Chapter 1

DSGE Modeling

Estimated dynamic stochastic general equilibrium (DSGE) models are now widely used by academics to conduct empirical research macroeconomics as well as by central banks to interpret the current state of the economy, to analyze the impact of changes in monetary or fiscal policy, and to generate predictions for key macroeconomic aggregates. The term DSGE model encompasses a broad class of macroeconomic models that span the real business cycle models of Kydland and Prescott (1982) and King, Plosser, and Rebelo (1988) as well as the New Keynesian models of Rotemberg and Woodford (1997) or Christiano, Eichenbaum, and Evans (2005), which feature nominal price and wage rigidities and a role for central banks to adjust interest rates in response to inflation and output fluctuations. A common feature of these models is that decision rules of economic agents are derived from assumptions about preferences and technologies by solving intertemporal optimization problems. Moreover, agents potentially face uncertainty with respect to aggregate variables such as total factor productivity or nominal interest rates set by a central bank. This uncertainty is generated by exogenous stochastic processes that may shift technology or generate unanticipated deviations from a central bank's interest-rate feedback rule.

The focus of this book is the Bayesian estimation of DSGE models. Conditional on distributional assumptions for the exogenous shocks, the DSGE model generates a likelihood function, that is, a joint probability distribution for the endogenous model variables such as output, consumption, investment, and inflation that depends on the structural parameters of the model. These structural parameters characterize agents' preferences, production technologies, and the law of motion of the exogenous shocks. In a Bayesian framework, this likelihood function can be used to transform a prior distribution for the structural parameters of the DSGE model into a posterior distribution. This posterior is the basis for substantive inference and decision making. Unfortunately, it is not feasible to characterize moments and quantiles of the posterior distribution analytically. Instead, we have to use computational techniques to generate draws from the posterior and then approximate posterior expectations by Monte Carlo averages.

In Section 1.1 we will present a small-scale New Keynesian DSGE model and describe the decision problems of firms and households and the behavior of the monetary and fiscal authorities. We then characterize the resulting equilibrium conditions. This model is subsequently used in many of the numerical illustrations. Section 1.2 briefly sketches two other DSGE models that will be estimated in subsequent chapters.

1.1 A Small-Scale New Keynesian DSGE Model

We begin with a small-scale New Keynesian DSGE model that has been widely studied in the literature (see Woodford (2003) or Gali (2008) for textbook treatments). The particular specification presented below is based on An and Schorfheide (2007). The model economy consists of final goods producing firms, intermediate goods producing firms, households, a central bank, and a fiscal authority. We will first describe the decision problems of these agents, then describe the law of motion of the exogenous processes, and finally summarize the equilibrium conditions. The likelihood function for a linearized version of this model can be quickly evaluated, which makes the model an excellent showcase for the computational algorithms studied in this book.

1.1.1 Firms

Production takes place in two stages. There are monopolistically competitive intermediate goods producing firms and perfectly competitive final goods producing firms that aggregate the intermediate goods into a single good that is used for household and government consumption. This two-stage production process makes it fairly straightforward to introduce price stickiness, which in turn creates a real effect of monetary policy.

The perfectly competitive final good producing firms combine a continuum of intermediate goods indexed by $j \in [0, 1]$ using the technology

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}. \tag{1.1}$$

The final good producers take input prices $P_t(j)$ and output prices P_t as given. The revenue from the sale of the final good is $P_t Y_t$ and the input costs incurred to produce Y_t are $\int_0^1 P_t(j) Y_t(j) dj$. Maximization of profits

$$\Pi_t = P_t \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}} - \int_0^1 P_t(j) Y_t(j) dj, \tag{1.2}$$

with respect to the inputs $Y_t(j)$ implies that the demand for intermediate good j is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t. \tag{1.3}$$

Thus, the parameter $1/\nu$ represents the elasticity of demand for each intermediate good. In the absence of an entry cost, final good producers will enter the market until profits are equal to zero. From the zero-profit condition, it is possible to derive the following relationship between the intermediate goods prices and the price of the final good:

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}. \tag{1.4}$$

Intermediate good j is produced by a monopolist who has access to the following linear production technology:

$$Y_t(j) = A_t N_t(j), \tag{1.5}$$

where A_t is an exogenous productivity process that is common to all firms and $N_t(j)$ is the labor input of firm j. To keep the model simple, we abstract from capital as a factor or production for now. Labor is hired in a perfectly competitive factor market at the real wage W_t.

In order to introduce nominal price stickiness, we assume that firms face quadratic price adjustment costs

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \pi\right)^2 Y_t(j), \qquad (1.6)$$

where ϕ governs the price rigidity in the economy and π is the steady state inflation rate associated with the final good. Under this adjustment cost specification it is costless to change prices at the rate π. If the price change deviates from π, the firm incurs a cost in terms of lost output that is a quadratic function of the discrepancy between the price change and π. The larger the adjustment cost parameter ϕ, the more reluctant the intermediate goods producers are to change their prices and the more rigid the prices are at the aggregate level. Firm j chooses its labor input $N_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - W_{t+s}N_{t+s}(j) - AC_{t+s}(j)\right)\right]. \qquad (1.7)$$

Here, $Q_{t+s|t}$ is the time t value of a unit of the consumption good in period $t+s$ to the household, which is treated as exogenous by the firm.

1.1.2 Households

The representative household derives utility from consumption C_t relative to a habit stock (which is approximated by the level of technology A_t)[1] and real money balances M_t/P_t.

[1] This assumption ensures that the economy evolves along a balanced growth path even if the utility function is additively separable in consumption, real money balances, and leisure.

The household derives disutility from hours worked H_t and maximizes

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{(C_{t+s}/A_{t+s})^{1-\tau}-1}{1-\tau}\right.\right. \\ \left.\left.+\chi_M \ln\left(\frac{M_{t+s}}{P_{t+s}}\right)-\chi_H H_{t+s}\right)\right], \tag{1.8}$$

where β is the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, and χ_M and χ_H are scale factors that determine steady state money balances and hours worked. We will set $\chi_H = 1$. The household supplies perfectly elastic labor services to the firms, taking the real wage W_t as given. The household has access to a domestic bond market where nominal government bonds B_t are traded that pay (gross) interest R_t. Furthermore, it receives aggregate residual real profits D_t from the firms and has to pay lump-sum taxes T_t. Thus, the household's budget constraint is of the form

$$P_tC_t + B_t + M_t + T_t \\ = P_tW_tH_t + R_{t-1}B_{t-1} + M_{t-1} + P_tD_t + P_tSC_t, \tag{1.9}$$

where SC_t is the net cash inflow from trading a full set of state-contingent securities.

1.1.3 Monetary and Fiscal Policy

Monetary policy is described by an interest rate feedback rule of the form

$$R_t = R_t^{*\,1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}}, \tag{1.10}$$

where $\epsilon_{R,t}$ is a monetary policy shock and R_t^* is the (nominal) target rate:

$$R_t^* = r\pi^*\left(\frac{\pi_t}{\pi^*}\right)^{\psi_1}\left(\frac{Y_t}{Y_t^*}\right)^{\psi_2}. \tag{1.11}$$

Here r is the steady state real interest rate (defined below), π_t is the gross inflation rate defined as $\pi_t = P_t/P_{t-1}$, and π^* is the target inflation rate. Y_t^* in (1.11) is the level of output that would prevail in the absence of nominal rigidities.

We assume that the fiscal authority consumes a fraction ζ_t of aggregate output Y_t, that is $G_t = \zeta_t Y_t$, and that $\zeta_t \in [0, 1]$

follows an exogenous process specified below. The government levies a lump-sum tax T_t (subsidy) to finance any shortfalls in government revenues (or to rebate any surplus). The government's budget constraint is given by

$$P_t G_t + R_{t-1} B_{t-1} + M_{t-1} = T_t + B_t + M_t. \tag{1.12}$$

1.1.4 Exogenous Processes

The model economy is perturbed by three exogenous processes. Aggregate productivity evolves according to

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \quad \ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t}. \tag{1.13}$$

Thus, on average technology grows at the rate γ and z_t captures exogenous fluctuations of the technology growth rate. Define $g_t = 1/(1 - \zeta_t)$, where ζ_t was previously defined as the fraction of aggregate output purchased by the government. We assume that

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}. \tag{1.14}$$

Finally, the monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. The three innovations are independent of each other at all leads and lags and are normally distributed with means zero and standard deviations σ_z, σ_g, and σ_R, respectively.

1.1.5 Equilibrium Relationships

We consider the symmetric equilibrium in which all intermediate goods producing firms make identical choices so that the j subscript can be omitted. The market clearing conditions are given by

$$Y_t = C_t + G_t + AC_t \quad \text{and} \quad H_t = N_t. \tag{1.15}$$

Because the households have access to a full set of state-contingent claims, it turns out that $Q_{t+s|t}$ in (1.7) is

$$Q_{t+s|t} = (C_{t+s}/C_t)^{-\tau} (A_t/A_{t+s})^{1-\tau}. \tag{1.16}$$

Thus, in equilibrium households and firms are using the same stochastic discount factor. Moreover, it can be shown that output, consumption, interest rates, and inflation have to satisfy the following optimality conditions:

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] & (1.17) \\ 1 &= \phi(\pi_t - \pi) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\pi}{2\nu} \right] & (1.18) \\ & \quad -\phi \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{Y_{t+1}/A_{t+1}}{Y_t/A_t} (\pi_{t+1} - \pi) \pi_{t+1} \right] \\ & \quad + \frac{1}{\nu} \left[1 - \left(\frac{C_t}{A_t} \right)^{\tau} \right]. \end{aligned}$$

Equation (1.17) is the consumption Euler equation which reflects the first-order condition with respect to the government bonds B_t. In equilibrium, the household equates the marginal utility of consuming a dollar today with the discounted marginal utility from investing the dollar, earning interest R_t, and consuming it in the next period. Equation (1.18) characterizes the profit maximizing choice of the intermediate goods producing firms. The first-order condition for the firms' problem depends on the wage W_t. We used the households' labor supply condition to replace W_t by a function of the marginal utility of consumption. In the absence of nominal rigidities ($\phi = 0$) aggregate output is given by

$$Y_t^* = (1 - \nu)^{1/\tau} A_t g_t, \qquad (1.19)$$

which is the target level of output that appears in the monetary policy rule (1.11).

In Section 2.1 of Chapter 2 we will use a solution technique for the DSGE model that is based on a Taylor series approximation of the equilibrium conditions. A natural point around which to construct this approximation is the steady state of the DSGE model. The steady state is attained by setting the innovations $\epsilon_{R,t}$, $\epsilon_{g,t}$, and $\epsilon_{z,t}$ to zero at all times. Because technology $\ln A_t$ evolves according to a random walk with drift $\ln \gamma$, consumption and output need to be detrended for a steady state to exist. Let $c_t = C_t/A_t$ and $y_t = Y_t/A_t$, and $y_t^* = Y_t^*/A_t$. Then the steady state is given by

$$\begin{aligned} \pi &= \pi^*, \quad r = \frac{\gamma}{\beta}, \quad R = r\pi^*, \\ c &= (1-\nu)^{1/\tau}, \quad y = gc = y^*. \end{aligned} \tag{1.20}$$

Steady state inflation equals the targeted inflation rate π_*; the real rate depends on the growth rate of the economy γ and the reciprocal of the households' discount factor β; and finally steady state output can be determined from the aggregate resource constraint. The nominal interest rate is determined by the Fisher equation; the dependence of the steady state consumption on ν reflects the distortion generated by the monopolistic competition among intermediate goods producers. We are now in a position to rewrite the equilibrium conditions by expressing each variable in terms of percentage deviations from its steady state value. Let $\hat{x}_t = \ln(x_t/x)$ and write

$$1 = \beta \mathbb{E}_t \left[e^{-\tau \hat{c}_{t+1} + \tau \hat{c}_t + \hat{R}_t - \hat{z}_{t+1} - \hat{\pi}_{t+1}} \right] \tag{1.21}$$

$$\begin{aligned} 0 &= \left(e^{\hat{\pi}_t} - 1\right) \left[\left(1 - \frac{1}{2\nu}\right) e^{\hat{\pi}_t} + \frac{1}{2\nu} \right] \\ &\quad - \beta \mathbb{E}_t \left[\left(e^{\hat{\pi}_{t+1}} - 1\right) e^{-\tau \hat{c}_{t+1} + \tau \hat{c}_t + \hat{y}_{t+1} - \hat{y}_t + \hat{\pi}_{t+1}} \right] \\ &\quad + \frac{1-\nu}{\nu \phi \pi^2} \left(1 - e^{\tau \hat{c}_t}\right) \end{aligned} \tag{1.22}$$

$$e^{\hat{c}_t - \hat{y}_t} = e^{-\hat{g}_t} - \frac{\phi \pi^2 g}{2} \left(e^{\hat{\pi}_t} - 1\right)^2 \tag{1.23}$$

$$\begin{aligned} \hat{R}_t &= \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1 \hat{\pi}_t \\ &\quad + (1-\rho_R)\psi_2 \left(\hat{y}_t - \hat{g}_t\right) + \epsilon_{R,t} \end{aligned} \tag{1.24}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_{g,t} \tag{1.25}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z,t}. \tag{1.26}$$

The equilibrium law of motion of consumption, output, interest rates, and inflation has to satisfy the expectational difference equations (1.21) to (1.26).

1.2 Other DSGE Models Considered in This Book

In addition to the small-scale New Keynesian DSGE model, we consider two other models: the widely used Smets-Wouters (SW) model, which is a more elaborate version of the small-scale DSGE model that includes capital accumulation as well as wage rigidities, and a real business cycle model with a detailed characterization of fiscal policy. We will present a brief overview of these models below and provide further details as needed in Chapter 6.

1.2.1 The Smets-Wouters Model

The Smets and Wouters (2007) model is a more elaborate version of the small-scale DSGE model presented in the previous section. In the SW model capital is a factor of intermediate goods production, and in addition to price stickiness the model features nominal wage stickiness. In order to generate a richer autocorrelation structure, the model also includes investment adjustment costs, habit formation in consumption, and partial dynamic indexation of prices and wages to lagged values. The model is based on work by Christiano, Eichenbaum, and Evans (2005), who added various forms of frictions to a basic New Keynesian DSGE model in order to capture the dynamic response to a monetary policy shock as measured by a structural vector autoregression (VAR). In turn (the publication dates are misleading), Smets and Wouters (2003) augmented the Christiano-Eichenbaum-Evans model by additional exogenous structural shocks (among them price markup shocks, wage markup shocks, preference shocks, and others) to be able to capture the joint dynamics of Euro Area output, consumption, investment, hours, wages, inflation, and interest rates.

The Smets and Wouters (2003) paper has been highly influential, not just in academic circles but also in central banks because it demonstrated that a modern DSGE model that is usable for monetary policy analysis can achieve a time series fit that is comparable to a less restrictive vector autoregression (VAR). The 2007 version of the SW model contains a number of minor modifications of the 2003 model in order to optimize its fit on U.S. data. We will use the 2007 model exactly as it is presented in Smets and Wouters (2007) and refer the reader to that article for details. The log-linearized equilibrium conditions are reproduced in Appendix A.1. By now, the SW model has become one of the workhorse models in the DSGE model literature and in central banks around the world. It forms the core of most large-scale DSGE models that augment the SW model with additional features such as a multi-sector production structure or financial frictions. Because of its widespread use, we will consider its estimation in this book.

1.2.2 A DSGE Model for the Analysis of Fiscal Policy

In the small-scale New Keynesian DSGE model and in the SW model, fiscal policy is passive and non-distortionary. The level of government spending as a fraction of GDP is assumed to evolve exogenously and an implicit money demand equation determines the amount of seignorage generated by the interest rate feedback rule. Fiscal policy is passive in the sense that the government raises lump-sum taxes (or distributes lump-sum transfers) to ensure that its budget constraint is satisfied in every period. These lump-sum taxes are non-distortionary, because they do not affect the decisions of households and firms. The exact magnitude of the lump-sum taxes and the level of government debt are not uniquely determined, but they also do not matter for macroeconomic outcomes. Both the small-scale DSGE model and the SW model were explicitly designed for the analysis of monetary policy and abstract from a realistic representation of fiscal policy.

In order to provide a careful and realistic assessment of the effects of exogenous changes in government spending and tax rates, a more detailed representation of the fiscal sector is necessary. An example of such a model is the one studied by Leeper, Plante, and Traum (2010). While the authors abstract from monetary policy, they allow for capital, labor, and consumption tax rates that react to the state of the economy, in particular the level of output and debt, and are subject to exogenous shocks, which reflect unanticipated changes in fiscal policy. In addition to consumption, investment, and hours worked, the model is also estimated based on data on tax revenues, government spending, and government debt to identify the parameters of the fiscal policy rules. The estimated model can be used to assess the effect of counterfactual fiscal policies.

We selected the fiscal policy model because during and after the 2007–09 recession, the DSGE model-based analysis of government spending and tax changes has received considerable attention and because the model gives rise to complicated, multi-modal posterior distributions which require sophisticated posterior samplers – such as the ones discussed in this book – to implement the Bayesian estimation. For a detailed model description we refer the reader to Leeper, Plante, and Traum (2010). The log-linearized equilibrium conditions are reproduced in Appendix A.2.