1

SHOULD YOU BE HAPPY?

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The following puzzle was tested on students from high school up to graduate level. What do you think?

> You are a rabid baseball fan and, miraculously, your team has won the pennant—thus, it gets to play in the World Series. Unfortunately, the opposition is a superior team whose probability of winning any given game against your team is 60%.
>
> Sure enough, your team loses the first game in the best-of-seven series, and you are so unhappy that you drink yourself into a stupor. When you regain consciousness, you discover that two more games have been played.
>
> You run out into the street and grab the first passer-by. "What happened in games two and three of the World Series?"
>
> "They were split," she says. "One game each."
>
> Should you be happy?

In an experiment, about half of respondents answered "Yes—if those games *hadn't* been split, your team would probably have lost them both."

The other half argued: "No—if your team keeps splitting games, they will lose the series. They have to do better."

Which argument is correct—and how do you verify the answer without a messy computation?

1 Comparing Probabilities

If "should you be happy" means anything at all, it should mean "are you better off than you were before?" In the above puzzle, the question comes down to: "Is your team's probability of winning the series better now, when it needs three of the next four games, than it was before, when it needed four out of six?"

Computing and comparing tails of binomial distributions is messy but not difficult; don't bother doing it, I'll give you the results later. The aim here is to suggest another method of attack, which is called *coupling*.

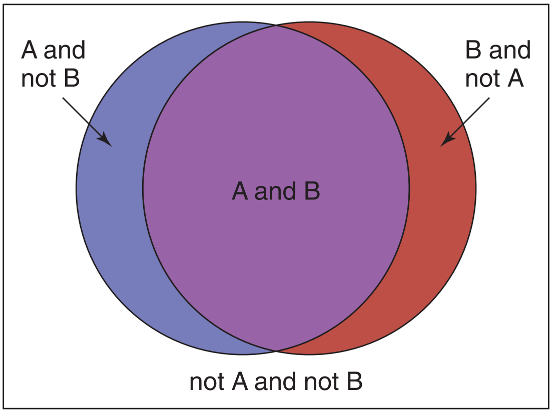

Figure 1.1. Comparing areas of the two crescents is equivalent to comparing areas of the disks.

The idea is, when you need to compare probabilities of two events A and B, to try to put them into the same experiment. All you need do is compare Pr(A but not B) with Pr(B but not A). This might be quite easy, especially if most of the time either both A and B occur or neither. The Venn diagram of Figure 1.1 illustrates the desired situation. If the blue region is larger than the red region, you deduce that A is more likely than B.

2 A Chess Problem, of Sorts

Let's try this on a problem adapted from Martin Gardner's legendary "Mathematical Games" column in *Scientific American*.[1] You want to join a certain chess club, but admission requires that you play three games against Ioana, the current club champion, and win two in a row.

Since it is an advantage to play the white pieces (which move first), you alternate playing white and black.

A coin is flipped, and the result is that you will be white in the first and third games, black in the second.

Should you be happy?

Gardner gave a (correct) algebraic proof of the answer ("no"), but, acknowledging the value of a proof by reasoning, he also provided two arguments that you'd be better off playing black first: (1) You must win the crucial middle

[1] A great source for this and lots of other thought-provoking puzzles is M. Gardner, *The Colossal Book of Short Puzzles and Problems*, W. W. Norton & Co., New York, 2006. See especially Problems 2.10 and 2.12.

game, thus you want to be playing white second; and (2) You must win a game as black, so you're better off with two chances to do so.

In fact, neither argument is convincing, and even together they are not a proof.

Using coupling, you can get the answer without algebra—even if the problem is modified so that you have to win two in a row out of *seventeen* games, or m in a row out of n. (If n is even, it doesn't matter who plays white first; if n is odd, you want to be black first when m is odd, white first when m is even.)

The coupling argument in the original two-out-of-three puzzle goes like this: Imagine that you are going to play four games against Ioana, playing white, then black, then white, then black. You still need to win two in a row, but you must decide in advance whether to discount the first game, or the last.

Obviously turning the first game into a "practice game" is equivalent to playing BWB in the original problem, and failing to count the last game is equivalent to playing WBW, so the new problem is equivalent to the old one.

But now the events are on the same space. For it to make a difference which game you discounted, the results must be either WWLX or XLWW. In words: if you win the first two games, and lose (or draw) the third, you will wish that you had discounted the last game; if you lose the second but win the last two, you will wish that you had discounted the first game.

But it is easy to see that XLWW is more likely than WWLX. The two wins in each case are one with white and one with black, so those cases balance; but the loss in XLWW is with black, more likely than the loss in WWLX with white. So you want to discount the first game (i.e., start as black in the original problem).

A slightly more challenging version of this argument works if you change the number of games played, and/or the number of wins needed in a row.

3 Back to Baseball

Let's first "do the math" and see whether you should be happy about splitting games two and three. Before the news, your team needed to win four, five, or six of the next six games. (Wait, what if fewer than seven games are played? Not to worry; we are safe in imagining that all seven games are played no matter what. It doesn't make any difference if the series is stopped when one team registers its fourth win; that, in fact, is why the rest of the games are canceled.)

The probability that your team wins exactly four of six games is "6 choose 4" (the number of ways that can happen) times $(2/5)^4$ (the probability that your team wins a particular four games) times $(3/5)^2$ (the probability that the other guys win the other two). Altogether, the probability that your team wins at least four of six is

$$\binom{6}{4}\left(\frac{2}{5}\right)^4\left(\frac{6}{5}\right)^2+\binom{6}{5}\left(\frac{2}{5}\right)^5\left(\frac{6}{5}\right)+\binom{6}{6}\left(\frac{2}{5}\right)^6=\frac{112}{625}.$$

After the second and third games are split, your team needs at least three of the remaining four. The probability of winning is now

$$\binom{4}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)+\binom{4}{4}\left(\frac{2}{5}\right)^4=\frac{112}{625}.$$

So you should be entirely indifferent to the news! The two arguments (one backward-looking, suggesting that you should be happy, the other forward-looking, suggesting that you should be unhappy) seem to have canceled each other precisely. Can this be a coincidence? Is there any way to get this result "in your head?"

Of course there is. To do the coupling a little imagination helps. Suppose that, after game three, it is discovered that an umpire who participated in games two and three had lied on his application. There is a movement to void those two games, and a countermovement to keep them. The commissioner of baseball, wise man that he is, appoints a committee to decide what to do with the results of games two and three; and in the meantime, he tells the teams to keep playing.

Of course, the commissioner hopes—and so do we puzzle-solvers—that *by the time the committee makes its decision, the question will be moot.*

Suppose five(!) more games are played before the committee is ready to report. If your team wins four or five of them, it has won the series regardless of the disposition of the second and third game results. On the other hand, if the opposition has won three or more, they have won the series regardless. The only case in which the committee can make a difference is when your team has won exactly three of those five new games.

In that case, if the results of games two and three are voided, one more game needs to be played; your team wins the series if they win that game, which happens with probability 2/5.

If, on the other hand, the committee decides to count games two and three, the series ended before the fifth game, and whoever *lost* that game is the World Series winner. Sounds good for your team, no? Oops, remember that your team won three of those five new games, so the probability that the last game is among the two they lost is again 2/5. Voilà !

4 Coin-Flipping and Dishwashing

Here's another probability-comparison puzzle, again adapted from Martin Gardner. This one yields to a variation of the coupling arguments you've seen above.

You and your spouse flip a coin to see who washes the dishes each evening. "Heads" he washes, "tails" you wash.

Tonight he tells you he is imposing a different scheme. You flip the coin thirteen times, then he flips it twelve times. If you get more heads than he does, he washes; if you get the same number of heads or fewer, you wash.

Should you be happy?

Here, the easy route is to imagine that first you and your spouse flip just twelve times each. If you get different numbers of heads, the one with fewer will be washing dishes regardless of the outcome of the next flip; so those scenarios cancel. The rest of the time, when you tie, the final flip will determine the washer. So it's still a fifty-fifty proposition, and you should be indifferent to the change in procedure (unless you dislike flipping coins).

5 Application to Squash Strategy

Squash, or "squash racquets," is a popular sport in Britain and its former colonies, probably familiar to some readers and not to others. The game is played by two players (usually) in a rectangular walled box, with slender racquets and a small black ball. As in tennis, table tennis, and racquetball, one player puts a ball in play by serving, and then the last to make a legal shot wins the rally.

In squash (using English scoring, sometimes known as "hand-out" scoring), a point is added to the rally-winner's score only when he (or she) has served. If the rally is won by the receiver, no point is scored, but the right to serve changes hands. The game is won normally by the first player to score nine points, but there is an exception: if the score reaches eight-all, the player *not* serving has the right to "set two," meaning that he may change the target score to ten points instead of nine. If he does not exercise this right, he has "set one" and the game continues to be played to nine. This choice is final; that is, even though the score may remain at eight-all for a while, no further decisions are called for.

The question (asked by Pradeep Mutalik at the recent Eleventh Gathering for Gardner, but no doubt asked by many before him) is: if you are in that position, should you set two, or just continue to play for nine points?

Simplifying the situation here (as opposed to tennis) is the fact that in squash, especially in the British ("soft ball") form of the game, having the service has almost no effect on the probability of winning the subsequent rally. Serving usually just puts the ball in play, and many strokes are likely to follow.

Thus, it is reasonable here to assume that you (the player not serving if and when eight-all is reached) have some fixed probability p of winning any given rally, regardless of who is serving. Further, the outcome of each rally can be assumed to be independent of all other rallies, and of the outcome of your decision to set one or set two.

The intuition here is quite similar to that in the World Series problem. If $p = \frac{1}{2}$, it seems clear that you want to set two, in order to minimize the service advantage. To put it another way, you are in imminent danger of losing the next rally and thus the game if you set one; this would appear to be the dominant factor in your decision.

However, if p is small, the fact that the longer game favors your (superior) opponent comes into play, and it should be better to keep the target at nine points and try to get lucky. If these arguments are correct, there ought to be some threshold value p_c (like the 40% of the World Series problem) at which you are indifferent: when $p > p_c$ you should set two, and when $p < p_c$ you should set one. What is the value of p_c?

You can solve this problem in principle by computing your probability of winning (as a function of p) when you set one, and again when you set two, then comparing those two values. This takes some work; there are infinitely many ways the squash game can continue, so you'll need to sum some infinite series or solve some equations. As before, however, we can minimize the work (and perhaps gain some insight) by coupling the two scenarios and concentrating on the circumstances in which your decision makes a difference.

Accordingly, let us assume that the game is played until someone scores ten points, even though the winner will have been determined earlier if you chose to set one. If "set one" beats "set two," that is, if you would have won if you had set one but lost if you had set two, it must be that you scored the next point but ended up with nine points to your opponent's ten. Call this event S_1.

For set two to beat set one, your opponent must be the one to reach 9-8 but then yield the next two points to you; call this event S_2.

It will be useful to have another parameter. There are several choices here, all about equally good. Let f be the probability that you "flip": that is, you score the next point even though you are not currently serving. (Notice that f applies to you only; your opponent's probability of flipping will be different unless $p = \frac{1}{2}$.)

To flip you must win the next rally, then either win the second as well, or lose it and flip. Thus

$$f = p(p + (1-p)f),$$

which we could solve for f, but let's leave it in this form for now.

Event S_1 requires that you flip, then lose the next rally, then fail twice to flip: in symbols,

$$\Pr(S_1) = f(1-p)(1-f)^2.$$

Event S_2 requires that you fail to flip, then flip, then either win the next rally or lose and then flip; thus

$$\Pr(S_2) = (1 - f) f (p + (1-p) f).$$

Both events require a flip and a failure to flip, so we can divide out by $f(1-f)$ and just compare $(1-p)(1-f)$ with $p + (1-p)f$, but notice that the latter is just f/p (from our equation for f), and the former is one minus the latter. Thus $f/p = \frac{1}{2}$, from which we get $2p = 1 - p + p^2$, $p = (3 - \sqrt{5})/2 \sim .381966011$.

Thus, you should set two unless your probability of winning a rally is less than 38%. Since most games between players as mismatched as 62% : 38% will not reach the score of eight-all, you would not be far wrong simply to make it a policy to set two. Squash players' intuitions are apparently trustworthy: in tournament play, at least, set one has been a fairly rare choice.

English scoring is being gradually replaced by "PARS" (point-a-rally scoring) in which a point is scored regardless of whether the rally was won by the server or receiver. In PARS the game is played to eleven points, but perhaps partly in acknowledgment of the strategic facts explicated above, set two is in effect made automatic by requiring the winner to win by two points.

An important advantage to PARS is that the number of rallies in a game is somewhat less variable than in English scoring, thus it is easier to schedule matches in a tournament. If you are the underdog looking to register an upset and are permitted to select the scoring system for the game, which of the two should you choose?

I'll leave that to you.

Acknowledgement

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