D)^d} + \|\boldsymbol{\psi}\|_{L^2(\partial D)^d} \leq C\|\boldsymbol{H}\|_{H^1(\partial D)}. \tag{1.65}$$

For any integer n, there exists a positive constant C_n depending only on $c_0 :=$

dist$(D, \partial\Omega)$ *and* λ, μ *(not on* $\widetilde{\lambda}, \widetilde{\mu}$*) such that*

$$\|\boldsymbol{H}\|_{\mathcal{C}^n(\overline{D})} \leq C_n \|\boldsymbol{g}\|_{L^2(\partial\Omega)}. \tag{1.66}$$

Moreover,

$$\boldsymbol{H}(\boldsymbol{x}) + \mathcal{S}_D[\boldsymbol{\psi}](\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}. \tag{1.67}$$

We will also need the following lemma.

LEMMA 1.10. *Let* $\boldsymbol{\varphi} \in \Psi$. *Let* D *be a bounded Lipschitz domain. If the pair* $(\boldsymbol{f}, \boldsymbol{g}) \in L^2(\partial D) \times L^2_\Psi(\partial D)$ *is the solution of*

$$\begin{cases} \widetilde{\mathcal{S}}_D[\boldsymbol{f}]\big|_- - \mathcal{S}_D[\boldsymbol{g}]\big|_+ = \boldsymbol{\varphi}|_{\partial D}, \\ \dfrac{\partial}{\partial \widetilde{\boldsymbol{\nu}}} \widetilde{\mathcal{S}}_D[\boldsymbol{f}]\Big|_- - \dfrac{\partial}{\partial \boldsymbol{\nu}} \mathcal{S}_D[\boldsymbol{g}]\Big|_+ = 0, \end{cases} \tag{1.68}$$

then $\boldsymbol{g} = 0$.

PROOF. Define $\boldsymbol{u}$ by

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{cases} \mathcal{S}_D[\boldsymbol{g}](\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{D}, \\ \widetilde{\mathcal{S}}_D[\boldsymbol{f}](\boldsymbol{x}) - \boldsymbol{\varphi}(\boldsymbol{x}), & \boldsymbol{x} \in D. \end{cases}$$

Since $\boldsymbol{g} \in L^2_\Psi(\partial D)$, then $\displaystyle\int_{\partial D} \boldsymbol{g}\, d\sigma = 0$, and hence

$$\mathcal{S}_D[\boldsymbol{g}](\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d}) \quad \text{as } |\boldsymbol{x}| \to +\infty.$$

Therefore, $\boldsymbol{u}$ is the unique solution of

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{D}, \\ \mathcal{L}^{\widetilde{\lambda},\widetilde{\mu}}\boldsymbol{u} = 0 \quad \text{in } D, \\ \boldsymbol{u}|_+ = \boldsymbol{u}|_- \quad \text{on } \partial D, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\Big|_+ = \dfrac{\partial \boldsymbol{u}}{\partial \widetilde{\boldsymbol{\nu}}}\Big|_- \quad \text{on } \partial D, \\ \boldsymbol{u}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d}) \quad \text{as } |x| \to +\infty. \end{cases} \tag{1.69}$$

Using the fact that the trivial solution $\boldsymbol{u} = 0$ is the unique solution to (1.69), we see that

$$\mathcal{S}_D[\boldsymbol{g}](\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{D}.$$

It then follows that $\mathcal{L}^{\lambda,\mu}\mathcal{S}_D[\boldsymbol{g}](\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in D$ and $\mathcal{S}_D[\boldsymbol{g}](\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial D$. Thus, $\mathcal{S}_D[\boldsymbol{g}](\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in D$. Since by (1.42)

$$\boldsymbol{g} = \frac{\partial(\mathcal{S}_D[\boldsymbol{g}])}{\partial \boldsymbol{\nu}}\bigg|_+ - \frac{\partial(\mathcal{S}_D[\boldsymbol{g}])}{\partial \boldsymbol{\nu}}\bigg|_- \quad \text{on } \partial D,$$

we have that $\boldsymbol{g} = 0$. □

1.5 HELMHOLTZ-KIRCHHOFF IDENTITIES

We now discuss the reciprocity property and derive the Helmholtz-Kirchhoff identities for elastic media. Some of the results presented in this section can be found in [171, 172] in the context of elastodynamic seismic interferometry. Indeed, the elastodynamic reciprocity theorems (Propositions 1.11 and 1.15) will be the key ingredients to understand the relation between the cross correlations of signals emitted by uncorrelated noise sources and the Green function between the observation points.

Note first that the conormal derivative tensor $\partial \mathbf{\Gamma}^\omega / \partial \boldsymbol{\nu}$ means that for all constant vectors $\boldsymbol{p}$,

$$\left[\frac{\partial \mathbf{\Gamma}^\omega}{\partial \boldsymbol{\nu}}\right] \boldsymbol{p} := \frac{\partial \left[\mathbf{\Gamma}^\omega \boldsymbol{p}\right]}{\partial \boldsymbol{\nu}}.$$

From now on, we set $\mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y}) := \mathbf{\Gamma}^\omega(\boldsymbol{x} - \boldsymbol{y})$ for $\boldsymbol{x} \neq \boldsymbol{y}$.

1.5.1 Reciprocity Property and Helmholtz-Kirchhoff Identities

An important property satisfied by the fundamental solution $\mathbf{\Gamma}^\omega$ is the reciprocity property. If the medium is not homogeneous, then the following holds:

$$\mathbf{\Gamma}^\omega(\boldsymbol{y}, \boldsymbol{x}) = \left[\mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y})\right]^t, \quad \boldsymbol{x} \neq \boldsymbol{y}. \tag{1.70}$$

If the medium is homogeneous, then one can see from (1.31) and (1.35) that $\mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y})$ is symmetric and

$$\mathbf{\Gamma}^\omega(\boldsymbol{y}, \boldsymbol{x}) = \mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x} \neq \boldsymbol{y}. \tag{1.71}$$

Identity (1.70) states that the nth component of the displacement at $\boldsymbol{x}$ due to a point source excitation at $\boldsymbol{y}$ in the mth direction is identical to the mth component of the displacement at $\boldsymbol{y}$ due to a point source excitation at $\boldsymbol{x}$ in the nth direction.

The following result is from [172, Eq. (73)]. It is the first building block of the resolution analysis in Chapters 5 and 6. Moreover, elements of the proof are used in Proposition 1.12.

PROPOSITION 1.11. *Let Ω be a bounded Lipschitz domain. For all $\boldsymbol{x}, \boldsymbol{z} \in \Omega$, we have*

$$\int_{\partial\Omega} \left[\frac{\partial \mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \overline{\mathbf{\Gamma}}_\omega(\boldsymbol{y}, \boldsymbol{z}) - \mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial \overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{\nu}(\boldsymbol{y})}\right] d\sigma(\boldsymbol{y}) = -2\mathrm{i}\Im m \left\{\mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{z})\right\}. \tag{1.72}$$

PROOF. Our goal is to show that for all real constant vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, we have

$$\begin{aligned}\int_{\partial\Omega} \left[\boldsymbol{q} \cdot \frac{\partial \mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y}, \boldsymbol{z}) \boldsymbol{p} - \boldsymbol{q} \cdot \mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial \overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \boldsymbol{p}\right] d\sigma(\boldsymbol{y}) \\ = -2\mathrm{i}\boldsymbol{q} \cdot \Im m\{\mathbf{\Gamma}^\omega(\boldsymbol{x}, \boldsymbol{z})\}\boldsymbol{p}.\end{aligned}$$

Taking scalar products of equations

$$(\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{\Gamma}^\omega(\boldsymbol{y}, \boldsymbol{x})\boldsymbol{q} = \delta_{\boldsymbol{x}}(\boldsymbol{y})\boldsymbol{q} \quad \text{and} \quad (\mathcal{L}^{\lambda,\mu} + \omega^2)\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y}, \boldsymbol{z})\boldsymbol{p} = \delta_{\boldsymbol{z}}(\boldsymbol{y})\boldsymbol{p}$$

with $\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}$ and $\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}$ respectively, subtracting the second result from the first, and integrating with respect to $\boldsymbol{y}$ over Ω, we obtain

$$\begin{aligned}&\int_\Omega \Big[(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot\mathcal{L}^{\lambda,\mu}(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}) - \mathcal{L}^{\lambda,\mu}(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\Big]\,d\boldsymbol{y}\\ &= -\boldsymbol{p}\cdot(\mathbf{\Gamma}^\omega(\boldsymbol{z},\boldsymbol{x})\boldsymbol{q}) + \boldsymbol{q}\cdot(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{x},\boldsymbol{z})\boldsymbol{p}) = -2\mathrm{i}\boldsymbol{q}\cdot\Im m\{\mathbf{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z})\}\boldsymbol{p},\end{aligned}$$

where we have used the reciprocity relation (1.70).

Using the form of the operator $\mathcal{L}^{\lambda,\mu}$, this gives

$$\begin{aligned}-2\mathrm{i}\boldsymbol{q}\cdot\Im m\{\mathbf{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z})\}\boldsymbol{p} \;=\;& \int_\Omega \lambda\Big[(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot\{\nabla\nabla\cdot(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\}\\ &\qquad -(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\cdot\Big\{\nabla\nabla\cdot(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\Big\}\Big]d\boldsymbol{y}\\ &+\int_\Omega \mu\Big[(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot\{(\Delta+\nabla\nabla\cdot)(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\}\\ &\qquad -(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\cdot\Big\{(\Delta+\nabla\nabla\cdot)(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\Big\}\Big]d\boldsymbol{y}.\end{aligned}$$

We recall that, for two functions $\boldsymbol{u},\boldsymbol{v}:\mathbb{R}^d\to\mathbb{R}^d$, we have

$$\begin{aligned}(\Delta\boldsymbol{u}+\nabla(\nabla\cdot\boldsymbol{u}))\cdot\boldsymbol{v} &= 2\nabla\cdot\big[\nabla^s\boldsymbol{u}\boldsymbol{v}\big] - 2\nabla^s\boldsymbol{u}:\nabla^s\boldsymbol{v},\\ \nabla(\nabla\cdot\boldsymbol{u})\cdot\boldsymbol{v} &= \nabla\cdot\big[(\nabla\cdot\boldsymbol{u})\boldsymbol{v}\big] - (\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{v}),\end{aligned}$$

where $\nabla\boldsymbol{u} = (\partial_j u_i)_{i,j=1}^d$. Therefore, we find

$$\begin{aligned}&-2\mathrm{i}\boldsymbol{q}\cdot\Im m\,\{\mathbf{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z})\}\boldsymbol{p}\\ &= \int_\Omega \lambda\Big[\nabla\cdot\Big\{[\nabla\cdot(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})](\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\Big\}\\ &\qquad -\nabla\cdot\Big\{[\nabla\cdot(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})](\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\Big\}\Big]\,d\boldsymbol{y}\\ &\quad+\int_\Omega \mu\Big[\nabla\cdot\Big\{\big((\nabla\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})+\nabla(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})^t\big)\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}\Big\}\\ &\qquad-\nabla\cdot\Big\{\big(\nabla(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}+\nabla(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}))^t\big)\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}\Big\}\Big]\,d\boldsymbol{y}.\end{aligned}$$

Now, we use divergence theorem to get

$$\begin{aligned}&-2\mathrm{i}\boldsymbol{q}\cdot\Im m\,\{\mathbf{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z})\}\boldsymbol{p}\\ &= \int_{\partial\Omega} \lambda\Big[\boldsymbol{n}\cdot\Big\{[\nabla\cdot(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})](\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\Big\}\\ &\qquad -\boldsymbol{n}\cdot\Big\{[\nabla\cdot(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})](\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\Big\}\Big]\,d\sigma(\boldsymbol{y})\\ &\quad+\int_{\partial\Omega} \mu\Big[\boldsymbol{n}\cdot\Big\{\big(\nabla(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})+(\nabla(\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}))^t\big)\,\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}\Big\}\\ &\qquad-\boldsymbol{n}\cdot\Big\{\Big(\nabla(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})+(\nabla(\overline{\mathbf{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}))^t\Big)\,\mathbf{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}\Big\}\Big]\,d\sigma(\boldsymbol{y}).\end{aligned}$$

Hence,

$$
\begin{aligned}
&-2i\boldsymbol{q}\cdot\Im m\,\{\boldsymbol{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z})\}\boldsymbol{p}\\
&=\int_{\partial\Omega}\lambda\Big[(\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot\{\nabla\cdot(\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\boldsymbol{n}\}\\
&\qquad\qquad -(\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\cdot\Big\{\nabla\cdot(\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\boldsymbol{n}\Big\}\Big]\,d\sigma(\boldsymbol{y})\\
&\quad+\int_{\partial\Omega}\mu\Big[(\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot\big\{\big(\nabla(\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})+(\nabla(\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}))^t\big)\,\boldsymbol{n}\big\}\\
&\qquad\qquad -(\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\cdot\Big\{\Big(\nabla(\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})+(\nabla(\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}))^t\Big)\,\boldsymbol{n}\Big\}\Big]\,d\sigma(\boldsymbol{y}),
\end{aligned}
$$

and therefore, using the definition of the conormal derivative,

$$
\begin{aligned}
&-2i\boldsymbol{q}\cdot\Im m\,\{\boldsymbol{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z})\}\boldsymbol{p}\\
&=\int_{\partial\Omega}\left[(\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p})\cdot\frac{\partial\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q}}{\partial\boldsymbol{\nu}(\boldsymbol{y})}-(\boldsymbol{\Gamma}^\omega(\boldsymbol{y},\boldsymbol{x})\boldsymbol{q})\cdot\frac{\partial\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\right]d\sigma(\boldsymbol{y})\\
&=\int_{\partial\Omega}\left[\boldsymbol{q}\cdot\frac{\partial\boldsymbol{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}-\boldsymbol{q}\cdot\boldsymbol{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{y})\frac{\partial\overline{\boldsymbol{\Gamma}}^\omega(\boldsymbol{y},\boldsymbol{z})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\boldsymbol{p}\right]d\sigma(\boldsymbol{y}),
\end{aligned}
$$

which is the desired result. Note that for establishing the last equality we have used the reciprocity relation (1.70). □

The proof of Proposition 1.11 uses only the reciprocity relation and the divergence theorem. Consequently, Proposition 1.11 also holds in a heterogeneous medium, as shown in [172].

The following proposition from [15] is an important ingredient in the analysis of elasticity imaging.

PROPOSITION 1.12. *Let Ω be a bounded Lipschitz domain. For all $\boldsymbol{x},\boldsymbol{z}\in\Omega$, we have*

$$
\int_{\partial\Omega}\left[\frac{\partial\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\overline{\boldsymbol{\Gamma}_p^\omega}(\boldsymbol{y},\boldsymbol{z})-\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\frac{\partial\overline{\boldsymbol{\Gamma}_p^\omega}(\boldsymbol{y},\boldsymbol{z})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\right]d\sigma(\boldsymbol{y})=0. \tag{1.73}
$$

PROOF. First, we note that $\boldsymbol{\Gamma}_p^\omega(\boldsymbol{y},\boldsymbol{x})$ and $\boldsymbol{\Gamma}_s^\omega(\boldsymbol{y},\boldsymbol{x})$ are solutions of

$$
(\mathcal{L}^{\lambda,\mu}+\omega^2\rho)\boldsymbol{\Gamma}_p^\omega=\mathcal{H}^p\left[\delta_{\mathbf{0}}\mathbf{I}\right]\quad\text{and}\quad(\mathcal{L}^{\lambda,\mu}+\omega^2\rho)\boldsymbol{\Gamma}_s^\omega=\mathcal{H}^s\left[\delta_{\mathbf{0}}\mathbf{I}\right]. \tag{1.74}
$$

Here,

$$
\mathcal{H}^p\left[\delta_{\mathbf{0}}\mathbf{I}\right]=\nabla\nabla\cdot(\Gamma\mathbf{I}),\quad \mathcal{H}^s\left[\delta_{\mathbf{0}}\mathbf{I}\right]=\nabla\times\nabla\times(\Gamma\mathbf{I}),
$$

$\Gamma(\boldsymbol{x})=-1/(4\pi|\boldsymbol{x}|)$ for $d=3$, and $\Gamma(\boldsymbol{x})=1/(2\pi)\ln|\boldsymbol{x}|$ for $d=2$ [155].

Then we proceed as in the proof of the previous proposition to find:

$$
\begin{aligned}
&\int_{\partial\Omega}\left[\frac{\partial\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\overline{\boldsymbol{\Gamma}_p^\omega}(\boldsymbol{y},\boldsymbol{z})-\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\frac{\partial\overline{\boldsymbol{\Gamma}_p^\omega}(\boldsymbol{y},\boldsymbol{z})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\right]d\sigma(\boldsymbol{y})\\
&\qquad=\int_{\Omega}\left[\mathcal{H}^s[\delta_{\boldsymbol{x}}\mathbf{I}](\boldsymbol{y})\overline{\boldsymbol{\Gamma}_p^\omega}(\boldsymbol{y},\boldsymbol{z})-\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\mathcal{H}^p[\delta_{\boldsymbol{z}}\mathbf{I}](\boldsymbol{y})\right]\,d\boldsymbol{y}\\
&\qquad=[\mathcal{H}^s[\delta_{\mathbf{0}}\mathbf{I}]*\overline{\boldsymbol{\Gamma}_p^\omega}(\cdot,\boldsymbol{z})](\boldsymbol{x})-[\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\cdot)*\mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}]](\boldsymbol{z}),
\end{aligned}
$$

where $*$ denotes the convolution product. Using the fact that $\mathbf{\Gamma}_p^\omega = \mathcal{H}^p[\mathbf{\Gamma}^\omega]$ and (1.16) we get

$$\mathcal{H}^s[\mathcal{H}^s[\delta_{\mathbf{0}}\mathbf{I}] * \overline{\mathbf{\Gamma}_p^\omega}(\cdot, \boldsymbol{z})] = 0 \quad \text{and} \quad \mathcal{H}^p[\mathcal{H}^s[\delta_{\mathbf{0}}\mathbf{I}] * \overline{\mathbf{\Gamma}_p^\omega}(\cdot, \boldsymbol{z})] = 0.$$

Therefore, we conclude

$$[\mathcal{H}^s[\delta_{\mathbf{0}}\mathbf{I}] * \overline{\mathbf{\Gamma}_p^\omega}(\cdot, \boldsymbol{z})](\boldsymbol{x}) = 0.$$

Similarly, we have

$$[\mathbf{\Gamma}_s^\omega(\boldsymbol{x}, \cdot) * \mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}]](\boldsymbol{z}) = 0,$$

which gives the desired result. □

Finally the following proposition shows that the elastodynamic reciprocity theorem (Proposition 1.11) holds for each wave component in a homogeneous medium.

PROPOSITION 1.13. *Let Ω be a bounded Lipschitz domain. For all $\boldsymbol{x}, \boldsymbol{z} \in \Omega$ and $\alpha = p, s$,*

$$\int_{\partial\Omega} \left[\frac{\partial \mathbf{\Gamma}_\alpha^\omega(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \overline{\mathbf{\Gamma}_\alpha^\omega}(\boldsymbol{y}, \boldsymbol{z}) - \mathbf{\Gamma}_\alpha^\omega(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial \overline{\mathbf{\Gamma}_\alpha^\omega}(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \right] d\sigma(\boldsymbol{y}) = -2i\Im m \left\{ \mathbf{\Gamma}_\alpha^\omega(\boldsymbol{x}, \boldsymbol{z}) \right\}. \tag{1.75}$$

PROOF. As both cases, $\alpha = p$ and $\alpha = s$, are similar, we only provide a proof for $\alpha = p$. For $\alpha = p$, we have as in the previous proof

$$\begin{aligned}\int_{\partial\Omega} &\left[\frac{\partial \mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \overline{\mathbf{\Gamma}_p^\omega}(\boldsymbol{y}, \boldsymbol{z}) - \mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial \overline{\mathbf{\Gamma}_p^\omega}(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \right] d\sigma(\boldsymbol{y}) \\ &= [\mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}] * \overline{\mathbf{\Gamma}_p^\omega}(\cdot, \boldsymbol{z})](\boldsymbol{x}) - [\mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \cdot) * \mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}]](\boldsymbol{z}).\end{aligned}$$

We can write

$$[\mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}] * \overline{\mathbf{\Gamma}_p^\omega}(\cdot, \boldsymbol{z})](\boldsymbol{x}) = [\mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}] * \overline{\mathbf{\Gamma}_p^\omega}(\cdot)](\boldsymbol{x} - \boldsymbol{z})$$

and

$$[\mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \cdot) * \mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}]](\boldsymbol{z}) = [\mathbf{\Gamma}_p^\omega(\cdot) * \mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}]](\boldsymbol{z} - \boldsymbol{x}) = [\mathcal{H}^p[\delta_{\mathbf{0}}\mathbf{I}] * \mathbf{\Gamma}_p^\omega(\cdot)]](\boldsymbol{x} - \boldsymbol{z}).$$

Therefore,

$$\int_{\partial\Omega} \left[\frac{\partial \mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \overline{\mathbf{\Gamma}_p^\omega}(\boldsymbol{y}, \boldsymbol{z}) - \mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial \overline{\mathbf{\Gamma}_p^\omega}(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} \right] d\sigma(\boldsymbol{y}) = -2i\Im m \left\{ \mathbf{\Gamma}_p^\omega(\boldsymbol{x}, \boldsymbol{z}) \right\},$$

where the last equality results from (1.16). □

We emphasize that the proofs of Propositions 1.12 and 1.13 require the medium to be homogeneous (so that $\mathcal{H}^s$ and $\mathcal{H}^p$ commute with $\mathcal{L}^{\lambda,\mu}$), and we cannot expect these propositions to be true in a heterogeneous medium because of mode conversion between pressure and shear waves.

1.5.2 Approximation of the Conormal Derivative

In this subsection, we derive an approximation of the conormal derivative

$$\partial \mathbf{\Gamma}^{\omega}(\boldsymbol{x},\boldsymbol{y})/\partial \boldsymbol{\nu}(\boldsymbol{y}), \quad \boldsymbol{y} \in \partial\Omega, \boldsymbol{x} \in \Omega.$$

In general this approximation involves the angles between the pressure and shear rays and the normal direction on $\partial\Omega$. This approximation becomes simple when Ω is a ball with very large radius, since in this case all rays are normal to $\partial\Omega$ (Proposition 1.14). It allows us to use a simplified version of the Helmholtz-Kirchhoff identities in order to analyze elasticity imaging.

PROPOSITION 1.14. *If* $\boldsymbol{n}(\boldsymbol{y}) = \widehat{\boldsymbol{y}-\boldsymbol{x}}$ $(:= (\boldsymbol{y}-\boldsymbol{x})/|\boldsymbol{x}-\boldsymbol{y}|)$ *and* $|\boldsymbol{x}-\boldsymbol{y}| \gg 1$, *then, for* $\alpha = p, s$,

$$\frac{\partial \mathbf{\Gamma}_{\alpha}^{\omega}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}} = \mathrm{i}\omega c_{\alpha} \mathbf{\Gamma}_{\alpha}^{\omega}(\boldsymbol{x},\boldsymbol{y}) + o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{(d-1)/2}}\right). \tag{1.76}$$

PROOF. We only prove here the proposition for $d = 3$. The case $d = 2$ follows from exactly the same arguments. Moreover, it is enough to show that for all constant vectors $\boldsymbol{q}$,

$$\frac{\partial \mathbf{\Gamma}_{\alpha}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}}{\partial \boldsymbol{\nu}} = \mathrm{i}\omega c_{\alpha} \mathbf{\Gamma}_{\alpha}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q} + o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right), \quad \alpha = p, s.$$

Pressure component: Recall from (1.25) and (1.27) that

$$\mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{\omega^2}\mathbf{D}\Gamma_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{c_p^2}\Gamma_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\widehat{\boldsymbol{y}-\boldsymbol{x}} \otimes \widehat{\boldsymbol{y}-\boldsymbol{x}} + o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right),$$

so we have

$$\mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q} = \frac{1}{c_p^2}\Gamma_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\ (\widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \boldsymbol{q})\ \widehat{\boldsymbol{y}-\boldsymbol{x}} + o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right).$$

Therefore,

$$\begin{aligned}
\frac{\partial \mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}}{\partial \boldsymbol{\nu}} &= \lambda \nabla_{\boldsymbol{y}} \cdot \big(\mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}\big)\, \boldsymbol{n}(\boldsymbol{y}) \\
&\quad + \mu \left[\nabla_{\boldsymbol{y}}(\mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}) + (\nabla_{\boldsymbol{y}}(\mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}))^t\right] \boldsymbol{n}(\boldsymbol{y}) \\
&= \frac{\widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \boldsymbol{q}}{c_p^3} \mathrm{i}\omega \Gamma_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y}) \left[\lambda\ \widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}-\boldsymbol{x}}\boldsymbol{n} + 2\mu(\widehat{\boldsymbol{y}-\boldsymbol{x}} \otimes \widehat{\boldsymbol{y}-\boldsymbol{x}})\boldsymbol{n}\right] \\
&\quad + o\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right) \\
&= \frac{\widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \boldsymbol{q}}{c_p^3} \mathrm{i}\omega \Gamma_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y}) \left[\lambda \boldsymbol{n} + 2\mu(\widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \boldsymbol{n})\widehat{\boldsymbol{y}-\boldsymbol{x}}\right] + o\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right) \\
&= \frac{\widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \boldsymbol{q}}{c_p^3} \mathrm{i}\omega \Gamma_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y}) \left[\lambda\big(\boldsymbol{n} - \widehat{\boldsymbol{y}-\boldsymbol{x}}\big) + 2\mu\big((\widehat{\boldsymbol{y}-\boldsymbol{x}} \cdot \boldsymbol{n}) - 1\big)\widehat{\boldsymbol{y}-\boldsymbol{x}}\right] \\
&\quad + \mathrm{i}\omega c_p \mathbf{\Gamma}_{p}^{\omega}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q} + o\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right).
\end{aligned}$$

In particular, when $\boldsymbol{n} = \widehat{\boldsymbol{y}-\boldsymbol{x}}$, we have

$$\frac{\partial \boldsymbol{\Gamma}_p^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}}{\partial \boldsymbol{\nu}} = \mathrm{i}\omega c_p \boldsymbol{\Gamma}_p^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q} + o\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right).$$

Shear components: As

$$\begin{aligned}\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y}) &= \frac{1}{\omega^2}\left(\kappa_s^2\mathbf{I}+\mathbf{D}\right)\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\\ &= \frac{1}{c_s^2}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\left(\mathbf{I}-\widehat{\boldsymbol{y}-\boldsymbol{x}}\otimes\widehat{\boldsymbol{y}-\boldsymbol{x}}\right)+o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right),\end{aligned}$$

we have

$$\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q} = \frac{1}{c_s^2}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\ \left(\boldsymbol{q}-(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{q})\widehat{\boldsymbol{y}-\boldsymbol{x}}\right)+o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right).$$

Therefore,

$$\begin{aligned}\frac{\partial \boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}}{\partial \boldsymbol{\nu}} &= \lambda\nabla_{\boldsymbol{y}}\cdot(\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q})\,\boldsymbol{n}(\boldsymbol{y})+\mu\left[\nabla_{\boldsymbol{y}}(\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q})\right.\\ &\quad\left.+(\nabla_{\boldsymbol{y}}(\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}))^t\right]\boldsymbol{n}(\boldsymbol{y}).\end{aligned}$$

Now, remark that

$$\begin{aligned}\lambda\nabla\cdot(\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q})\,\boldsymbol{n} &= \lambda\frac{\mathrm{i}\omega}{c_s^3}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\left[\left(\boldsymbol{q}-(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{q})\widehat{\boldsymbol{y}-\boldsymbol{x}}\right)\cdot\widehat{\boldsymbol{y}-\boldsymbol{x}}\right]\boldsymbol{n}\\ &\quad+o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\\ &= o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right),\end{aligned}$$

and

$$\begin{aligned}&\mu\left[\nabla(\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q})+\nabla(\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q})^t\right]\boldsymbol{n}\\ &=\mu\frac{\mathrm{i}\omega}{c_s^3}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\left[\boldsymbol{q}\otimes\widehat{\boldsymbol{y}-\boldsymbol{x}}+\widehat{\boldsymbol{y}-\boldsymbol{x}}\otimes\boldsymbol{q}-2\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{q}\big)\widehat{\boldsymbol{y}-\boldsymbol{x}}\otimes\widehat{\boldsymbol{y}-\boldsymbol{x}}\right]\boldsymbol{n}\\ &\quad+o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\\ &=\mu\frac{\mathrm{i}\omega}{c_s^3}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\left[\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{n}\big)\boldsymbol{q}+\big(\boldsymbol{q}\cdot\boldsymbol{n}\big)\widehat{\boldsymbol{y}-\boldsymbol{x}}-2\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{q}\big)\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{n}\big)\widehat{\boldsymbol{y}-\boldsymbol{x}}\right]\\ &\quad+o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\\ &=\mu\frac{\mathrm{i}\omega}{c_s^3}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\left[\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{n}\big)-1\right]\left[\boldsymbol{q}-\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{q}\big)\widehat{\boldsymbol{y}-\boldsymbol{x}}\right]\\ &\quad+\mu\frac{\mathrm{i}\omega}{c_s^3}\Gamma_s^\omega(\boldsymbol{x},\boldsymbol{y})\left[\left(\boldsymbol{q}\cdot\boldsymbol{n}-\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{q}\big)\big(\widehat{\boldsymbol{y}-\boldsymbol{x}}\cdot\boldsymbol{n}\big)\right)\widehat{\boldsymbol{y}-\boldsymbol{x}}\right]\mathrm{i}\omega c_s\boldsymbol{\Gamma}_s^\omega(\boldsymbol{x},\boldsymbol{y})\\ &\quad+o\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right).\end{aligned}$$

In particular, when $\boldsymbol{n} = \widehat{\boldsymbol{y} - \boldsymbol{x}}$, we have

$$\frac{\partial \mathbf{\Gamma}_s^\omega(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{q}}{\partial \boldsymbol{\nu}} = \mathrm{i}\omega c_s \mathbf{\Gamma}_s^\omega(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{q} + o\left(\frac{1}{|\boldsymbol{y} - \boldsymbol{x}|}\right).$$

This completes the proof. □

The following is a direct consequence of Propositions 1.12, 1.13, and 1.14.

PROPOSITION 1.15 (Helmholtz-Kirchhoff Identities). *Let $\Omega \subset \mathbb{R}^d$ be a ball with radius R. Then, for all $\boldsymbol{x}, \boldsymbol{z} \in \Omega$, we have*

$$\lim_{R \to +\infty} \int_{\partial \Omega} \mathbf{\Gamma}_\alpha^\omega(\boldsymbol{x}, \boldsymbol{y}) \overline{\mathbf{\Gamma}_\alpha^\omega}(\boldsymbol{y}, \boldsymbol{z}) d\sigma(\boldsymbol{y}) = -\frac{1}{\omega c_\alpha} \Im m \left\{ \mathbf{\Gamma}_\alpha^\omega(\boldsymbol{x}, \boldsymbol{z}) \right\}, \ \alpha = p, s, \tag{1.77}$$

and

$$\lim_{R \to +\infty} \int_{\partial \Omega} \mathbf{\Gamma}_s^\omega(\boldsymbol{x}, \boldsymbol{y}) \overline{\mathbf{\Gamma}_p^\omega}(\boldsymbol{y}, \boldsymbol{z}) d\sigma(\boldsymbol{y}) = 0. \tag{1.78}$$

1.6 EIGENVALUE CHARACTERIZATIONS AND NEUMANN AND DIRICHLET FUNCTIONS

1.6.1 Eigenvalue Characterizations

In this subsection we characterize the eigenvalues of the elasticity operator on a bounded domain with Neumann or Dirichlet boundary conditions as the characteristic values of certain layer potentials which are meromorphic operator-valued functions. For doing so, we first recall the notions of characteristic values and root functions of analytic operator-valued functions. We refer, for instance, to [132] for the details.

If $\mathcal{B}$ and $\mathcal{B}'$ are two Banach spaces, we denote by $\mathcal{L}(\mathcal{B}, \mathcal{B}')$ the space of bounded linear operators from $\mathcal{B}$ into $\mathcal{B}'$.

Let $\mathfrak{U}(\omega_0)$ be the set of all operator-valued functions with values in $\mathcal{L}(\mathcal{B}, \mathcal{B}')$ which are holomorphic in some neighborhood of ω_0, except possibly at ω_0.

The point ω_0 is called a *characteristic value* of $\mathbf{A}(\omega) \in \mathfrak{U}(\omega_0)$ if there exists a vector-valued function $\boldsymbol{\varphi}(\omega)$ with values in $\mathcal{B}$ such that

(i) $\boldsymbol{\varphi}(\omega)$ is holomorphic at ω_0 and $\boldsymbol{\varphi}(\omega_0) \neq 0$,

(ii) $\mathbf{A}(\omega)[\boldsymbol{\varphi}(\omega)]$ is holomorphic at ω_0 and vanishes at this point.

Here, $\boldsymbol{\varphi}(\omega)$ is called a *root* function of $\mathbf{A}(\omega)$ associated with the characteristic value ω_0. The vector $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}(\omega_0)$ is called an eigenvector. The closure of the linear set of eigenvectors corresponding to ω_0 is denoted by $\mathrm{Ker}\mathbf{A}(\omega_0)$.

Let κ be an eigenvalue of $-\mathcal{L}^{\lambda,\mu}$ in Ω with the Neumann condition on $\partial\Omega$ and let $\boldsymbol{u}$ denote an eigenvector associated with κ; *i.e.*,

$$\begin{cases} \mathcal{L}^{\lambda,\mu} \boldsymbol{u} + \kappa \boldsymbol{u} = 0 & \text{in } \ \Omega, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{1.79}$$

We note that since $-\mathcal{L}^{\lambda,\mu}$ is elliptic, it has discrete eigenvalues of finite multiplicities. The following proposition from [120, Chapter 7] is of importance to us.

PROPOSITION 1.16 (Neumann eigenvalue characterization). *The necessary and sufficient condition for (1.79) to have a nontrivial solution is that κ is nonnegative and $\sqrt{\kappa}$ coincides with one of the characteristic values of*

$$(1/2)\,\mathbf{I} - \mathcal{K}_\Omega^\omega : L^2(\partial\Omega)^d \to L^2(\partial\Omega)^d.$$

If $\kappa = \omega_0^2\rho$ is a Neumann eigenvalue of (1.79) with multiplicity m, then

$$\big((1/2)\,\mathbf{I} - \mathcal{K}_\Omega^{\omega_0}\big)[\boldsymbol{\varphi}_0] = 0$$

has m linearly independent solutions. Moreover, for every eigenvalue $\kappa > 0$, $\sqrt{\kappa}$ is a simple pole of the operator-valued function $\omega \mapsto ((1/2)\,\mathbf{I} - \mathcal{K}_\Omega^\omega)^{-1}$.

For the Dirichlet eigenvalue problem, the following characterization holds.

PROPOSITION 1.17 (Dirichlet eigenvalue characterization). *Consider the eigenvalue problem with Dirichlet boundary conditions*

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{v} + \tau\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.80}$$

The necessary and sufficient condition for (1.80) to have a nontrivial solution is that τ is nonnegative and $\sqrt{\tau}$ coincides with one of the characteristic values of

$$(1/2)\,\mathbf{I} + (\mathcal{K}_\Omega^\omega)^* : L^2(\partial\Omega)^d \to L^2(\partial\Omega)^d.$$

If $\tau = \omega_0^2\rho$ is a Dirichlet eigenvalue of (1.80) with multiplicity m, then

$$\big((1/2)\,\mathbf{I} + (\mathcal{K}_\Omega^{\omega_0})^*\big)[\boldsymbol{\psi}_0] = 0$$

has m linearly independent solutions. Moreover, for every eigenvalue $\tau > 0$, $\sqrt{\tau}$ is a simple pole of the operator-valued function $\omega \mapsto ((1/2)\,\mathbf{I} + (\mathcal{K}_\Omega^\omega)^*)^{-1}$.

1.6.2 Neumann Function

Let $0 = \kappa_1 \le \kappa_2 \le \dots$ be the eigenvalues of $-\mathcal{L}^{\lambda,\mu}$ in Ω with the Neumann condition on $\partial\Omega$. Note that $\kappa_1 = 0$ is of multiplicity $d(d+1)/2$, the eigenspace being Ψ. For $\omega\sqrt{\rho} \notin \{\sqrt{\kappa_j}\}_{j\ge1}$, let $\mathbf{N}_\Omega^\omega(\boldsymbol{x},\boldsymbol{z})$ be the Neumann function for $\mathcal{L}^{\lambda,\mu} + \omega^2\rho$ in Ω corresponding to a Dirac mass at $\boldsymbol{z}$. That is, for $\boldsymbol{z} \in \Omega$, $\mathbf{N}_\Omega^\omega(\cdot,\boldsymbol{z})$ is the matrix-valued solution to

$$\begin{cases} (\mathcal{L}^{\lambda,\mu} + \omega^2\rho)\mathbf{N}_\Omega^\omega(\boldsymbol{x},\boldsymbol{z}) = -\delta_{\boldsymbol{z}}(\boldsymbol{x})\mathbf{I}, & \boldsymbol{x} \in \Omega, \\ \dfrac{\partial \mathbf{N}_\Omega^\omega}{\partial \boldsymbol{\nu}}(\boldsymbol{x},\boldsymbol{z}) = 0, & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{1.81}$$

Then the following relation holds (see [26]):

$$\Big(-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega^\omega\Big)[\mathbf{N}_\Omega^\omega(\cdot,\boldsymbol{z})](\boldsymbol{x}) = \mathbf{\Gamma}^\omega(\boldsymbol{x},\boldsymbol{z}), \quad \boldsymbol{x} \in \partial\Omega, \quad \boldsymbol{z} \in \Omega. \tag{1.82}$$

Let $(\boldsymbol{u}_j)_{j\geq 1}$ denote the set of orthogonal eigenvectors associated with $(\kappa_j)_{j\geq 1}$, with $\|\boldsymbol{u}_j\|_{L^2(\Omega)} = 1$. Then we have the following spectral decomposition:

$$\mathbf{N}_\Omega^\omega(\boldsymbol{x}, \boldsymbol{z}) = \sum_{j=1}^{+\infty} \frac{\boldsymbol{u}_j(\boldsymbol{x})\boldsymbol{u}_j(\boldsymbol{z})^t}{\kappa_j - \omega^2\rho}. \tag{1.83}$$

Here we regard $\boldsymbol{u}_j$ as a column vector, and hence $\boldsymbol{u}_j(\boldsymbol{x})\, \boldsymbol{u}_j(\boldsymbol{z})^t$ is a $d \times d$ matrix-valued function. We refer the reader to [158] for a proof of (1.83). Note that the eigenvectors $(\boldsymbol{u}_j)_{j\geq 1}$ have in general a nontrivial decomposition in terms of p- and s-waves.

Let $\mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z})$ be the Neumann function for the Lamé system on Ω, namely, for $\boldsymbol{z} \in \Omega$, $\mathbf{N}(\boldsymbol{x}, \boldsymbol{z})$ is the solution to

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z}) = -\delta_{\boldsymbol{z}}(\boldsymbol{x})\mathbf{I}, & \boldsymbol{x} \in \Omega, \\ \dfrac{\partial \mathbf{N}_\Omega}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, \boldsymbol{z}) = -\dfrac{1}{|\partial\Omega|}\mathbf{I}, & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{1.84}$$

subject to the orthogonality condition:

$$\int_{\partial\Omega} \mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{\psi}(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) = 0 \quad \forall\, \boldsymbol{\psi} \in \Psi. \tag{1.85}$$

We have

$$\mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z}) = \sum_{j=1}^{+\infty} \frac{1}{\kappa_j}\boldsymbol{u}_j(\boldsymbol{x})\boldsymbol{u}_j(\boldsymbol{z})^t, \quad \boldsymbol{x} \neq \boldsymbol{z}.$$

Moreover, the following identity holds:

$$(-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)^{-1}\left[\boldsymbol{\Gamma}(\cdot - \boldsymbol{z})\right](\boldsymbol{x}) = \mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z}), \quad \boldsymbol{x} \in \partial\Omega, \quad \boldsymbol{z} \in \Omega, \tag{1.86}$$

modulo a function in Ψ. See [28, 32, 120] for properties of the Neumann function and a proof of (1.86).

Using $\mathbf{N}_\Omega$ one can derive a representation formula for the solution to (1.61) in terms of the background solution, *i.e.*, the solution to

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\boldsymbol{u}_0 = 0 \quad \text{in } \Omega, \\ \left.\dfrac{\partial \boldsymbol{u}_0}{\partial \boldsymbol{\nu}}\right|_{\partial\Omega} = \boldsymbol{g}, \\ \boldsymbol{u}_0\big|_{\partial\Omega} \in L^2_\Psi(\partial\Omega). \end{cases} \tag{1.87}$$

We need to fix some notation. Let $D \Subset \Omega \subset \mathbb{R}^d$ be two bounded Lipschitz domains. Let us define for $\boldsymbol{f} \in L^2_\Psi(\partial D)$

$$\mathcal{N}_D[\boldsymbol{f}](\boldsymbol{x}) := \int_{\partial D} \mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \partial\Omega. \tag{1.88}$$

THEOREM 1.18. *Let $\boldsymbol{u}$ be the solution to (1.61) and $\boldsymbol{u}_0$ the background solution.*

Then the following holds:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) - \mathcal{N}_D[\boldsymbol{\psi}](\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega, \tag{1.89}$$

where $\boldsymbol{\psi}$ *is defined by (1.64).*

PROOF. By substituting (1.63) into the equation (1.62), we obtain

$$\boldsymbol{H}(\boldsymbol{x}) = -\mathcal{S}_\Omega[\boldsymbol{g}](\boldsymbol{x}) + \mathcal{D}_\Omega\Big[\boldsymbol{H}|_{\partial\Omega} + \mathcal{S}_D[\boldsymbol{\psi}]\big|_{\partial\Omega}\Big](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

By using (1.43), we see that

$$(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega)[\boldsymbol{H}|_{\partial\Omega}] = -\mathcal{S}_\Omega[\boldsymbol{g}]\Big|_{\partial\Omega} + (\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)[\mathcal{S}_D[\boldsymbol{\psi}]\big|_{\partial\Omega}] \quad \text{on } \partial\Omega. \tag{1.90}$$

Since $\boldsymbol{u}_0(\boldsymbol{x}) = -\mathcal{S}_\Omega[\boldsymbol{g}](\boldsymbol{x}) + \mathcal{D}_\Omega[\boldsymbol{u}_0|_{\partial\Omega}](\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$, we have

$$(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega)[\boldsymbol{u}_0]|_{\partial\Omega} = -\mathcal{S}_\Omega[\boldsymbol{g}]\big|_{\partial\Omega}. \tag{1.91}$$

By Theorem 1.7 and (1.86), we have

$$(-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)[(\mathcal{N}_D[\boldsymbol{\psi}])|_{\partial\Omega})(\boldsymbol{x}) = \mathcal{S}_D[\boldsymbol{\psi}](\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega, \tag{1.92}$$

since $\boldsymbol{\psi} \in L^2_\Psi(\partial D)$. We see from (1.90), (1.91), and (1.92) that

$$(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega)\Big[\boldsymbol{H}|_{\partial\Omega} - \boldsymbol{u}_0|_{\partial\Omega} + (\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)[(\mathcal{N}_D[\boldsymbol{\psi}])|_{\partial\Omega}]\Big] = 0 \quad \text{on } \partial\Omega,$$

and hence, by Lemma 1.3, we obtain that

$$\boldsymbol{H}|_{\partial\Omega} - \boldsymbol{u}_0|_{\partial\Omega} + (\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)[(\mathcal{N}_D[\boldsymbol{\psi}])|_{\partial\Omega}] \in \Psi.$$

Note that

$$(\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)[(\mathcal{N}_D[\boldsymbol{\psi}])|_{\partial\Omega}] = (\mathcal{N}_D[\boldsymbol{\psi}])|_{\partial\Omega} + (\mathcal{S}_D[\boldsymbol{\psi}])|_{\partial\Omega},$$

which follows from (1.86). Thus, (1.63) gives

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_0|_{\partial\Omega} - (\mathcal{N}_D[\boldsymbol{\psi}])|_{\partial\Omega} \quad \text{modulo } \Psi. \tag{1.93}$$

Since all the functions in (1.93) belong to $L^2_\Psi(\partial\Omega)$, we have (1.89). This completes the proof. □

In Chapter 3 we will be dealing with elastic inclusions of the form $D = \epsilon B + \boldsymbol{z}$, where B is a bounded Lipschitz domain in $\mathbb{R}^d$ and ϵ is a small positive parameter. For the purpose of use in Chapter 3, we consider the asymptotic expansion of $\mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z} + \epsilon\boldsymbol{y})$ for $\boldsymbol{x} \in \partial\Omega, \boldsymbol{z} \in \Omega$ and $\boldsymbol{y} \in \partial B$ as $\epsilon \to 0$.

Recall that if $j = (j_1, \ldots, j_d)$ is a multi-index (an ordered d-tuple of nonnegative integers), then we write $j! = j_1! \ldots j_d!$, $\boldsymbol{y}^j = y_1^{j_1} \ldots y_d^{j_d}$, $|j| = j_1 + \ldots + j_d$, and $\partial^j = \partial^{|j|}/\partial y_1^{j_1} \ldots \partial y_d^{j_d}$.

Since

$$\mathbf{\Gamma}(\boldsymbol{x} - \epsilon \boldsymbol{y}) = \sum_{|\beta|=0}^{+\infty} \frac{1}{\beta!} \epsilon^{|\beta|} \partial^\beta (\mathbf{\Gamma}(\boldsymbol{x})) \boldsymbol{y}^\beta,$$

we get from (1.86) that, modulo Ψ,

$$(-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega)\Big[\mathbf{N}_\Omega(\cdot, \epsilon \boldsymbol{y} + \boldsymbol{z})\Big](\boldsymbol{x}) = \sum_{|\beta|=0}^{+\infty} \frac{1}{\beta!} \epsilon^{|\beta|} \partial^\beta (\mathbf{\Gamma}(\boldsymbol{x} - \boldsymbol{z})) \boldsymbol{y}^\beta$$

$$= (-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega) \left[\sum_{|\beta|=0}^{+\infty} \frac{1}{\beta!} \epsilon^{|\beta|} \partial_{\boldsymbol{z}}^\beta \mathbf{N}_\Omega(\cdot, \boldsymbol{z}) \boldsymbol{y}^j \right] (\boldsymbol{x}).$$

Since $\mathbf{N}_\Omega(\cdot, \boldsymbol{w}) \in L^2_\Psi(\partial\Omega)$ for all $\boldsymbol{w} \in \Omega$, we have the following asymptotic expansion of the Neumann function.

LEMMA 1.19. *For $\boldsymbol{x} \in \partial\Omega$, $\boldsymbol{z} \in \Omega$, $\boldsymbol{y} \in \partial B$, and $\epsilon \to 0$,*

$$\mathbf{N}_\Omega(\boldsymbol{x}, \epsilon \boldsymbol{y} + \boldsymbol{z}) = \sum_{|\beta|=0}^{+\infty} \frac{1}{\beta!} \epsilon^{|\beta|} \partial_{\boldsymbol{z}}^\beta \mathbf{N}_\Omega(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{y}^\beta. \tag{1.94}$$

A proof of Lemma 1.19 can be found in [28].

1.6.3 Dirichlet Function

Now we turn to the properties of the Dirichlet function. Let $0 \le \tau_1 \le \tau_2 \le \ldots$ be the eigenvalues of $-\mathcal{L}^{\lambda,\mu}$ in Ω with the Dirichlet condition on $\partial\Omega$. For $\omega\sqrt{\rho} \notin \{\sqrt{\tau_j}\}_{j\ge1}$, let $\mathbf{G}^\omega_\Omega(\boldsymbol{x}, \boldsymbol{z})$ be the Dirichlet function for $\mathcal{L}^{\lambda,\mu} + \omega^2\rho$ in Ω corresponding to a Dirac mass at $\boldsymbol{z}$. That is, for $\boldsymbol{z} \in \Omega$, $\mathbf{G}^\omega_\Omega(\cdot, \boldsymbol{z})$ is the matrix-valued solution to

$$\begin{cases} (\mathcal{L}^{\lambda,\mu} + \omega^2\rho)\mathbf{G}^\omega_\Omega(\boldsymbol{x}, \boldsymbol{z}) = -\delta_{\boldsymbol{z}}(\boldsymbol{x})\mathbf{I}, & \boldsymbol{x} \in \Omega, \\ \mathbf{G}^\omega_\Omega(\boldsymbol{x}, \boldsymbol{z}) = 0, & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{1.95}$$

Then for any $\boldsymbol{x} \in \partial\Omega$, and $\boldsymbol{z} \in \Omega$ we can prove in the same way as (1.82) that

$$\Big(\frac{1}{2}\mathbf{I} + (\mathcal{K}^\omega_\Omega)^*\Big)[\frac{\partial \mathbf{G}^\omega_\Omega}{\partial \boldsymbol{\nu}}(\cdot, \boldsymbol{z})](\boldsymbol{x}) = -\frac{\partial \mathbf{\Gamma}^\omega}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, \boldsymbol{z}). \tag{1.96}$$

Moreover, we mention the following important properties of $\mathbf{G}^\omega_\Omega$:

(i) Let $(\boldsymbol{v}_j)_{j\ge1}$ denote the set of orthogonal eigenvectors associated with $(\tau_j)_{j\ge1}$, with $\|\boldsymbol{v}_j\|_{L^2(\Omega)} = 1$. Then we have the following spectral decomposition:

$$\mathbf{G}^\omega_\Omega(\boldsymbol{x}, \boldsymbol{z}) = \sum_{j=1}^{+\infty} \frac{\boldsymbol{v}_j(\boldsymbol{x})\boldsymbol{v}_j(\boldsymbol{z})^t}{\tau_j - \omega^2\rho}. \tag{1.97}$$

(ii) For $\boldsymbol{x} \in \partial\Omega$, $\boldsymbol{z} \in \Omega$, $\boldsymbol{y} \in \partial B$, and $\epsilon \to 0$,

$$\mathbf{G}^\omega_\Omega(\boldsymbol{x}, \epsilon \boldsymbol{y} + \boldsymbol{z}) = \sum_{|\beta|=0}^{+\infty} \frac{1}{\beta!} \epsilon^{|\beta|} \partial_{\boldsymbol{z}}^\beta \mathbf{G}^\omega_\Omega(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{y}^\beta. \tag{1.98}$$

1.7 A REGULARITY RESULT

We state a generalization of Meyer's theorem concerning the regularity of solutions to systems with bounded coefficients. This result will be useful in Chapter 11. For $p > 1$, define $H^{1,p}(\Omega)$ by

$$H^{1,p}(\Omega) := \left\{ v \in L^p(\Omega), \nabla v \in L^p(\Omega)^d \right\}$$

and let $H^{-1,q}(\Omega)$ with $q = (p-1)/p$ be its dual. Introduce

$$H^{1,p}_{\text{loc}}(\Omega) := \left\{ v \in H^{1,p}(K) \ \forall\, K \Subset \Omega \right\}.$$

THEOREM 1.20. *Let $\mathbb{C} \in L^\infty(\Omega)$ be a strongly convex tensor,* i.e., *there exists a positive constant C such that*

$$\mathbb{C}\mathbf{A} : \mathbf{A} \geq C \|\mathbf{A}\|^2$$

for every $d \times d$ symmetric matrix $\mathbf{A} = (a_{ij})$, where $\|\mathbf{A}\| = \sqrt{\sum_{i,j} a_{ij}^2}$. There exists $\eta > 0$ such that if $\boldsymbol{u} \in H^1(\Omega)^d$ is solution to

$$\nabla \cdot (\mathbb{C}\nabla^s \boldsymbol{u}) = \boldsymbol{f} \quad in \quad \Omega,$$

where $\boldsymbol{f} \in H^{-1,2+\eta}(\Omega)^d$, then $\boldsymbol{u} \in H^{1,2+\eta}_{\text{loc}}(\Omega)^d$ and for any two disks $B_\rho \subset B_{2\rho} \Subset \Omega$

$$\|\nabla \boldsymbol{u}\|_{L^{2+\eta}(B_\rho)} \leq C' (\|\boldsymbol{f}\|_{H^{-1,2+\eta}(B_{2\rho})} + \rho^{\frac{2}{2+\eta}} \|\nabla \boldsymbol{u}\|_{L^2(B_{2\rho})})$$

for some positive constant C'.

The above theorem was proved in [60, 129].

CONCLUDING REMARKS

In this chapter, we have briefly reviewed layer potential techniques associated with the static and time-harmonic elasticity equations. Our main concern has been to represent the solutions of these equations and to establish Helmholtz-Kirchhoff identities in the time-harmonic case. In the next chapters, these results will be used to establish an asymptotic theory for the perturbations in the displacement measurements due to inclusions or cracks. Further, these results will be indispensible to analyze the stability and resolution properties of the imaging algorithms which we will design for locating the defects and characterizing them.