Chapter One

Buildings

We use this chapter to assemble a few standard definitions, fix some notation and review a few of the results about buildings and Moufang polygons which will be used most frequently in these notes.

A summary of the basic facts about Coxeter groups and buildings with which we expect the reader to have some familiarity can be found, with references to proofs, in [65, 29.1-29.15]. These include the basic properties of roots, residues, apartments and projection maps. (We emphasize, however, that although we assume some familiarity with this background material, we have made every effort throughout these notes to include explicit references to the results in [60], [62], [65] and elsewhere each time they are applied.)

When we refer to the *type* of a building Δ, we mean either the corresponding Coxeter diagram or, equivalently, the corresponding Coxeter system (W, S); see 19.2. The cardinality $|S|$, which we always assume to be finite, is called the *rank* of Δ. More generally, the *rank* of a J-residue of Δ is $|J|$ for each subset J of S.

Root groups and the Moufang condition play a central role in this monograph. A *root* of a building is a root of one of its apartments. For a given root α of a building Δ, the corresponding *root group* U_α is the subgroup of $\mathrm{Aut}(\Delta)$ consisting of all elements that act trivially on each panel containing two chambers of α.

Definition 1.1. As in [62, 11.7], we say that a building Δ is *Moufang* (or *satisfies the Moufang condition*) if

(i) it is thick, irreducible and spherical as defined in [62, 1.6 and 7.10];

(ii) its rank is at least 2; and

(iii) for every root α, the root group U_α acts transitively on the set of all apartments containing α.

We emphasize that if we say that a building is Moufang, we are implying that it is spherical, thick, irreducible and of rank at least 2. Nevertheless, when we say that a building is Moufang, we will sometimes also say explicitly that the building is spherical just to avoid any possible confusion. (In Chapter 24 we introduce the more general notion of a Moufang *structure* on a spherical building. See also [1, 8.3] and [44, Chapter 6, §4, and Chapter 11, §7] for other notions of a Moufang building. These other notions will not play any role in these notes.)

Definition 1.2. Let Δ be a building, let R be a residue of Δ and let c be an arbitrary chamber of Δ. By [62, 8.21], there is a unique chamber z in R nearest c and

$$\mathrm{dist}(x, c) = \mathrm{dist}(x, z) + \mathrm{dist}(z, c) \tag{1.3}$$

for every chamber $x \in R$. This unique nearest chamber z is called the *projection* of c to R and is denoted by $\mathrm{proj}_R(c)$. The *projection map* $\mathrm{proj}_R \colon \Delta \to R$ will play an important role in these notes starting in Chapter 21.

Remark 1.4. A fundamental result of Tits says that an irreducible thick spherical building of rank at least 3 satisfies the Moufang condition as do all the irreducible residues of rank at least 2 of such a building. For a proof, see [62, 11.6 and 11.8].

Moufang sets.

A building of type A_1—in other words, a building of rank 1—is only a set of cardinality at least 2 without any further structure, but the buildings of type A_1 we will encounter come endowed with a group of permutations having special properties which led to the following definition introduced by Tits in [58]:

Definition 1.5. A *Moufang set* is a pair $(X, \{U_x \mid x \in X\})$, where X is a set with $|X| \geq 3$ and for each $x \in X$, U_x is a subgroup of $\mathrm{Sym}(X)$ (where we compose from right to left) such that the following hold:

(i) For each $x \in X$, U_x fixes x and acts sharply transitively on $X \backslash \{x\}$.

(ii) For all $x, y \in X$ and each $g \in U_x$, $gU_yg^{-1} = U_{g(y)}$.

The groups U_x for $x \in X$ are called the *root groups* of the Moufang set.

Definition 1.6. Let $\mathbb{M} = (X, \{U_x \mid x \in X\})$ be a Moufang set and let

$$G = \langle U_x \mid x \in X \rangle.$$

By 1.5(i), the group G acts 2-transitively on X and by 1.5(ii), the root groups are all conjugate to each other in G. Let x, y be distinct elements of X. For each $g \in U_x^* := U_x \backslash \{1\}$, there exist a unique element $\mu_{xy}(g)$ in the double coset

$$U_y g U_y$$

that interchanges x and y. Thus $\mu := \mu_{xy}$ is a map from U_x^* to G which depends on the choice of x and y. By [19, 3.1(ii)], the stabilizer G_{xy} is generated by the set

$$\{\mu(g_1)\mu(g_2) \mid g_1, g_2 \in U_x\}.$$

Since U_x acts sharply transitively on $X \backslash \{x\}$, the subgroup G_{xy} is isomorphic to the subgroup of $\mathrm{Aut}(U_x)$ it induces. The *tori* of $\mathbb{M}$ are the conjugates in G of the subgroup G_{xy}. Since G acts 2-transitively on X, the tori are precisely the 2-point stabilizers in G.

Definition 1.7. Two Moufang sets

$$(X, \{U_x \mid x \in X\}) \text{ and } (X', \{U_x \mid x \in X'\})$$

are *isomorphic* if there exists a bijection from X to X' that transports root groups to root groups (and such a bijection is called an *isomorphism*).

Definition 1.8. Let

$$\mathbb{M} = (X, \{U_x \mid x \in X\}) \text{ and } \mathbb{M}' = (X', \{U_x \mid x \in X'\})$$

be two Moufang sets and let x, y be an ordered pair of distinct elements of X. An *xy-isomorphism* from $\mathbb{M}$ to $\mathbb{M}'$ is a bijection ψ from X to X' inducing an isomorphism φ from U_x to $U'_{\psi(x)}$ such that

$$\varphi(u^{\mu(a)\mu(b)}) = \varphi(u)^{\mu'(\varphi(a))\mu'(\varphi(b))} \tag{1.9}$$

for all $u \in U_x$ and all $a, b \in U_x^*$, where $\mu = \mu_{xy}$ and $\mu' = \mu_{\varphi(x)\varphi(y)}$ are as in 1.6 with respect to $\mathbb{M}$, respectively, $\mathbb{M}'$. If x_1, y_1 is another ordered pair of distinct elements of X, then there is an element g in the group G defined in 1.6 mapping the ordered pair x, y to the ordered pair x_1, y_1 and the composition of g with an xy-isomorphism from $\mathbb{M}$ to $\mathbb{M}'$ is an x_1y_1-isomorphism from $\mathbb{M}$ to $\mathbb{M}'$. We will say that $\mathbb{M}$ and $\mathbb{M}'$ are *weakly isomorphic* (and write $\mathbb{M} \approx \mathbb{M}'$) if there is an xy-isomorphism from $\mathbb{M}$ to $\mathbb{M}'$ for some choice of x, y in X (and hence for all choices of x, y in X), and we define a *weak isomorphism* from $\mathbb{M}$ to $\mathbb{M}'$ to be an xy-isomorphism for some choice of x, y in X. The inverse of a weak isomorphism is a weak isomorphism as is the composition of two weak isomorphisms, and every isomorphism of Moufang sets is also a weak isomorphism.

Remark 1.10. Let $\mathbb{M}$, $\mathbb{M}'$, etc., be as in 1.8, let x, y be an ordered pair of distinct elements of X, let x', y' be an ordered pair of distinct elements of X' and suppose that φ is an isomorphism from U_x to $U'_{x'}$ such that (1.9) holds for all $u \in U_x$ and all $a, b \in U_x^*$ with $\mu = \mu_{xy}$ and $\mu' = \mu_{x'y'}$. Then the map ψ from X to X' which sends x to x' and y^u to $(y')^{\varphi(u)}$ for all $u \in U_x$ is an xy-isomorphism from $\mathbb{M}$ to $\mathbb{M}'$.

Notation 1.11. Let $\mathbb{M} = (X, \{U_x \mid x \in X\})$ be a Moufang set, choose distinct points x, y in X, let $\mu = \mu_{xy}$ be the map from U_x^* to $\mathrm{Aut}(\mathbb{M})$ defined in 1.6, choose $a \in U_x^*$ and let $m = \mu(a)$. There exists a unique permutation ρ of U_x^* such that

$$y^{u^\rho} = x^{m^{-1}um}$$

for all $u \in U_x^*$. Therefore

$$y^{u^\rho} = (y^u)^m \tag{1.12}$$

for all $u \in U_x^*$ since m interchanges x and y. We identify U_x with the set $X\backslash\{x\}$ via the map $u \mapsto y^u$, then we identify ρ with the permutation $y^u \mapsto y^{u^\rho}$ of $X\backslash\{x, y\}$ and finally we extend ρ to a permutation of X by declaring that it interchanges x and y. Given these identifications, it follows from (1.12) that the permutations m and ρ of X are the same. In particular $\langle U_x, \rho\rangle = \langle U_x, U_y\rangle$. Since this group acts transitively on X, it acts transitively on the set of root groups $\{U_z \mid z \in X\}$. It follows that $\mathbb{M}$ is uniquely determined by U_x and ρ (although ρ, of course, depends on the choice of a). We can thus set

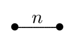
$$\mathbb{M} = \mathbb{M}(U_x, \rho). \tag{1.13}$$

This is the point of view taken in [17] and [19].

See 3.9 for examples of various families of Moufang sets described in terms of a single root group and a permutation of its non-trivial elements as in 1.11.

Moufang polygons and root group sequences.

A *generalized n-gon* (for $n \geq 2$) is a building of type



and a *generalized polygon* is a generalized n-gon for some n. See [62, 7.14 and 7.15] for an equivalent definition in terms of bipartite graphs. The classification of generalized n-gons satisfying the Moufang conditions (i.e. of *Moufang polygons*) was carried out in [60]. Moufang n-gons exist, in particular, only for $n = 3$, 4, 6 and 8. The classification says that each Moufang n-gon is uniquely determined by a root group sequence Ω as defined in [60, 8.7], and these root group sequences are, in turn, determined by certain algebraic data and isomorphisms $x_1, \ldots, x_n$ from this algebra data to the root groups from which Ω is composed according to one of the nine recipes [60, 16.1–16.9].

Notation 1.14. In accordance with [65, 30.8], we will use the following names for the root group sequences obtained by applying the recipes [60, 16.1–16.9]:

(i) $\mathcal{T}(K)$, where K is a field or a skew field or an octonion division algebra as defined in [60, 9.11].

(ii) $\mathcal{Q}_{\mathcal{I}}(\Lambda)$, where $\Lambda = (K, K_0, \sigma)$ is an involutory set as defined in [60, 11.1].

(iii) $\mathcal{Q}_{\mathcal{Q}}(\Lambda)$, where $\Lambda = (K, L, q)$ is a non-trivial anisotropic quadratic space as defined in 2.1 (see 2.14).

(iv) $\mathcal{Q}_{\mathcal{D}}(\Lambda)$, where $\Lambda = (K, K_0, L_0)$ is an indifferent set as defined in [60, 10.1].

(v) $\mathcal{Q}_{\mathcal{P}}(\Lambda)$, where $\Lambda = (K, K_0, \sigma, L, q)$ is an anisotropic pseudo-quadratic space as defined in [60, 11.17].

(vi) $\mathcal{Q}_{\mathcal{E}}(\Lambda)$, where $\Lambda = (K, L, q)$ is a quadratic space of type E_6, E_7 or E_8 as defined in 8.1.

(vii) $\mathcal{Q}_{\mathcal{F}}(\Lambda)$, where $\Lambda = (K, L, q)$ is a quadratic space of type F_4 as defined in 9.1.

(viii) $\mathcal{H}(\Lambda)$, where $\Lambda = (J, F, \#)$ is an hexagonal system as defined in [60, 15.15].

(ix) $\mathcal{O}(\Lambda)$, where $\Lambda = (K, \sigma)$ is an octagonal system as defined in [60, 10.11].

Notation 1.15. We will say that a root group sequence is of *of type $\mathcal{T}$* if it is isomorphic to a root group sequence in case (i) of 1.14, *of type $\mathcal{Q}_{\mathcal{I}}$* or *of involutory type* if it is isomorphic to a root group sequence in case (ii), *of type $\mathcal{Q}_{\mathcal{Q}}$* or *of quadratic form type* if it is isomorphic to a root group sequence in case (iii), etc.

Among all the Moufang polygons, the *exceptional Moufang quadrangles*—those corresponding to a root group sequence of type $\mathcal{Q}_{\mathcal{E}}$ or $\mathcal{Q}_{\mathcal{F}}$—are the most extraordinary. They will be the focus of our attention in Parts 2 and 5 of this monograph.

Let c be a chamber of a Moufang spherical building Δ and let $E_2(c)$ denote the subgraph spanned by all the irreducible rank 2 residues of Δ. Another fundamental result of Tits ([62, 10.16]) says that Δ is uniquely determined by $E_2(c)$. The irreducible rank 2 residues containing c, which are in one-to-one correspondence with the edges of the Coxeter diagram of Δ, are Moufang polygons. Thus each of these residues is determined by a root group sequence. This leads to the notion of a *root group labeling* of the Coxeter diagram Π. In a root group labeling, the edges of Π are decorated with root group sequences and the vertices with isomorphisms identifying certain root groups of the root group sequences decorating the different adjacent edges. A description of the results of Tits' classification of Moufang spherical buildings in terms of root group labelings is given in [65, 30.14]. In these notes we will apply the corresponding notation for these buildings as given in [65, 30.15]. Thus, in particular:

Remark 1.16. In the notion in [65, 30.15], the Moufang quadrangles corresponding to the first eight cases of 1.14 are, in order, called: $\mathsf{A}_2(K)$, $\mathsf{B}_2^{\mathcal{I}}(\Lambda)$ or $\mathsf{C}_2^{\mathcal{I}}(\Lambda)$, $\mathsf{B}_2^{\mathcal{Q}}(\Lambda)$ or $\mathsf{C}_2^{\mathcal{Q}}(\Lambda)$, $\mathsf{B}_2^{\mathcal{D}}(\Lambda)$ or $\mathsf{C}_2^{\mathcal{D}}(\Lambda)$, $\mathsf{B}_2^{\mathcal{P}}(\Lambda)$ or $\mathsf{C}_2^{\mathcal{P}}(\Lambda)$, $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$ or $\mathsf{C}_2^{\mathcal{E}}(\Lambda)$, $\mathsf{B}_2^{\mathcal{F}}(\Lambda)$ or $\mathsf{C}_2^{\mathcal{F}}(\Lambda)$, and $\mathsf{G}_2(\Lambda)$.

Remark 1.17. Let Ω' be a subsequence of a root group sequence Ω as defined in [60, 8.17]. By [60, 7.4 and 8.1], the generalized polygon associated with Ω' is a subbuilding of the generalized polygon associated with Ω. Suppose, for example, that $\Lambda' = (F, A, B)$ is an indifferent set. Then $\Lambda := (F, F, F)$ is an indifferent set containing Λ' canonically as a "sub"-indifferent set and by [60, 8.12] and the formulas in [60, 32.8], $\mathcal{Q}_{\mathcal{D}}(\Lambda')$ is a subsequence of the root group sequence $\mathcal{Q}_{\mathcal{D}}(\Lambda)$. Hence $\mathsf{B}_2^{\mathcal{D}}(\Lambda')$ is a subbuilding of $\mathsf{B}_2^{\mathcal{D}}(\Lambda)$. As a second example, let Λ be the involutory set (E, F, σ), where E/F is a separable quadratic extension and σ is the non-trivial element of $\mathrm{Gal}(E/F)$. Then $\Lambda' := (F, F, \mathrm{id}_F)$ is canonically a "sub"-involutory set of Λ and by the formulas in [60, 32.6 and 32.9] $\mathcal{Q}_{\mathcal{I}}(\Lambda')$ is a subsequence of the root group sequence $\mathcal{Q}_{\mathcal{I}}(\Lambda)$. Hence $\mathsf{B}_2^{\mathcal{I}}(\Lambda')$ is a subbuilding of $\mathsf{B}_2^{\mathcal{I}}(\Lambda)$. It follows from [65, 30.16] that $\mathsf{B}_\ell^{\mathcal{I}}(\Lambda')$ is, in fact, a subbuilding of $\mathsf{B}_\ell^{\mathcal{I}}(\Lambda)$ for all $\ell \geq 3$.

Notation 1.18. Let Δ be a Moufang spherical building, let Σ be an apartment of Δ, let $\Upsilon = \Upsilon_\Sigma$ be the set of all roots of Σ and let $G = \mathrm{Aut}(\Delta)$. We denote by $G^\dagger$ the subgroup of G generated by all the root groups of Δ. By [62, 11.22], there exists a map

$$\mu_\Sigma\colon \bigcup_{\alpha\in\Upsilon} U_\alpha^* \to G^\dagger$$

such that for each $\alpha \in \Upsilon$ and for each non-trivial element g in the root group U_α, $\mu_\Sigma(g)$ is the unique element in the double coset

$$U_{-\alpha} g U_{-\alpha}$$

which maps Σ to itself. Here $-\alpha$ denotes the root of Σ opposite α (i.e. the complement of α in Σ regarded as a set of chambers). The *wall* of α is the set of all panels of Δ containing one chamber in α and one in $-\alpha$. If $\alpha \in \Upsilon$, then by [62, 3.13], there is a unique automorphism s_α of Σ that stabilizes every panel in the wall of α and interchanges α with $-\alpha$. We have $s_\alpha = s_{-\alpha}$ and $s_\alpha^2 = 1$ for all $\alpha \in \Upsilon$. A *reflection* of Σ is an automorphism of the form s_α for some $\alpha \in \Upsilon$. For each $\alpha \in \Upsilon$ and each $g \in U_\alpha^*$, the element $\mu_\Sigma(g)$ induces s_α on Σ (but is not necessarily of order 2). See 19.15 below.

Notation 1.19. Let Δ and $G^\dagger$ be as in 1.18, let P be a panel of Δ and let G_P be the stabilizer of P in $G^\dagger$. We choose a chamber x in P and an apartment Σ containing x and let α denote the unique root of Σ containing x but not $P \cap \Sigma$. By [62, 11.4], the root group U_α acts sharply transitively and, in particular, faithfully on $P\backslash\{x\}$. Let U_x denote the image of U_α in $\mathrm{Sym}(P)$ and let U_x^+ denote the group generated by U_β for all roots β of Σ containing x. If β is a root of Σ containing x other than α, then U_β acts trivially on P. By [62, 11.11(ii)], the group U_x^+ acts transitively on the set of apartments containing x. It follows that the permutation group U_x is independent of the choice of the apartment Σ. Thus $gU_xg^{-1} = U_{g(x)}$ for all $x \in P$ and all $g \in G_P$ and hence the pair

$$\mathbb{M}_{\Delta,P} := (P, \{U_x \mid x \in P\})$$

is a Moufang set as defined in 1.5 and

$$\overline{\mu_\Sigma(g)} = \mu_{xy}(\bar{g})$$

for all $g \in U_\alpha$, where μ_Σ is as in 1.18, y is the unique chamber of $P \cap \alpha$ other than x, μ_{xy} is as in 1.6, $\bar{g}$ denotes the image of g in U_x and $\overline{\mu_\Sigma(g)}$ denotes the image of $\mu_\Sigma(g)$ in $\mathrm{Sym}(P)$.

Bruhat-Tits buildings.

In these notes, we use the term "Bruhat-Tits building" in the sense introduced in [65]:

Definition 1.20. A *Bruhat-Tits building* is a thick irreducible affine building whose building at infinity is Moufang. The *building at infinity* of an affine building is constructed in [65, Chapter 8]. By 1.1, the building at infinity of a Bruhat-Tits building is spherical, irreducible and thick.

Assumption 1.21. By [65, 27.4], there is no loss in generality if we assume that the building at infinity Ξ^∞ of a Bruhat-Tits building Ξ is formed with respect to the complete system of apartments of Ξ (as defined in [65, 8.5]) and we will always do this in these notes.

Let Ξ be a Bruhat-Tits building. The type of Ξ is an irreducible affine Coxeter diagram $\tilde{X}_\ell$ for some $\ell \geq 2$ and for $X = A,\ B, \ldots,\ F$ or G (see 20.41) and the type of Ξ^∞ is X_ℓ. By [65, 27.5], the algebraic data corresponding to the Moufang building Ξ^∞ is defined over a field or a skew-field or an octonion division algebra K which is complete with respect to a discrete valuation. Tits showed (see [65, 27.6]) that Ξ is uniquely determined by Ξ^∞ and completed the classification of Bruhat-Tits buildings by determining exactly which Moufang buildings can appear as the building at infinity (see [65, 27.5]).

Notation 1.22. We will apply the notation for Bruhat-Tits buildings given in the fourth column of Table 27.2 in [65] except that we suppress the reference to the valuation of K since we are assuming that the system of apartments $\mathcal{A}$ is complete, hence that the field or skew-field or octonion division algebra K is complete and hence that the discrete valuation of K is unique (by [65, 23.15]). Thus, for example, $\tilde{\mathsf{A}}_2(K)$ denotes the unique Bruhat-Tits building whose building at infinity is $\mathsf{A}_2(K)$, $\tilde{\mathsf{B}}_2^{\mathcal{E}}(\Lambda) = \tilde{\mathsf{C}}_2^{\mathcal{E}}(\Lambda)$ denotes the unique Bruhat-Tits building whose building at infinity is $\mathsf{B}_2^{\mathcal{E}}(\Lambda) = \mathsf{C}_2^{\mathcal{E}}(\Lambda)$, etc.

Remark 1.23. If $\Xi = \tilde{\mathsf{X}}_\ell^*(\Lambda)$ in the notation described in 1.22, then Ξ^∞ is obtained by simply removing the tilde. Note, however, that the spherical Coxeter diagrams B_ℓ and C_ℓ are the same for all $\ell \geq 2$ as are the affine Coxeter diagrams $\tilde{B}_2$ and $\tilde{C}_2$, but that the affine Coxeter diagrams $\tilde{B}_\ell$ and $\tilde{C}_\ell$ are not the same when $\ell > 2$. As a consequence, the inverse of the process of "deleting the tilde" is not so straightforward when $X = B$ or C and $\ell > 2$. Suppose, for example, that

$$\Delta = \mathsf{B}_\ell^{\mathcal{Q}}(\Lambda) = \mathsf{C}_\ell^{\mathcal{Q}}(\Lambda)$$

for some $\ell \geq 2$ and some anisotropic quadratic space $\Lambda = (K, L, q)$ such that K is complete with respect to a discrete valuation ν and $1 \in q(L)$. Then by [65, 19.23], the unique Bruhat-Tits building whose building at infinity is Δ is $\tilde{\mathsf{X}}_\ell^{\mathcal{Q}}(\Lambda)$, where $\mathsf{X} = \mathsf{B}$ if $\nu(q(L^*)) = 2\mathbb{Z}$ and $\mathsf{X} = \mathsf{C}$ if $\nu(q(L^*)) = \mathbb{Z}$. Similar results hold in the other cases; see [65, 27.2].

Definition 1.24. As observed in [65, 30.33], a Moufang building can be *mixed* as defined in [65, 30.24] (see also 28.3), *algebraic* or *exceptional* as defined in [65, 30.32] or *classical* as defined in [65, 30.30]. (If it is exceptional, it is automatically algebraic, if it is algebraic, then it is either exceptional or classical and if it is not algebraic, then it is either classical or mixed.) We will say that a Bruhat-Tits building is mixed, exceptional, classical, respectively, algebraic if its building at infinity is mixed, exceptional, classical, respectively, algebraic.

Remark 1.25. Let F be a field complete with respect to a discrete valuation and let G be an absolutely simple algebraic group. If the F-rank of G is 1, then

$$(\Delta, \{U_x \mid x \in \Delta\})$$

is a Moufang set, where Δ is the set of parabolic subgroups of $G(F)$ and U_x is the unipotent radical of x for each $x \in \Delta$, and there exists a tree Ξ whose set Ξ^∞ of ends is Δ to which the action of the groups U_x can be extended. These trees together with Bruhat-Tits buildings in our sense are precisely the affine buildings that were investigated in [7] (together with certain non-discrete generalizations).

The following result should have been formulated explicitly in [65]:

Theorem 1.26. *Let Ξ be a Bruhat-Tits building. Then every automorphism of $\Delta := \Xi^\infty$ is induced by a unique automorphism of Ξ. In other words,* $\mathrm{Aut}(\Xi)$ *and* $\mathrm{Aut}(\Delta)$ *are canonically isomorphic.*

Proof. By [65, 13.10 and 13.31], it suffices to show that any two valuations of the root datum of Δ are equipollent. Let K (or $\{K, K^{\mathrm{op}}\}$ or $\{K, E\}$) be the defining field of Δ in the sense of [65, 30.29]. By [65, 27.5] K is complete with respect to a discrete valuation. As was observed in 1.22, the discrete valuation ν of K is unique. By [65, 19.4, 23.16, 24.9 and 25.5], the parameter system defining Δ is ν-compatible as defined in the references in the second column of [65, Table 27.2]. By [65, 3.41(iii) and 16.4] combined with the results [65, 20.2(ii), 21.27(ii) and 22.16(ii)], it follows that any two valuations of the root datum of Δ are equipollent as claimed. □

Remark 1.27. We allow ourselves, in light of 1.26, to identify the automorphism group of a Bruhat-Tits building with the automorphism group of its building at infinity. Note, however, that the Coxeter diagrams of Ξ and Δ are, of course, different, and it can happen (see [65, 18.1]) that the isomorphism in 1.26 carries non-type-preserving automorphisms of Ξ to type-preserving elements of Δ.

Simplicial complexes.

In the original definition given in [55], a building is a simplicial complex, but in these notes (as in [62] and [65]), we view buildings as certain edge-colored graphs and the residues as certain subgraphs. See [62, 1.2 and 7.1] for the precise definitions. The vertices of these graphs are called *chambers* and when we write, for example, $c \in \Delta$ or $c \in R$ or $c \in \Sigma$ or $c \in \alpha$, we mean that

c is a chamber of the building Δ or the residue R or the apartment Σ or the root α.

In Chapters 26 and 27, however, where we work more closely with the notion of the building at infinity of an affine building, the notion of a building as a simplicial complex plays an important role. We use the rest of this chapter to fix some notation which we will need (only in those two chapters).

Definition 1.28. A *simplicial complex* is a pair $(V, \mathcal{S})$, where V is a set whose elements are called *vertices* and $\mathcal{S}$ is a subset of the power set of V whose elements are called *simplices*, such that

(i) $\{v\} \in \mathcal{S}$ for all $v \in V$ and

(ii) all subsets of a simplex are also simplices.

The *dimension* of a simplex is its cardinality minus one. The set V is generally identified with the set of simplices of dimension 0.

Definition 1.29. Let $B = (V, \mathcal{S})$ be a simplicial complex. A *numbering* of B is a surjective map from V to a set I (which we call the *index set*) such that the restriction of this map to each simplex is injective. A *numbered simplicial complex* is a simplicial complex endowed with a numbering.

Definition 1.30. Let $B = (V, \mathcal{S})$ and $B' = (V', \mathcal{S}')$ be two simplicial complexes with numberings ν and ν' having index sets I and I'. A *morphism* from (B, ν) to (B', ν') is a pair (ξ, σ), where ξ is a map from V to V' carrying simplices to simplices and σ is a map from I to I' such that $\nu' \circ \xi = \sigma \circ \nu$. An *isomorphism* from (B, ν) to (B', ν') is a morphism (ξ, ν) such that ξ and ν are bijections and (ξ^{-1}, ν^{-1}) is a morphism from (B', ν') to (B, ν). We denote by $\text{Aut}(B, \nu)$ the group consisting of all isomorphisms from (B, ν) to itself. A *subcomplex* of (B, ν) is a numbered simplicial complex (B_1, ν_1) whose vertex set is a subset of V and whose index set is a subset of I such that $(\text{incl}, \text{incl})$ is a morphism from (B_1, ν_1) to (B, ν).

Notation 1.31. Let Π be a Coxeter diagram with vertex set S, let $n = |S|$, let Δ be a building of type Π, let V be the set of all maximal residues of Δ and let ν be the map from V to S which sends a maximal residue whose type is J to the unique element s of S such that $J = S\backslash\{s\}$. If R is a proper residue of Δ and $J \subset S$ is its type (so $J = \emptyset$ if R is a single chamber), then for each $s \in S\backslash J$, there exists a unique $(S\backslash\{s\})$-residue R_s such that $R \subset R_s$ and by [62, 7.25],

$$R = \bigcap_{s \in S \backslash J} R_s.$$

For each residue R, we denote by A_R the set of maximal residues containing R (so $A_\Delta = \emptyset$) and we set

$$\Delta^\# := \big((V, \mathcal{S}), \nu\big)$$

where $\mathcal{S}$ denotes the set of subsets A_R of V for all residues R of Δ (proper or not). Then $\Delta^\#$ is a numbered simplicial complex with index set S whose simplices of dimension k as defined in 1.28 correspond to residues of Δ of rank $n-k-1$ as defined in [65, 29.1]. In particular, every simplex of $\Delta^\#$ has dimension at most $n-1$ and the chambers of Δ (i.e. the minimal residues) correspond to the simplices of $\Delta^\#$ of dimension $n-1$ (i.e. the maximal simplices).

Remarks 1.32. Let Δ, $\Delta^\#$, S and n be as in 1.31. Then the following hold:

(a) The building Δ can be reconstructed from $\Delta^\#$: Two chambers are s-adjacent in Δ for some $s \in S$ precisely when the intersection of the corresponding maximal simplices has dimension $n-2$.

(b) The correspondence

$$\text{residues of } \Delta \rightsquigarrow \text{simplices of } \Delta^\#$$

is containment-reversing.

(c) There is a canonical isomorphism from $\text{Aut}(\Delta)$ to $\text{Aut}(\Delta^\#)$.

(d) Apartments of Δ correspond to certain subcomplexes of $\Delta^\#$. More precisely, an apartment Σ of Δ corresponds to the subcomplex $(V_\Sigma, \mathcal{S}_\Sigma)$, where V_Σ is the set of maximal residues of Δ containing a chamber of Σ and $\mathcal{S}_\Sigma$ is the set of simplices in $\mathcal{S}$ containing only elements of V_Σ.

In light of these observations, it is natural to think of Δ and $\Delta^\#$ as the same object, simply seen from two points of view.