

# 1

## Introduction

### 1.1 Motivation

Algebraic and arithmetic geometry in positive characteristic provide important examples of imperfect fields, such as (i) Laurent-series fields over finite fields and (ii) function fields of positive-dimensional varieties (even over an algebraically closed field of constants). Generic fibers of positive-dimensional algebraic families naturally lie over a ground field as in (ii).

For a smooth connected affine group  $G$  over a field  $k$ , the unipotent radical  $\mathcal{R}_u(G_{\bar{k}}) \subset G_{\bar{k}}$  may not arise from a  $k$ -subgroup of  $G$  when  $k$  is imperfect. (Examples of this phenomenon will be given shortly.) Thus, for the maximal smooth connected unipotent normal  $k$ -subgroup  $\mathcal{R}_{u,k}(G) \subset G$  (the  $k$ -unipotent radical), the quotient  $G/\mathcal{R}_{u,k}(G)$  may not be reductive when  $k$  is imperfect.

A *pseudo-reductive group* over a field  $k$  is a smooth connected affine  $k$ -group  $G$  such that  $\mathcal{R}_{u,k}(G)$  is trivial. For any smooth connected affine  $k$ -group  $G$ , the quotient  $G/\mathcal{R}_{u,k}(G)$  is pseudo-reductive. A pseudo-reductive  $k$ -group  $G$  that is perfect (i.e.,  $G$  equals its derived group  $\mathcal{D}(G)$ ) is called *pseudo-semisimple*. If  $k$  is perfect then pseudo-reductive  $k$ -groups are connected reductive  $k$ -groups by another name. For imperfect  $k$  the situation is completely different:

**Example 1.1.1.** Weil restrictions  $G = R_{k'/k}(G')$  for finite extensions  $k'/k$  and connected reductive  $k'$ -groups  $G'$  are pseudo-reductive [CGP, Prop. 1.1.10]. If  $G'$  is nontrivial and  $k'/k$  is not separable then such  $G$  are *never* reductive [CGP, Ex. 1.6.1]. A solvable pseudo-reductive group is necessarily commutative [CGP, Prop. 1.2.3], but the structure of commutative pseudo-reductive groups appears to be intractable (see [T]). The quotient of a pseudo-reductive  $k$ -group by a smooth connected normal  $k$ -subgroup or by a central closed  $k$ -subgroup scheme can fail to be pseudo-reductive, and a smooth connected normal  $k$ -subgroup of

a pseudo-semisimple  $k$ -group can fail to be perfect; see [CGP, Ex. 1.3.5, 1.6.4] for such examples over any imperfect field  $k$ .

A typical situation where the structure theory of pseudo-reductive groups is useful is in the study of smooth affine  $k$ -groups about which one has limited information but for which one wishes to prove a general theorem (e.g., cohomological finiteness); examples include the Zariski closure in  $\mathrm{GL}_n$  of a subgroup of  $\mathrm{GL}_n(k)$ , and the maximal smooth  $k$ -subgroup of a schematic stabilizer (as in local-global problems). For questions not amenable to study over  $\bar{k}$  when  $k$  is imperfect, this structure theory makes possible what had previously seemed out of reach over such  $k$ : to reduce problems for general smooth affine  $k$ -groups to the reductive and commutative cases (over finite extensions of  $k$ ). Such procedures are essential to prove finiteness results for degree-1 Tate-Shafarevich sets of arbitrary affine group schemes of finite type over global function fields, even in the general *smooth* affine case; see [C1, §1] for this and other applications.

A detailed study of pseudo-reductive groups was initiated by Tits; he constructed several instructive examples and his ultimate goal was a classification. The general theory developed in [CGP] by characteristic-free methods includes the open cell, root systems, rational conjugacy theorems, the Bruhat decomposition for rational points, and a structure theory “modulo the commutative case” (summarized in [C1, §2] and [R]). The lack of a concrete description of commutative pseudo-reductive groups is not an obstacle in applications (see [C1]).

In general, if  $G$  is a smooth connected affine  $k$ -group then  $\mathcal{R}_{u,k}(G)_K \subset \mathcal{R}_{u,K}(G_K)$  for any extension field  $K/k$ , and this inclusion is an equality when  $K$  is separable over  $k$  [CGP, Prop. 1.1.9] but generally not otherwise (e.g., equality fails with  $K = \bar{k}$  for any imperfect  $k$  and non-reductive pseudo-reductive  $G$ ). Taking  $K = k_s$  shows that  $G$  is pseudo-reductive if and only if  $G_{k_s}$  is pseudo-reductive (and also shows that if  $k$  is perfect then pseudo-reductive  $k$ -groups are precisely connected reductive  $k$ -groups). Hence, any smooth connected normal  $k$ -subgroup of a pseudo-reductive  $k$ -group is pseudo-reductive.

Every smooth connected affine  $k$ -group  $G$  is generated by  $\mathcal{D}(G)$  and a single Cartan  $k$ -subgroup. Since  $\mathcal{D}(G)$  is pseudo-semisimple when  $G$  is pseudo-reductive [CGP, Prop. 1.2.6], and Cartan  $k$ -subgroups of pseudo-reductive  $k$ -groups are commutative and pseudo-reductive, the main work in describing pseudo-reductive groups lies in the pseudo-semisimple case. A smooth affine  $k$ -group  $G$  is *pseudo-simple* (over  $k$ ) if it is pseudo-semisimple, nontrivial, and has no nontrivial smooth connected proper normal  $k$ -subgroup; it is *absolutely pseudo-simple* if  $G_{k_s}$  is pseudo-simple. (See [CGP, Def. 3.1.1, Lemma 3.1.2] for equivalent formulations.) A pseudo-reductive  $k$ -group  $G$  is *pseudo-split* if it

contains a split maximal  $k$ -torus  $T$ , in which case any two such tori are conjugate by an element of  $G(k)$  [CGP, Thm. C.2.3]

**Remark 1.1.2.** If  $G$  is a pseudo-semisimple  $k$ -group then the set  $\{G_i\}$  of its pseudo-simple normal  $k$ -subgroups is finite, the  $G_i$ 's pairwise commute and generate  $G$ , and every perfect smooth connected normal  $k$ -subgroup of  $G$  is generated by the  $G_i$ 's that it contains (see [CGP, Prop. 3.1.8]). The core of the study of pseudo-reductive groups  $G$  is the absolutely pseudo-simple case.

Although [CGP] gives general structural foundations for the study and application of pseudo-reductive groups over any imperfect field  $k$ , there are natural topics not addressed in [CGP] whose development requires new ideas, such as:

- (i) Are there versions of the Isomorphism and Isogeny Theorems for pseudo-split pseudo-reductive groups and of the Existence Theorem for pseudo-split pseudo-simple groups?
- (ii) The *standard construction* (see §2.1) is exhaustive when  $p := \text{char}(k) \neq 2, 3$ . Incorporating constructions resting on exceptional isogenies [CGP, Ch. 7–8] and birational group laws [CGP, §9.6–§9.8] gives an analogous result when  $p = 2, 3$  provided that  $[k : k^2] = 2$  if  $p = 2$ ; see [CGP, Thm. 10.2.1, Prop. 10.1.4]. More examples exist if  $p = 2$  and  $[k : k^2] > 2$  (see §1.3); can we generalize the standard construction for such  $k$ ?
- (iii) Is the automorphism functor of a pseudo-semisimple group representable? (Representability fails in the commutative pseudo-reductive case.) If so, what can be said about the structure of the identity component and component group of its maximal smooth closed subgroup  $\text{Aut}_{G/k}^{\text{sm}}$  (thereby defining a notion of “pseudo-inner”  $k_S/k$ -form via  $(\text{Aut}_{G/k}^{\text{sm}})^0$ )?
- (iv) What can be said about existence and uniqueness of pseudo-split  $k_S/k$ -forms, and of quasi-split pseudo-inner  $k_S/k$ -forms? (“Quasi-split” means the existence of a solvable pseudo-parabolic  $k$ -subgroup.)
- (v) Is there a Tits-style classification in the pseudo-semisimple case recovering the version due to Tits in the semisimple case? (Many ingredients in the semisimple case *break down* for pseudo-semisimple  $G$ ; e.g.,  $G$  may have no pseudo-split  $k_S/k$ -form, and the quotient  $G/Z_G$  of  $G$  modulo the scheme-theoretic center  $Z_G$  can be a proper  $k$ -subgroup of  $(\text{Aut}_{G/k}^{\text{sm}})^0$ .)

The special challenges of characteristic 2 are reviewed in §1.3–§1.4 and §4.2. Recent work of Gabber on compactification theorems for arbitrary linear algebraic groups uses the structure theory of pseudo-reductive groups over general (imperfect) fields. That work encounters additional complications in characteristic 2 which are overcome via the description of pseudo-reductive groups as

central extensions of groups obtained by the “generalized standard” construction given in Chapter 9 of this monograph (see the Structure Theorem in §1.6).

## 1.2 Root systems and new results

A maximal  $k$ -torus  $T$  in a pseudo-reductive  $k$ -group  $G$  is an almost direct product of the maximal central  $k$ -torus  $Z$  in  $G$  and the maximal  $k$ -torus  $T' := T \cap \mathcal{D}(G)$  in  $\mathcal{D}(G)$  [CGP, Lemma 1.2.5]. Suppose  $T$  is *split*, so the set  $\Phi := \Phi(G, T)$  of nontrivial  $T$ -weights on  $\text{Lie}(G)$  injects into  $X(T')$  via restriction.

The pair  $(\Phi, X(T')_{\mathbf{Q}})$  is always a root system (coinciding with  $\Phi(\mathcal{D}(G), T')$  since  $G/\mathcal{D}(G)$  is commutative) [CGP, Thm. 2.3.10], and can be canonically enhanced to a root datum [CGP, §3.2]. In particular, to every pseudo-semisimple  $k_s$ -group we may attach a *Dynkin diagram*. However,  $(\Phi, X(T')_{\mathbf{Q}})$  can be non-reduced when  $k$  is imperfect of characteristic 2 (the non-multipliable roots are the roots of the maximal geometric reductive quotient  $G_{\bar{k}}^{\text{red}}$ ). A pseudo-split pseudo-semisimple group is (absolutely) pseudo-simple precisely when its root system is irreducible [CGP, Prop. 3.1.6].

This monograph builds on earlier work [CGP] via new techniques and constructions to answer the questions (i)–(v) raised in §1.1. In so doing, we also simplify the proofs of some results in [CGP]. (For instance, the standardness of all pseudo-reductive  $k$ -groups if  $\text{char}(k) \neq 2, 3$  is recovered here by another method in Theorem 3.4.2.) Among the new results in this monograph are:

- (i) pseudo-reductive versions of the Existence, Isomorphism, and Isogeny Theorems (see Theorems 3.4.1, 6.1.1, and A.1.2),
- (ii) a *structure theorem* over arbitrary imperfect fields  $k$  (see §1.5–§1.6),
- (iii) existence of the automorphism scheme  $\text{Aut}_{G/k}$  for pseudo-semisimple  $G$ , and properties of the identity component and component group of its maximal smooth closed  $k$ -subgroup  $\text{Aut}_{G/k}^{\text{sm}}$  (see Chapter 6),
- (iv) uniqueness and optimal existence results for pseudo-split and “quasi-split”  $k_s/k$ -forms for imperfect  $k$ , including examples (in *every* positive characteristic) where existence *fails* (see §1.7),
- (v) a Tits-style classification of pseudo-semisimple  $k$ -groups  $G$  in terms of both the Dynkin diagram of  $G_{k_s}$  with  $*$ -action of  $\text{Gal}(k_s/k)$  on it and the  $k$ -isomorphism class of the embedded anisotropic kernel (see §1.7).

We illustrate (v) in Appendix D by using anisotropic quadratic forms over  $k$  to construct and classify absolutely pseudo-simple groups of type  $F_4$  with  $k$ -rank 2 (which never exist in the semisimple case).

### 1.3 Exotic groups and degenerate quadratic forms

If  $p = 2$  and  $[k : k^2] > 2$  then there exist families of *non-standard* absolutely pseudo-simple  $k$ -groups of types  $B_n$ ,  $C_n$ , and  $BC_n$  (for every  $n \geq 1$ ) with no analogue when  $[k : k^2] = 2$ . Their existence is explained by a construction with certain *degenerate* quadratic spaces over  $k$  that exist only if  $[k : k^2] > 2$ :

**Example 1.3.1.** Let  $(V, q)$  be a quadratic space over a field  $k$  with  $\text{char}(k) = 2$ ,  $d := \dim V \geq 3$ , and  $q \neq 0$ . Let  $B_q : (v, w) \mapsto q(v + w) - q(v) - q(w)$  be the associated symmetric bilinear form and  $V^\perp$  the defect space consisting of  $v \in V$  such that the linear form  $B_q(v, \cdot)$  on  $V$  vanishes. The restriction  $q|_{V^\perp}$  is 2-linear (i.e., additive and  $q(cv) = c^2q(v)$  for  $v \in V$ ,  $c \in k$ ) and  $\dim(V/V^\perp) = 2n$  for some  $n \geq 0$  since  $B_q$  induces a non-degenerate symplectic form on  $V/V^\perp$ .

Assume  $0 < \dim V^\perp < \dim V$ . Now  $q$  is non-degenerate (i.e., the projective hypersurface  $(q = 0) \subset \mathbf{P}(V^*)$  is  $k$ -smooth) if and only if  $\dim V^\perp = 1$ , which is to say  $d = 2n + 1$ . It is well-known that in such cases  $\text{SO}(q)$  is an absolutely simple group of type  $B_n$  with  $\text{O}(q) = \mu_2 \times \text{SO}(q)$ , so  $\text{SO}(q)$  is the maximal smooth closed  $k$ -subgroup of  $\text{O}(q)$  since  $\text{char}(k) = 2$ . Assume also that  $(V, q)$  is *regular*; i.e.,  $\ker(q|_{V^\perp}) = 0$ . Regularity is preserved by any separable extension on  $k$  (Lemma 7.1.1). For such (possibly degenerate)  $q$ , define  $\text{SO}(q)$  to be the maximal smooth closed  $k$ -subgroup of the  $k$ -group scheme  $\text{O}(q)$ ; i.e.,  $\text{SO}(q)$  is the  $k$ -descent of the Zariski closure of  $\text{O}(q)_{k_s}$  in  $\text{O}(q)_{k_s}$ . In §7.1–§7.3 we prove:  $\text{SO}(q)$  is absolutely pseudo-simple with root system  $B_n$  over  $k_s$  where  $2n = \dim(V/V^\perp)$ , the dimension of a root group of  $\text{SO}(q)_{k_s}$  is 1 for long roots and  $\dim V^\perp$  for short roots, and the minimal field of definition over  $k$  for the geometric unipotent radical of  $\text{SO}(q)$  is the  $k$ -finite subextension  $K \subset k^{1/2}$  generated over  $k$  by the square roots  $(q(v')/q(v))^{1/2}$  for nonzero  $v, v' \in V^\perp$ .

For any nonzero  $v_0 \in V^\perp$ , the map  $v \mapsto (q(v)/q(v_0))^{1/2}$  is a  $k$ -linear injection of  $V^\perp$  into  $k^{1/2}$  with image  $\mathcal{V}$  containing 1 and generating  $K$  as a  $k$ -algebra. If we replace  $v_0$  with a nonzero  $v_1 \in V^\perp$  then the associated  $k$ -subspace of  $K$  is  $\lambda\mathcal{V}$  where  $\lambda = (q(v_0)/q(v_1))^{1/2} \in K^\times$ . In particular, the case  $K \neq k$  occurs if and only if  $\dim V^\perp \geq 2$ , which is precisely when the regular  $q$  is *degenerate*, and always  $[k : k^2] = [k^{1/2} : k] \geq \dim V^\perp$ . If  $V^\perp = K$ , as happens whenever  $[k : k^2] = 2$ , then  $\text{SO}(q)$  is the quotient of a “basic exotic”  $k$ -group [CGP, §7.2] modulo its center. The  $\text{SO}(q)$ ’s with  $V^\perp \neq K$  (so  $[k : k^2] > 2$ ) are a new class of absolutely pseudo-simple  $k$ -groups of type  $B_n$  (with trivial center); for  $n = 1$  and isotropic  $q$  these are the type- $A_1$  groups  $PH_{V^\perp, K/k}$  built in §3.1.

In §7.2–§7.3 we show that every  $k$ -isomorphism  $\text{SO}(q') \simeq \text{SO}(q)$  arises from a conformal isometry  $q' \simeq q$  and use this to construct more absolutely

pseudo-simple  $k$ -groups of type B with trivial center via geometrically integral non-smooth quadrics in Severi–Brauer varieties associated to certain elements of order 2 in the Brauer group  $\text{Br}(k)$ . Remarkably, this accounts for *all* non-reductive pseudo-reductive groups whose Cartan subgroups are tori (see Proposition 7.3.7), and when combined with the exceptional isogeny  $\text{Sp}_{2n} \rightarrow \text{SO}_{2n+1}$  in characteristic 2 via a fiber product construction it yields (in §8.2) new absolutely pseudo-simple groups of type  $C_n$  when  $n \geq 2$  and  $[k : k^2] > 2$  (with short root groups over  $k_s$  of dimension  $[K : k]$  and long root groups over  $k_s$  of dimension  $\dim V^\perp$ ). A generalization in §8.3 gives *even more* such  $k$ -groups for  $n = 2$  if  $[k : k^2] > 8$  (using that  $B_2 = C_2$ ). In §1.5–§1.6 we provide a context for this zoo of constructions.

## 1.4 Tame central extensions

A new ingredient in this monograph is a generalization of the “standard construction” (from §2.1) that is better-suited to the peculiar demands of characteristic 2. Before we address that, it is instructive to recall the principle underlying the ubiquity of standardness *away from* the case  $\text{char}(k) = 2$  with  $[k : k^2] > 2$ , via splitting results for certain classes of central extensions. We now review the most basic instance of such splitting, to see why it breaks down completely (and hence new methods are required) when  $\text{char}(k) = 2$  with  $[k : k^2] > 2$  (see 1.4.2).

**1.4.1.** Let  $G$  be an absolutely pseudo-simple  $k$ -group with minimal field of definition  $K/k$  for  $\mathcal{R}_u(G_{\bar{k}}) \subset G_{\bar{k}}$ , and let  $G' := G_K^{\text{ss}}$  be the maximal semisimple quotient of  $G_K$ . For the simply connected central cover  $q : \tilde{G}' \rightarrow G'$  and  $\mu := \ker q \subset Z_{\tilde{G}'}$ , there is (as in [CGP, Def. 5.3.5]) a canonical  $k$ -homomorphism

$$\xi_G : G \rightarrow \mathcal{D}(\mathbf{R}_{K/k}(G')) = \mathbf{R}_{K/k}(\tilde{G}')/\mathbf{R}_{K/k}(\mu) \quad (1.4.1.1)$$

induced by the natural map  $i_G : G \rightarrow \mathbf{R}_{K/k}(G')$ . The map  $\xi_G$  makes sense for any pseudo-reductive  $G$  but (as in [CGP]) it is of interest only for absolutely pseudo-simple  $G$ . By Proposition 2.3.4,  $\ker \xi_G$  is central if  $\text{char}(k) \neq 2$ .

The key to the proof that  $G$  is standard if  $\text{char}(k) \neq 2, 3$  is the surjectivity of  $\xi_G$  for such  $k$ , as then (1.4.1.1) pulls back to a central extension  $E$  of  $\mathbf{R}_{K/k}(\tilde{G}')$  by  $\ker \xi_G$ . This central extension is *split* due to a general fact: if  $k'/k$  is an arbitrary finite extension of fields and  $\mathcal{G}'$  a connected semisimple  $k'$ -group that is *simply connected* then for any commutative affine  $k$ -group scheme  $Z$  of finite type with no nontrivial smooth connected  $k$ -subgroup (e.g.,  $Z = \ker \xi_G$  as

above), every central extension of  $k$ -group schemes

$$1 \rightarrow Z \rightarrow E \rightarrow \mathbf{R}_{k'/k}(\mathcal{G}') \rightarrow 1 \quad (1.4.1.2)$$

is (uniquely) split over  $k$  (see [CGP, Ex. 5.1.4]). In contrast, for many imperfect  $k$  and  $k$ -finite  $k' \subset k^{1/p}$  for  $p = \text{char}(k)$ , the  $k$ -group  $\mathbf{R}_{k'/k}(\text{SL}_n)$  seems to admit non-split central extensions by  $\mathbf{G}_a$  when  $n > 2$  [CGP, Rem. 5.1.5].

**1.4.2.** For an absolutely pseudo-simple  $k$ -group  $G$ , two substantial difficulties arise if  $\xi_G$  is not surjective (so  $\text{char}(k) = 2, 3$ ) or if  $G_{k_s}$  has a non-reduced root system (which can occur only if the field  $k$  is imperfect and of characteristic 2):

(i) Assume  $G_{k_s}$  has a *reduced* root system (so  $\ker \xi_G$  is central in  $G$ , by Proposition 2.3.4) but that  $\xi_G$  is not surjective. The possibilities for  $\xi_G(G)$  force us to go beyond the simply connected semisimple central cover  $\widetilde{G}'$  of  $G_K^{\text{ss}}$  and consider a wider class of absolutely pseudo-simple groups over finite extensions of  $k$ , called *generalized basic exotic* and *basic exceptional*, building on §1.3; see Chapter 8. (The maximal geometric semisimple quotient of these new groups is simply connected, and the basic exceptional case – which occurs over  $k$  if and only if  $\text{char}(k) = 2$  with  $[k : k^2] > 8$  – rests on the equality  $\mathbf{B}_2 = \mathbf{C}_2$ .)

(ii) Assume  $k$  is imperfect with  $\text{char}(k) = 2$ . If  $[k : k^2] = 2$  then every pseudo-reductive  $k$ -group uniquely has the form  $H \times \prod H_i$  where  $H_{k_s}$  has a reduced root system and each  $H_i$  is absolutely pseudo-simple over  $k$  with a non-reduced root system over  $k_s$  [CGP, Prop. 10.1.4, Prop. 10.1.6] (and each  $H_i$  is pseudo-split, has trivial center, and  $\text{Aut}_k(H_i) = H_i(k)$  [CGP, Thm. 9.9.3]).

In contrast, when  $[k : k^2] > 2$  it is generally impossible to split off (as a direct factor) the contribution from non-reduced irreducible components of the root system over  $k_s$ ; see Example 6.1.5. Moreover, as is explained in [CGP, §9.8–§9.9], the classification of pseudo-split absolutely pseudo-simple  $k$ -groups with a non-reduced root system over  $k_s$  rests on invariants of linear algebraic nature that do not arise (in nontrivial ways) when  $[k : k^2] = 2$ .

**1.4.3.** To classify absolutely pseudo-semisimple  $G$  over any  $k$  whatsoever, we shall use the following new construction. A commutative affine  $k$ -group scheme of finite type is *k-tame* if it does not contain a nontrivial unipotent  $k$ -subgroup scheme. For example, if  $k'/k$  is an extension of finite degree and  $\mu'$  is a  $k'$ -group scheme of multiplicative type then  $\mathbf{R}_{k'/k}(\mu')$  is  $k$ -tame. If  $G$  is a perfect

smooth connected affine group over a field  $k$  then a central extension

$$1 \rightarrow Z \rightarrow E \rightarrow G \rightarrow 1$$

with affine  $E$  of finite type is called  $k$ -tame if  $Z$  is  $k$ -tame. In Theorem 5.1.3 we show that for any such  $G$ , if  $K/k$  is the minimal field of definition for  $\mathcal{R}_u(G_{\bar{k}}) \subset G_{\bar{k}}$  then the category of  $k$ -tame central extensions  $E$  of  $G$  that are smooth, connected, and perfect is equivalent to the category of connected semisimple central extensions of  $G' := G_K^{\text{ss}}$  over  $K$  via  $E \rightsquigarrow E' := E_K/\mathcal{R}_{u,K}(E_K)$ .

The perfect smooth connected  $k$ -tame central extension  $\widetilde{G}$  of  $G$  for which the associated connected semisimple central extension of  $G'$  is simply connected is called the *universal smooth  $k$ -tame central extension* of  $G$  (it is initial among smooth  $k$ -tame central extensions of  $G$ ). It is elementary that if  $G$  is pseudo-reductive then so is  $\widetilde{G}$  and that if  $G := \mathcal{D}(\mathbf{R}_{k'/k}(G'))$  for a finite extension  $k'/k$  and connected semisimple  $k'$ -group  $G'$  then  $\widetilde{G} = \mathbf{R}_{k'/k}(\widetilde{G}')$  for the simply connected central cover  $\widetilde{G}' \rightarrow G'$ . In proofs of general theorems it is often possible to pass from  $G$  to  $\widetilde{G}$  (which has better properties), and by Theorem 9.2.1 (and Proposition 5.3.3) the  $k$ -group  $\widetilde{G}$  is described in terms of a known list of constructions when  $\widetilde{G}$  is of “minimal type” in a sense discussed in §1.6.

## 1.5 Generalized standard groups

In [CGP, Ch. 7–8], we constructed a class of pseudo-semisimple groups over any imperfect field  $k$  of characteristic  $p \in \{2, 3\}$  by using certain non-standard absolutely pseudo-simple groups  $G'$  – called “basic exotic” – over finite extensions  $k'/k$  such that  $(G'_{\bar{k}'})^{\text{ss}}$  is simply connected and the irreducible root system  $\Phi$  of  $G'_{k'_s}$  is reduced with an edge of multiplicity  $p$ : such  $\Phi$  can be  $F_4$ ,  $B_n$ , or  $C_n$  in characteristic 2 (with any  $n \geq 2$ ) and  $G_2$  in characteristic 3. Letting  $K'/k'$  be the minimal field of definition for  $\mathcal{R}_u(G'_{\bar{k}'}) \subset G'_{\bar{k}'}$ , we have  $k' \subsetneq K' \subset k'^{1/p}$  and over  $k'_s$  the long root groups are 1-dimensional whereas short root groups have dimension  $[K' : k'] > 1$  (short root groups are isomorphic to  $\mathbf{R}_{K'_s/k'_s}(\mathbf{G}_a)$ ).

Going beyond these constructions, in [CGP, Ch. 9] pseudo-split absolutely pseudo-simple groups  $G'$  with root system  $BC_n$  (for any  $n \geq 1$ ) are constructed over any imperfect field  $k'$  with characteristic 2. If  $[k' : k'^2] = 2$  then by [CGP, Thm. 9.9.3(1)] the  $k'$ -group  $G'$  is classified up to  $k'$ -isomorphism by the rank  $n \geq 1$  and the minimal field of definition  $K'/k'$  for  $\mathcal{R}_u(G'_{\bar{k}'}) \subset G'_{\bar{k}'}$ ; here,  $n$  can be arbitrary and  $K'/k'$  can be any nontrivial purely inseparable finite extension.

For imperfect  $k$  of characteristic 2 or 3, with  $[k : k^2] = 2$  in the  $BC_n$ -cases, Weil restrictions to  $k$  of groups  $G'$  as above over finite extensions  $k'/k$  are

perfect and satisfy the splitting result for central extensions as in (1.4.1.2); see [CGP, Prop. 8.1.2, Thm. 9.9.3(3)]. (That splitting result fails in some  $BC_n$ -cases with  $k' = k$  and  $n \geq 1$  whenever  $[k : k^2] > 2$ ; see Examples B.4.1 and B.4.3.)

If  $k$  is imperfect and either  $\text{char}(k) = 3$  or  $\text{char}(k) = 2$  with  $[k : k^2] = 2$  then the preceding constructions capture *all* deviations from standardness over  $k$  (see [CGP, Thm. 10.2.1]). However, over any field  $k$  of characteristic 2 with  $[k : k^2] > 2$  there exist many other pseudo-reductive groups, starting with:

**Example 1.5.1.** Consider imperfect  $k$  of characteristic 2 and a pseudo-split absolutely pseudo-simple  $k$ -group  $G$  with a *reduced* root system such that  $G_{\bar{k}}^{\text{ss}} \simeq \text{SL}_2$ . If  $[k : k^2] = 2$  then  $G \simeq \text{R}_{K/k}(\text{SL}_2)$  for a purely inseparable finite extension  $K/k$  (see [CGP, Prop. 9.2.4]). In contrast, as we review in §3.1, if  $[k : k^2] > 2$  then many more possibilities for  $G$  occur: in addition to the field invariant  $K/k$ , there are linear algebra invariants (such as certain  $K^\times$ -homothety classes of nonzero  $kK^2$ -subspaces  $V$  of  $K$ , with the case  $V \neq K$  occurring if  $[k : k^2] > 2$ ).

The groups in Example 1.5.1 can be used to “shrink” short root groups (for type B) or “fatten” long root groups (for type C) in pseudo-split basic exotic  $k$ -groups with rank  $n \geq 2$ . When  $[k : k^2] > 2$ , this relates the new classes of absolutely pseudo-simple groups  $G$  mentioned in 1.4.2(i) to the basic exotic cases. For these additional constructions (and the derived groups of their Weil restrictions through finite extensions of the ground field) we prove a splitting result for central extensions as in (1.4.1.2) when  $[k : k^2] \leq 8$  (see Proposition B.3.4), but this splitting result *fails* whenever  $[k : k^2] > 8$  (see §B.1–§B.2).

For any imperfect field  $k$  of characteristic 2, the data classifying pseudo-split absolutely pseudo-simple  $k$ -groups  $G$  with root system  $BC_n$  ( $n \geq 1$ ) is much more intricate when  $[k : k^2] > 2$  than when  $[k : k^2] = 2$ , and one encounters new behavior when  $[k : k^2] > 2$  that never occurs when  $[k : k^2] = 2$ . For instance, if  $K/k$  is the minimal field of definition for  $\mathcal{R}_u(G_{\bar{k}}) \subset G_{\bar{k}}$  and  $[k : k^2] > 2$  then there can be proper subfields  $F \subset K$  over  $k$  such that the non-reductive maximal pseudo-reductive quotient  $G_F/\mathcal{R}_{u,F}(G_F)$  of  $G_F$  has a *reduced* root system (see [CGP, Ex. 9.1.8]); this never happens if  $[k : k^2] = 2$ .

Weil restrictions to  $k$  of generalized basic exotic groups, basic exceptional groups, and the constructions in [CGP, Ch. 9] with non-reduced root systems are used to define a *generalized standard* construction over any field  $k$  in Definitions 9.1.5 and 9.1.7. This construction satisfies many nice properties (see §9.1). The information required to make a “generalized standard presentation” of a pseudo-reductive  $k$ -group  $G$  (if it admits such a presentation at all!) consists of data that is uniquely functorial with respect to isomorphisms in the pair  $(G, T)$  where  $T$  is a maximal  $k$ -torus of  $G$  (see Proposition 9.1.12); any  $T$  may be used.

## 1.6 Minimal type and general structure theorem

For a pseudo-reductive group  $G$  over any field  $k$ , the important notion of  $G$  being of *minimal type* was introduced in [CGP, Def. 9.4.4] and is reviewed in §2.3. Every pseudo-reductive  $k$ -group  $G$  admits a canonical pseudo-reductive central quotient  $\overline{G}$  of minimal type with the *same* root datum as  $G$  (over  $k_s$ ) [CGP, Prop. 9.4.2(iii)], and the central quotient  $G/Z_G$  is always pseudo-reductive and of minimal type (see Proposition 4.1.3).

Many of our results for general  $G$  rest on a classification and structure theorem for pseudo-split absolutely pseudo-simple groups of minimal type (over any field) given in Theorem 3.4.1 in the spirit of the Existence and Isomorphism Theorems for split connected semisimple groups. This classification in the pseudo-split minimal-type case supplements the root datum with additional field-theoretic data, as well as linear-algebraic data in characteristic 2. To prove theorems about general pseudo-reductive groups, it is often harmless and genuinely useful to pass to the minimal type case (e.g., see the proofs of Proposition 3.4.4, Proposition 6.1.4, Proposition B.3.1, and Theorem C.2.10).

Generalized basic exotic and basic exceptional  $k$ -groups from 1.4.2(i) are of minimal type and admit an intrinsic characterization via this condition; see Theorem 8.4.5 (and Definition 8.4.1). However, a pseudo-reductive central quotient of a pseudo-reductive group of minimal type is generally *not* of minimal type; absolutely pseudo-simple counterexamples that are standard exist over *every* imperfect field (see Example 2.3.5). Hence, a general structure theorem for pseudo-reductive groups must go beyond the minimal type case.

There is a weaker condition on a pseudo-reductive  $k$ -group  $G$  that we call *locally of minimal type*: for a maximal  $k$ -torus  $T \subset G$ , this is the property that for all non-divisible roots  $a$  of  $(G_{k_s}, T_{k_s})$ , the pseudo-simple  $k_s$ -group  $(G_{k_s})_a$  of rank 1 generated by the  $\pm a$ -root groups admits a pseudo-simple central extension of minimal type. This notion might appear to be ad hoc, but it is not because it admits an elegant global characterization in the pseudo-semisimple case: such a  $k$ -group  $G$  is locally of minimal type if and only if its universal smooth  $k$ -tame central extension  $\widetilde{G}$  is of minimal type (Proposition 5.3.3). In particular, for *every* pseudo-semisimple  $G$ , the universal smooth  $k$ -tame central extension of  $G/Z_G$  is always of minimal type; this is convenient in general proofs. By design, if  $G$  is pseudo-reductive and locally of minimal type then so is any pseudo-reductive central quotient of  $G$ . It is also easy to check that all generalized standard pseudo-reductive groups are locally of minimal type.

**Example 1.6.1.** Rank-1 absolutely pseudo-simple  $k_s$ -groups are classified in

Proposition 3.1.9 if  $\text{char}(k) \neq 2$  and in [CGP, Prop. 9.2.4, Thm. 9.9.3(1)] if  $\text{char}(k) = 2$  with  $[k : k^2] \leq 2$ . This classification implies that over such  $k$  every pseudo-reductive  $k$ -group is locally of minimal type.

Example 1.6.1 is optimal in the  $k$ -aspect: if  $\text{char}(k) = 2$  with  $[k : k^2] > 2$  then for any  $n \geq 1$  there are pseudo-split pseudo-simple  $k$ -groups with root system  $BC_n$  that are *not* locally of minimal type (see §B.4), and likewise (see 4.2.2 and §B.1–§B.2) for pseudo-split pseudo-simple  $k$ -groups with root systems  $B_n$  and  $C_n$  for any  $n \geq 1$  when  $[k : k^2] > 8$  (optimal by Proposition B.3.1); here  $B_1$  and  $C_1$  mean  $A_1$ . These examples suggest that there is no analogue of the “standard construction” beyond the locally minimal type class.

Since “locally of minimal type” is more robust than “minimal type”, we seek to describe all pseudo-reductive groups locally of minimal type. One of our main results (Theorem 9.2.1) is a converse to the elementary fact that generalized standard pseudo-reductive groups (see §1.5) are locally of minimal type:

**Structure Theorem.** *A pseudo-reductive group locally of minimal type is generalized standard. In particular, if  $G$  is an arbitrary pseudo-reductive group then  $G/Z_G$  is generalized standard.*

The novelty is that when  $\text{char}(k) = 2$  there is no restriction on  $[k : k^2]$ . The cases  $\text{char}(k) \neq 2$  or  $\text{char}(k) = 2$  with  $[k : k^2] \leq 2$  are part of [CGP, Thm. 10.2.1]; that earlier result is reproved in a new way in this monograph (using certain inputs from [CGP]) in the course of proving the above more general theorem.

## 1.7 Galois-twisted forms and Tits classification

A  $k_s/k$ -form of a pseudo-reductive group  $G$  over a field  $k$  is a pseudo-reductive  $k$ -group  $H$  such that  $H_{k_s} \simeq G_{k_s}$ . The theory of Chevalley groups ensures that if  $G$  is reductive then it admits a unique split  $k_s/k$ -form. Uniqueness of pseudo-split  $k_s/k$ -forms holds in the pseudo-reductive case (Proposition C.1.1). Existence of a pseudo-split  $k_s/k$ -form seems to be intractable for commutative pseudo-reductive  $G$ , so now consider pseudo-semisimple  $G$ .

Over many imperfect  $k$  (with arbitrary characteristic  $p > 0$ ) there are pseudo-semisimple  $G$  *without* a pseudo-split  $k_s/k$ -form, due to a field-theoretic obstruction that cannot arise if  $G$  is absolutely pseudo-simple or if  $[k : k^p] = p$ ; see Example C.1.2. Additional examples allowing  $[k : k^p] = p$  are given in Example C.1.6, but those are also not absolutely pseudo-simple.

**Existence result:** For any *absolutely* pseudo-simple  $G$ , a pseudo-split  $k_s/k$ -form exists if  $\text{char}(k) \neq 2$  and also if  $\text{char}(k) = 2$  with  $[k : k^2] \leq 4$  except

possibly (for the latter case) if  $G$  is *standard* of type  $D_{2n}$  with  $n \geq 2$  and  $k$  admits a quadratic Galois extension (or a cubic Galois extension when  $n = 2$ ); see Proposition C.1.3 and Corollary C.2.12. The same conclusion holds in the *standard* absolutely pseudo-simple case when  $\text{char}(k) = 2$  without restriction on  $[k : k^2]$  (subject to the same exceptions for type  $D_{2n}$ ).

Avoidance of type  $D_{2n}$  ( $n \geq 2$ ) is necessary because for all  $n \geq 2$  and imperfect  $k$  of characteristic 2 that admits a quadratic (or cubic when  $n = 2$ ) Galois extension there exists a standard absolutely pseudo-simple  $k$ -group  $G$  of type  $D_{2n}$  with no pseudo-split  $k_s/k$ -form; see Proposition C.1.4 and Remark C.1.5.

**More counterexamples:** What about the *non-standard* case if  $\text{char}(k) = 2$  and  $[k : k^2] > 4$ ? If  $[k : k^2] > 4$  and  $k$  has sufficiently rich Galois theory then in Example C.3.1 we make (non-standard) absolutely pseudo-simple groups of type  $A_1$  over  $k$  *without* a pseudo-split  $k_s/k$ -form. These are used in §C.4 to make many more non-standard absolutely pseudo-simple  $k$ -groups without a pseudo-split  $k_s/k$ -form: generalized basic exotic  $k$ -groups whose root system over  $k_s$  is  $B_n$  or  $C_n$  for any  $n \geq 2$ , and absolutely pseudo-simple  $k$ -groups of minimal type whose root system over  $k_s$  is  $BC_n$  for any  $n \geq 1$ .

Going beyond the study of pseudo-split  $k_s/k$ -forms, it is natural to seek a pseudo-reductive analogue of the existence and uniqueness of quasi-split *inner* forms for connected reductive groups. Recall that for connected reductive  $G$  the notion of inner form involves Galois-twisting against the action of the identity component  $G/Z_G$  of the automorphism scheme of  $G$ . Due to the mysterious nature of commutative pseudo-reductive groups, for an analogous result in the pseudo-reductive case we shall restrict attention to pseudo-semisimple  $G$ .

The analogue of “inner form” for pseudo-semisimple groups  $G$  is *not* defined via the action of  $G/Z_G$ , but rather via the identity component of the maximal smooth closed  $k$ -subgroup of the automorphism scheme of  $G$ . To be precise, in §6.2 we prove for pseudo-semisimple  $G$  that the automorphism functor of  $G$  on the category of  $k$ -algebras is represented by an affine  $k$ -group scheme  $\text{Aut}_{G/k}$  of finite type (this functor is often *not* representable for commutative  $G$ ; see Example 6.2.1). In general  $\text{Aut}_{G/k}$  is *not*  $k$ -smooth (Example 6.2.3), but its maximal smooth closed  $k$ -subgroup  $\text{Aut}_{G/k}^{\text{sm}}$  has structure analogous to the semisimple case (see Propositions 6.2.4 and 6.3.10):  $(\text{Aut}_{G/k}^{\text{sm}})^0$  is pseudo-reductive and its *derived group* is  $G/Z_G$ . (Absolutely pseudo-simple  $G$  with  $G/Z_G \neq (\text{Aut}_{G/k}^{\text{sm}})^0$  arise over every imperfect field; see Remark 6.2.5.)

Inspired by the semisimple case, for pseudo-semisimple  $G$  we prove that  $\pi_0(\text{Aut}_{G/k}^{\text{sm}})(k_s)$  is a subgroup of the automorphism group of the based root datum over  $k_s$  (Remark 6.3.6). For absolutely pseudo-simple  $G$  we show that

this subgroup inclusion is often an equality. Counterexamples to equality in the absolutely pseudo-simple case exist *precisely* for type  $D_{2n}$  ( $n \geq 2$ ) with  $k$  imperfect of characteristic 2 (for the same reason that such cases may not have pseudo-split  $k_s/k$ -form); see Proposition 6.3.10.

In §6.3 we use our study of the structure of  $\text{Aut}_{G/k}^{\text{sm}}$  (including its behavior under passage to pseudo-reductive central quotients of  $G$ ) to prove a Tits-style classification theorem in the general pseudo-semisimple case (no minimal-type hypothesis!), recovering the well-known result due to Tits in the semisimple case. As an illustration of the method, in Appendix D we show that if  $k$  is imperfect of characteristic 2 then absolutely pseudo-simple  $k$ -groups of type  $F_4$  that are not pseudo-split cannot have  $k$ -rank 3 whereas they *can* have  $k$ -rank 2 (in contrast with the semisimple case!); all instances of the latter are described via anisotropic quadratic forms over  $k$  (with examples given over specific  $k$ ).

In §C.2 the notion of *pseudo-inner* form of a pseudo-reductive  $k$ -group  $G$  is defined in terms of  $(\text{Aut}_{\mathcal{D}(G)/k}^{\text{sm}})^0$ . We use the structure of  $(\text{Aut}_{\mathcal{D}(G)/k}^{\text{sm}})^0$  to prove uniqueness of pseudo-inner  $k_s/k$ -forms that are *quasi-split* (i.e., admit a solvable pseudo-parabolic  $k$ -subgroup). The existence of such  $k_s/k$ -forms is proved assuming when  $\text{char}(k) = 2$  that  $[k : k^2] \leq 4$  or  $G$  is standard (Theorem C.2.10). This is optimal because if  $\text{char}(k) = 2$ ,  $[k : k^2] > 4$ , and  $k$  has sufficiently rich Galois theory (more precisely,  $k$  admits a quadratic Galois extension  $k'$  such that  $\ker(\text{Br}(k) \rightarrow \text{Br}(k')) \neq 1$ ) then for every  $n \geq 1$  there exist non-standard absolutely pseudo-simple  $k$ -groups of types  $B_n$ ,  $C_n$ , and  $BC_n$  over  $k_s$  *without* a quasi-split  $k_s/k$ -form: examples without a pseudo-split  $k_s/k$ -form (see §C.3–§C.4) do the job, by Lemma C.2.2.

## 1.8 Background, notation, and acknowledgments

In this monograph we use many constructions and results from [CGP]. Familiarity with Chapters 1–5, §7.1–§7.2, §8.1, Chapter 9, parts of Appendix A (A.5, A.7, A.8), Theorem B.3.4, and Appendix C.2 (especially Theorem C.2.29) of [CGP] is sufficient for understanding our main techniques. Chapter 9 in the first edition of [CGP] has been completely rewritten in the second edition, incorporating significant improvements that are used throughout this monograph, and some results outside Chapter 9 of [CGP] are improved in the second edition and used here. We provide many cross-references to aid the reader. (All numerical labeling in the first edition of [CGP] is unchanged in the second edition except in Chapter 9, apart from an equation label in [CGP, Ex. 1.6.4].)

For a scheme  $X$  of finite type over a field  $k$  and a closed subscheme  $Z$  of

$X_K$  for an extension field  $K/k$ , the intersection of all subfields  $k' \subset K$  over  $k$  such that  $Z$  descends (necessarily uniquely) to a closed subscheme of  $X_{k'}$  is also such a subfield, called the *minimal field of definition* for  $Z \subset X_K$  relative to  $k$ ; see [EGA, IV<sub>2</sub>, §4.8] for a detailed discussion of the existence of such a field. The behavior of  $K/k$  with respect to extension of  $k$  is addressed in [CGP, Lemma 1.1.8]. (In [CGP] the phrase “field of definition” is understood to require minimality, but in this monograph we keep “minimality” in the terminology.)

For an automorphism  $\sigma$  of a field  $k$  and a  $k$ -scheme  $X$ ,  ${}^\sigma X$  denotes the  $k$ -scheme  $k \otimes_{\sigma, k} X$ . For a map  $f : X \rightarrow Y$  of  $k$ -schemes,  ${}^\sigma f$  denotes the induced map  ${}^\sigma X \rightarrow {}^\sigma Y$  over  $k$ .

The maximal smooth closed  $k$ -subgroup of a  $k$ -group scheme  $H$  of finite type is denoted  $H^{\text{sm}}$  (see [CGP, Rem. C.4.2]). For a smooth connected affine  $k$ -group  $G$  and closed  $k$ -subgroup scheme  $H \subset G$  that is either smooth or of multiplicative type, the *scheme-theoretic centralizer*  $Z_G(H)$  for the  $H$ -action on  $G$  via conjugation is defined as in [CGP, A.1.9ff., A.8.10]; the *scheme-theoretic center* is  $Z_G := Z_G(G)$  [CGP, A.1.10]. The  *$k$ -unipotent radical*  $\mathcal{R}_{u, k}(G)$  is the maximal unipotent smooth connected normal  $k$ -subgroup of  $G$ . The  *$k$ -radical*  $\mathcal{R}_k(G)$  is the maximal solvable smooth connected normal  $k$ -subgroup of  $G$ .

If  $K/k$  denotes the minimal field of definition for  $\mathcal{R}_u(G_{\bar{k}}) \subset G_{\bar{k}}$  then we define  $G_K^{\text{red}}$  to be the quotient  $G_K/\mathcal{R}_{u, K}(G_K)$  that is a  $K$ -descent of the maximal reductive quotient  $G_{\bar{k}}^{\text{red}}$  of  $G_{\bar{k}}$ . Taking  $K/k$  instead to be the minimal field of definition for  $\mathcal{R}(G_{\bar{k}}) \subset G_{\bar{k}}$  yields the quotient  $G_K^{\text{ss}} := G_K/\mathcal{R}_K(G_K)$  of  $G_K$  as a  $K$ -descent of the maximal semisimple quotient of  $G_{\bar{k}}$ . A *Levi  $k$ -subgroup* of  $G$  is a smooth closed  $k$ -subgroup  $L \subset G$  such that  $L_{\bar{k}} \rightarrow G_{\bar{k}}^{\text{red}}$  is an isomorphism.

The *Weyl group* of a root system  $\Phi$  is denoted  $W(\Phi)$ . If  $\Phi$  is irreducible and not simply laced then for any basis  $\Delta$  of  $\Phi$  we denote by  $\Delta_{>}$  and  $\Delta_{<}$  the respective subsets of longer and shorter roots in  $\Delta$ .

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