# 1. How's Your Math?

What would life be without arithmetic, but a
scene of horrors?
— *Sydney Smith*[1] *(in a letter dated July 22, 1835)*

In this opening chapter I'll discuss several examples of the kind of mathematics we'll encounter in "simple physics" questions that may (or anyway *could*) occur in "ordinary life." These are questions whose intent, I think, anybody can understand but that require at least some analytical thinking to answer. The math examples are very different from one another, with their only "unifying" (if I may use that word) feature being a progressively increasing sophistication. The central question to ask yourself as you read each example is, do I follow the arguments? If you can say yes, even if you can't initially work through the detailed analysis yourself, then your understanding is sufficient for the book.

## Example 1

Our first example of analytical thinking requires *no* formal math but, rather, *logic* and a bit of everyday knowledge (lit lightbulbs get hot). Think about it as you work through the rest of the examples, and, as with the wind-and-airplane problem at the end of the preface, I'll give you the answer at the end of the chapter.

Imagine that you are in a multistory house with three electrical switches in the basement and a 100-watt lightbulb in the attic. All three

switches have two positions, labeled ON and OFF, but only one of the switches controls the lightbulb. You don't know which one. All three switches are initially OFF. One way to determine the controlling switch is with the following obvious procedure: Flip any one of the switches to ON, and then go up to the attic to see if the bulb is lit. If it is, you are done. If it isn't, go back to the basement, turn one of the other OFF switches to ON, and then go back up to the attic to see if the bulb is lit. If it is, then the switch you just turned ON controls the bulb. If the bulb is not lit, then the switch that has never been ON controls the bulb. So, you can obviously figure out which switch controls the lightbulb with at most two trips to the attic.

There is, however, another procedure that *guarantees* your being able to make that determination with just *one* trip up to the attic. What is it?

## Example 2

This question also requires no real math but, again, logical reasoning (although it does require an *elementary* understanding of kinetic and potential energy). Suppose you fire a gun, sending a bullet straight up into the air. Taking air resistance into account, is the time interval during which the bullet is traveling upward greater than or less than the time interval during which the bullet falls back to Earth? You might think you need to know the details of the air-resistance drag-force law, but that is not so. All you need to know is that air resistance exists.[2] You may assume that the Earth's gravitational field is constant over the entire up-down path of the bullet (it remains the same, independent of the bullet's altitude). As in Example 1, think about this question as you work through the rest of the examples, and I'll give you the answer at the end of the chapter. *Hint*: Potential energy is the energy of position (taking the Earth's surface as the zero reference level, a mass $m$ at height $h$ above the surface has potential energy $mgh$, where $g$ is the acceleration of gravity, about 32 feet/seconds-squared), and kinetic energy is the energy of motion (a mass $m$ moving at speed $v$ has kinetic energy $\frac{1}{2}mv^2$).

## Example 3

This question does require some math, but it's really only arithmetic involving a lot of multiplying and dividing of really big numbers. In the 1956 science fiction story "Expedition" by Fredric Brown (1906–1972), the following situation is the premise. There are 30 seats available on the first rocket ship trip to colonize Mars, with the seats to be filled by selecting at random 30 people from a pool of 500 men and 100 women. What is the probability that the result (as in the story) is one man and 29 women?

We start by imagining the 30 seats lined up, side by side, from left to right, and then we compute the total number of distinguishable (we assume each person is uniquely identifiable) ways to fill the seats *without regard to gender* from the pool of 600 people. That number, $N_1$, is[3]

$$N_1 = (600)\,(599)\,(598)\ldots\big(571\big) = \frac{600!}{570!}.$$

Next, if $N_2$ is the total number of distinguishable ways to fill the 30 seats with exactly one man and 29 women, then the probability we seek is $P = \frac{N_2}{N_1}$. We calculate $N_2$ as follows:

there are 30 different ways to pick the seat for the one man

and

there are 500 different ways to pick the one man for that seat.

So,

$$N_2 = (30)(500)(100)(99)(98)\ldots(72) = 15{,}000\frac{100!}{71!}.$$

The *formal* answer to our question is then

$$P = \frac{15{,}000\frac{100!}{71!}}{\frac{600!}{570!}} = 15{,}000\frac{(100!)\,\big(570!\big)}{\big(71!\big)\,(600!)}.$$

I use the word *formal* because we still don't have a single number for $P$.

The factorials in this expression are all huge numbers, numbers that are far too large for direct calculation on a hand calculator (my calculator first fails at 70!). So, to make things more manageable, I'll use Stirling's asymptotic[4] approximation for $n!$: $n! \sim \sqrt{2\pi n}e^{-n}n^n$. Then,

$$P = 15{,}000\frac{\left(\sqrt{2\pi}\sqrt{100}e^{-100}100^{100}\right)\left(\sqrt{2\pi}\sqrt{570}e^{-570}570^{570}\right)}{\left(\sqrt{2\pi}\sqrt{71}e^{-71}71^{71}\right)\left(\sqrt{2\pi}\sqrt{600}e^{-600}600^{600}\right)}$$

$$= \left\{15{,}000e\sqrt{\frac{(100)(570)}{(71)(600)}}\right\}\left\{\frac{\left(100^{100}\right)\left(570^{570}\right)}{\left(71^{71}\right)\left(600^{600}\right)}\right\}$$

$$= \left\{15{,}000e\sqrt{\frac{(100)(570)}{(71)(600)}}\right\}\left(\frac{100}{71}\right)^{71}100^{29}\left(\frac{570}{600}\right)^{570}\frac{1}{600^{30}}$$

$$= \left\{15{,}000e\sqrt{\frac{(100)(570)}{(71)(600)}}\right\}\left\{\left(\frac{100}{71}\right)^{71}\right\}$$

$$\times\left\{\left(\frac{570}{600}\right)^{570}\right\}\left\{\left(\frac{100}{600}\right)^{29}\right\}\left\{\frac{1}{600}\right\}.$$

Each of the factors in the curly brackets is easily computed on a hand calculator, and the result is

$$P = 1.55 \times 10^{-23}.$$

The premise in Brown's story is therefore *highly unlikely*. No matter, though, because while it is *so* unlikely as to be verging on the "just can't happen," it is not impossible, and besides, it's a very funny story and well worth a willing suspension of disbelief.[5]

## Example 4

Quadratic equations are routinely encountered in mathematical physics (you'll see an example of this in Chapter 9), and here's an example of a quadratic in the form of a type of problem that many

readers will recall from a high school algebra class. Readers may take some comfort in learning that it was incorrectly solved by Marilyn vos Savant in her *Parade Magazine* column of June 22, 2014 (but, to her credit, she quickly admitted her slip in the July 13 column after some attentive readers set her straight).

Brad and Angelina, working together, take 6 hours to complete a project. Working alone, Brad would take 4 hours longer to do the project than would Angelina if she did it by herself. How long would it take each to do the project by themselves?

If we denote Angelina's time by $x$, then Brad's time is $x+4$. Thus, Angelina's *rate* of clearing the project is $\frac{1}{x}$ per hour, and Brad's rate is $\frac{1}{x+4}$. So, in six hours Angelina finishes the fraction $\frac{6}{x}$ of the project, and Brad finishes the fraction $\frac{6}{x+4}$ of the project. These two fractions must total the finished project (that is, must add to 1), and so $\frac{6}{x}+\frac{6}{x+4}=1$. Cross-multiplying, we get $6(x+4)+6x=x(x+4)=x^2+4x$ or, $12x+24=x^2+4x$ or,

$$x^2-8x-24=0.$$

The well-known formula for the quadratic equation gives

$$x=\frac{8\pm\sqrt{64+96}}{2}=\frac{8\pm\sqrt{160}}{2}=\frac{8\pm4\sqrt{10}}{2}=4\pm2\sqrt{10}.$$

Since $x$ must be positive, we use the $+$ sign (the minus sign gives $x<0$), and so $x=4+2\sqrt{10}=10.32$. Thus, Angelina can do the project by herself in 10.32 hours, and Brad can do the project by himself in 14.32 hours.

The underlying assumption in this analysis is that when working together, Brad and Angelina work independently and without interference. This is not necessarily the case, depending on the nature of the project. For example, suppose "the project" is making a truck delivery. If Brad can drive a truck from A to B by himself in one hour, and if Angelina can drive the same truck from A to B by herself in one hour, how long does it take for the two of them together to drive that same truck from A to B? It's still one hour! An even more outrageous abuse of logic is the belief that if one soldier can dig a foxhole in 30 minutes, then 1,800 soldiers can dig a foxhole in one second!

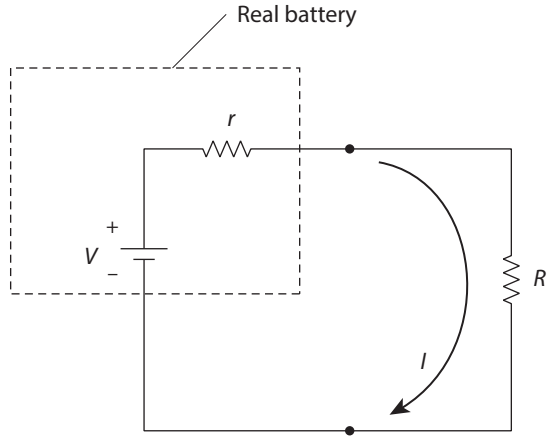


Figure 1.1. What value of $R$ dissipates maximum power?

## Example 5

A real battery (with internal resistance $r > 0$ ohms), with a potential difference between its terminals of $V$ volts (when no current is flowing in the battery), is connected to a resistor of $R$ ohms as shown in Figure 1.1. What should $R$ be so that maximum power is delivered to $R$? This problem is usually solved in textbooks with differential calculus, but that's mathematical overkill, because simple algebra is all that is required.

The current $I$ that flows is (by Ohm's law—see note 1 in Chapter 8 if this isn't clear)

$$I = \frac{V}{r + R}.$$

The power $P$ dissipated (as heat) in $R$ is (where $E$ is the voltage drop across $R$)

$$P = EI = (IR)\,I = I^2 R,$$

and so

$$P = V^2 \frac{R}{(r+R)^2}.$$

Obviously, $P = 0$ when $R = 0$, and $P = 0$ when $R = \infty$. Thus, there is some $R$ between zero and infinity for which $P$ reaches its greatest value. This value can easily be found with calculus (differentiate $P$ with respect to $R$ and set the result to zero), but all that is needed is algebra. Here's how:

$$P = V^2 \frac{R}{r^2 + 2Rr + R^2} = V^2 \frac{R}{r^2 - 2Rr + R^2 + 4Rr}$$

$$= V^2 \frac{R}{(r-R)^2 + 4Rr} = V^2 \frac{1}{\frac{(r-R)^2}{R} + 4r}.$$

We clearly maximize $P$ by minimizing the denominator of the rightmost side of this equation, which just as clearly occurs for $R = r$ (because that makes the first term in the denominator—which is never negative—as small as possible, that is, equal to zero). Thus, $R = r$, and the maximum power in $R$ is $\frac{V^2}{4R}$.

## Example 6

In this example you'll see how simple geometry, combined with physics, allows measuring the distance from the Earth to the Moon with fantastic precision. To establish the physics first, all we'll need is the idea that a ray of light incident on a mirror reflects from that mirror at an angle equal to the angle of incidence, as shown in Figure 1.2. This phenomenon was first noted by Euclid, in the third century BC; however, it was not explained until a few hundred years later, in the first century AD, when Heron of Alexandria (in his book on mirrors, *Catoptrica*) observed that the reflection law is a consequence of assuming the ray path *ARB* is the *minimum reflected length path*. That is, if the point R on the mirror was such that $\theta_i \neq \theta_r$, then the resulting total path length would be increased. Heron's observation was the

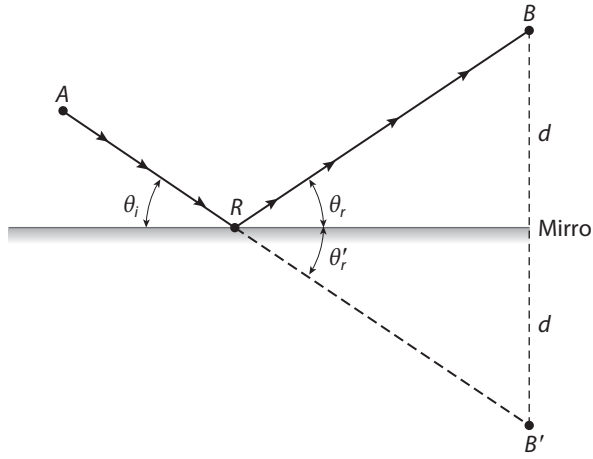


Figure 1.2. Geometry of Heron's reflection law

first occurrence of a *minimum principle* in mathematical physics; such principles play central roles in modern theoretical physics.

Here's a simple geometric proof of Heron's explanation of the mirror reflection law. If B, the destination point, is distance $d$ above the mirror, then B's reflected point (B′) is distance $d$ "below" the mirror. $RB$ and $RB'$ are therefore the equal-length hypotenuses of two congruent right triangles, which means that $\theta'_r = \theta_r$ (referring again to Figure 1.2). Now, the total light path is $AR + RB = AR + RB'$, and this last sum is the path length from A to B′. The shortest path from A to B′ (and so the shortest length for the reflected path, too) is along a straight line, and so $\theta'_r = \theta_i$, which says that $\theta_i = \theta_r$. That's it!

The law of reflection has the following application in an optical device called a *corner reflector* (see Figure 1.3). This gadget allowed the *Apollo 11* astronauts to participate in the 1969 measurement of the distance from the Earth to the Moon to within 2.5 meters! The path of an incoming ray of light to mirror 1 has the vector description $(r_x, r_y)$, and the path of the reflected ray has the vector description $(r_x, -r_y)$.[6] That is, one component of the path vector is reversed, while the other is not; mirror 1, lying along the $x$-axis, reverses the $y$-component. The reflected ray continues on to mirror 2, lying along the $y$-axis, and there the $x$-component of the path vector is reversed, giving a path vector

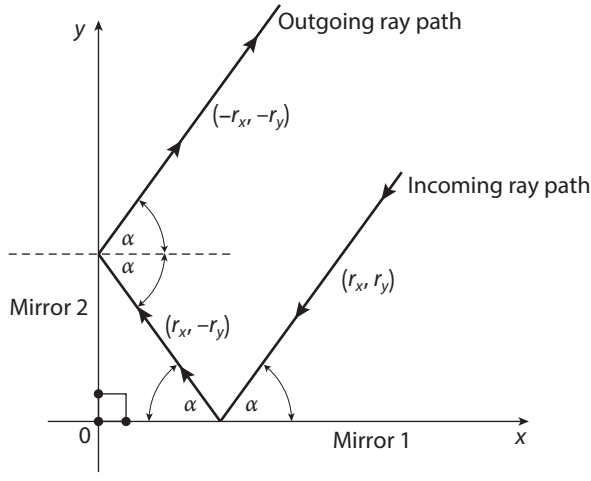


Figure 1.3. A two-dimensional corner reflector

description of the reflected ray off mirror 2 of $(-r_x, -r_y) = -(r_x, r_y)$, which is the *total reversal* of the original incoming ray's path vector. Notice that this means the reflected ray from mirror 2 is *perfectly parallel* to the incident ray on mirror 1, is laterally offset and *reversed* in direction, and these conditions are *independent* of the value of the angle $\alpha$.

Can the same thing be done in three dimensions? The answer is yes, and that is easily seen once we give the following interpretation to what a reflecting mirror does: the mirror reverses the incident ray's path vector component that is *normal* to the mirror and leaves the other component(s) unaltered. (Look back at the two-dimensional discussion and you'll see that's what happened there.) So, in the case of a *three*-dimensional corner reflector (think of the inside corner of a cube made of three mutually perpendicular mirrors, with the corner of the cube defining the origin of an $x, y, z$-coordinate system), imagine that mirrors 1, 2, and 3 lie along the $xy$-, the xz-, and the yz-surfaces, respectively. Then, a ray reflecting off mirror 1 has its z-component reversed, a ray reflecting off mirror 2 has its $y$-component reversed, and a ray reflecting off mirror 3 has its $x$-component reversed.

After an incident ray has completed three reflections it emerges from the corner cube reflector in an exactly reversed direction.

The special cases where the incoming ray hits only one (or two) of the mirrors are simply the cases where the incident ray arrives parallel to one (or two) of the mirrors, and so one (or two) of the path vector components happen to be zero (and, of course, the reversal of zero is zero). The *Apollo 11* astronauts placed multiple corner cube reflectors on the Moon's surface, which were then targets for *very* brief (picosecond[7]) laser pulses from Earth. The corner cube reflector sent reflected pulses back to almost precisely where their transmission had occurred, and the elapsed time for the Earth-to-Moon-to-Earth round trip then gave the separation distance. Such measurements have shown that the Moon is *very* slowly moving away from Earth (just an inch and a half per year), and in Chapter 10 you'll learn why.

## Example 7

Here's a simple example of high school trigonometry at work in an interesting physics setting. In Robert Serber's book on the U.S. atomic bomb project (see note 12 in the preface), mention is made of the occurrence of the equation

$$x\cos(x) = (1-a)\sin(x)$$

in one of the theoretical problems studied by the Los Alamos scientists, where $a$ is a given constant. For any particular value of $a$, what are the *positive* solutions for $x$ ($x \leq 0$ solutions were not *physically* interesting to the bomb designers)?

The most direct way to answer this question is simply to plot both sides of the equation and see where the two plots intersect. This is done for the case $a = \frac{1}{2}$ in Figure 1.4, and we see that the first approximate positive solution is at $x \approx 1.2$, and the next one is at $x \approx 4.6$. There are, of course, an infinite number of positive solutions for the plots beyond $x = 6$ in Figure 1.4. I used a computer to easily generate this figure, but you can appreciate how a technician with just a high school education and a set of math tables could easily make such plots by hand. It would be a laborious task, to be sure, and after a while processing lots of different values for the $a$-parameter wouldn't be very interesting, but

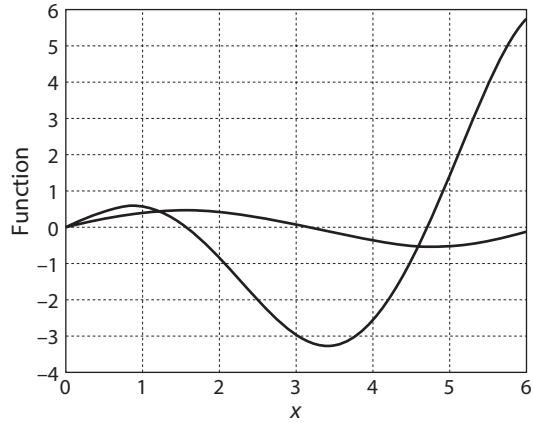


Figure 1.4. Solving $x\cos(x) = \frac{1}{2}\sin(x)$

the Los Alamos scientists had a large number of personnel available who did this sort of thing for them all day long.

## Example 8

If pi wasn't around, there would be no round pies!
— *The author, at age 10, has his first "scientific" revelation.*

Everybody "knows" that pi is a number a bit larger than 3 (pretty close to 22/7, as Archimedes showed more than 2,000 years ago) and, more accurately, is 3.14159265.... But *how* do we know the value of pi? It's the ratio of the circumference of a circle to a diameter, yes, but how can that explain how we know pi to hundreds of millions, even *trillions*, of decimal digits?[8] We can't *measure* lengths with that precision. Well then, just how *do* we *calculate* the value of pi? The symbol $\pi$ (for pi) occurs in countless formulas used by physicists and other scientists and engineers, and so this is an important question.

The short answer is, through the use of an infinite series expansion. For example, we know (after taking freshman calculus) that

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}.$$

TABLE 1.1
Calculating pi *slowly*.

| *Number of terms* | *Sum* |
|---|---|
| 100 | 3.1. . . . . |
| 1,000 | 3.14 . . . . . |
| 10,000 | 3.141 . . . . . |
| 100,000 | 3.1415. . . . . |

But since

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots,$$

which you can either derive by doing the implied long division or just confirm by simply cross-multiplying, then

$$\frac{\pi}{4} = \int_0^1 \left(1 - x^2 + x^4 - x^6 + \ldots\right) dx = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)\Big|_0^1,$$

and so

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right).$$

This famous result[9] is theoretically correct, but, alas, it is also next to useless for *calculating* $\pi$ because it converges *very* slowly. As the great Swiss-born mathematician Leonhard Euler (1707–1783) wrote (in 1737) about this way of calculating $\pi$, to get just 50 digits would be to "labor fere in aeternum" ("work almost forever"). To illustrate that claim, Table 1.1 shows some partial sums for several values of the number of terms used in the sum. As you can see, we have to increase that number by a factor of 10 (!) to determine *each* additional correct digit for pi (the ellipsis dots represent where the sum first fails to give correct digits). We clearly need a series that converges a lot faster (that is, uses far fewer terms to achieve a given number of correct digits).

As it turns out, this is not at all hard to do, as all that is required is a *minor* variation on what we've just done. Writing

$$\int_0^{1/\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}(x)|_0^{1/\sqrt{3}} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

we have

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3}\frac{1}{\sqrt{3}}\frac{1}{3} + \frac{1}{5}\frac{1}{\sqrt{3}}\frac{1}{3^2} - \frac{1}{7}\frac{1}{\sqrt{3}}\frac{1}{3^3} + \ldots,$$

and so,

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{3^2\cdot 5} - \frac{1}{3^3\cdot 7} + \ldots\right).$$

This series converges pretty quickly, and the sum of just the first 10 terms correctly gives the first five digits. The English astronomer Abraham Sharp (1651–1699) used the first 150 terms of the series (in 1699) to calculate the first 72 digits of pi. That's more than enough for physicists!

## Example 9

One day a math-deficient frog was sitting on a tiny lily pad in a big pond—a lily pad that doubled in size each night—and on this day the pad covered just one-eighth of the pond. The frog still saw the vast majority of his beloved water and so was unconcerned.

Then, just three days later, he woke to find the pond had vanished while he slept.

—*a sad cautionary tale for frogs with their heads in the sand*

Here's a simple application of calculus to a real, present-day concern. Suppose we have a *finite*, nonrenewable resource that is being steadily consumed at an increasing rate. That is, the depletion of the resource is growing exponentially. Specifically, if the quantity of the resource

consumed *today* is $r_0$, and the rate at which it is being consumed increases at a constant rate, then for some $k$ we have

$$r(t) = r_0 e^{kt}, \quad t \geq 0.$$

Such a resource is, for example, oil. If we know $r_0$, $k$, and $V$ (how much of the resource is left), then we can calculate how long ($T$) it will be until the resource is exhausted. The values of $r_0$ and $k$ are not hard to measure in the case of oil, but the value of $V$ is mostly a guessing game. Just how much oil *is* left in the world? Ten different "experts" will give 10 different answers.

For oil, let's take the present consumption as $r_0 = 6 \times 10^7 \frac{\text{cubic meters}}{\text{day}}$, and $k = 7\%$ per year. Now, no matter what we pick for $V$ there will always be somebody who thinks we are being too conservative. So, let's pick a value that *nobody* could claim to be an underestimate. Let's assume that the *entire planet* is nothing but oil. Nobody could say, then, that there are "undiscovered reserves"! Thus, taking the radius of the earth as $6.37 \times 10^6$ meters, we have the volume of the earth as

$$V = \frac{4}{3}\pi \left(6.37 \times 10^6\right)^3 \text{ cubic meters} = 1.083 \times 10^{21} \text{ cubic meters}.$$

That's a lot of oil—but it's still a finite amount—and so we ask: how long until the planet has vanished out the tailpipe of the last car?

The differential amount of oil consumed in differential time $dt'$ is $r(t')dt'$. So, the amount consumed from time $t' = 0$ to $t' = t$ is

$$\int_0^t r(t')dt' = \int_0^t r_0 e^{kt'} dt' = r_0 \left(\frac{e^{kt'}}{k}\right)\Big|_0^t = \frac{r_0}{k}(e^{kt} - 1).$$

At $t = T$ the consumed amount is, by definition, equal to *all* the oil, $V$, that we started with at $t = 0$, and so

$$V = \frac{r_0}{k}\left(e^{kT} - 1\right),$$

which we easily solve for $T$ by inspection:

$$T = \frac{1}{k}\ln\left(\frac{kV}{r_0} + 1\right).$$

Since $k = 0.07$ per year $= 1.92 \times 10^{-4}$ per day, we have

$$\begin{aligned} T &= \frac{1}{1.92 \times 10^{-4}}\ln\left(\frac{1.92 \times 10^{-4} \times 1.083 \times 10^{21}}{6 \times 10^{7}} + 1\right) \text{ days} \\ &= (5{,}208)\ln\left(0.3466 \times 10^{10}\right) \text{ days} \\ &= (5{,}208)(21.966) \text{ days} = 114{,}399 \text{ days} = 313+ \text{ years.} \end{aligned}$$

Just three more centuries and the whole planet is gone. Holy cow, this could be bad.

But wait! A returning astronaut has just discovered that there *is* more oil. The Moon! The Moon is all oil, too! Cities worldwide echo with the cheers of car owners who thought they would have to learn how to ride a bike. The world is saved—or is it? What we need to now calculate is, how much more time does the Moon oil extend our ability to consume oil?

Taking the radius of the Moon as $1.74 \times 10^6$ meters, we have the volume of the Moon as

$$\frac{4}{3}\pi\left(1.74 \times 10^{6}\right)^3 \text{ cubic meters} = 0.022 \times 10^{21} \text{cubic meters.}$$

Thus, starting with the Earth *and* the Moon, we have

$$V = \left(1.083 \times 10^{21} + 0.022 \times 10^{21}\right) \text{ cubic meters} = 1.105 \times 10^{21} \text{ cubic meters}$$

and

$$\begin{aligned} T &= \frac{1}{1.92 \times 10^{-4}}\ln\left(\frac{1.92 \times 10^{-4} \times 1.105 \times 10^{21}}{6 \times 10^{7}} + 1\right) \text{days} \\ &= (5{,}208)\ \ln\left(0.3536 \times 10^{10}\right)\ \text{days} \\ &= (5{,}208)(21.986) \text{ days} = 114{,}503 \text{ days.} \end{aligned}$$

So, if we consume not only the earth but the entire Moon, too, we'll get an extra 104 days. And then we really will be "outa gas."

The little math story I've just told you reminds me of a funny anecdote told of the great American inventor Thomas Edison. A practical man with little formal education, Edison nevertheless understood the value of education but also never missed a chance to show how a clever person could often work around a technical deficiency. For example, after hiring a young mathematician Edison assigned him the task of determining the volume of a new lightbulb, a bulb designed with an undulating shape. The mathematician carefully reduced the shape to a complicated equation and then laboriously, over a period of hours, integrated the equation over three dimensions to get the volume enclosed. Then, he proudly showed the result to Edison.

Edison congratulated the man on being a fine mathematician, as his computed answer agreed quite well with Edison's own value, which he had arrived at in less than 30 seconds. When the astonished mathematician asked how Edison had done that, the inventor (without saying a word) simply filled the bulb with water and then poured the water out of the bulb into a glass beaker with volume levels marked on the side.

Edison had made his point: math is great, but use it as a tool and not as a crutch.

## Solution to Example 1

Flip any switch ON, leave it ON for a minute or so, and then flip it OFF. Then, flip either one of the other two switches ON, and go up to the attic. If the bulb is lit, then the switch you left ON controls the bulb. If the bulb is not lit, feel it. If it's hot, then the switch you turned ON and then OFF controls the bulb. If the bulb is cold, then the third switch (the one you didn't touch) controls the bulb.

This problem, and the Edison lightbulb anecdote, remind me of a goofy "tech joke" that mathematicians like to tell: How many mathematicians does it take to change a lightbulb? The answer is *one*. That's because he or she simply hands the problem off to a group of physicists for whom (claim the mathematicians with lots of snickers

and snorts) it is already known that the answer is greater than one. The chief merit of this otherwise outrageous slander of physicists is that it illustrates the powerful trick of reducing an unsolved problem to one whose solution is already known.

## Solution to Example 2

On its upward path, the bullet trades kinetic energy for potential energy, as well as irreversibly losing energy because of air drag. So, as the bullet reaches its maximum altitude, it will begin its fall with less potential energy than the kinetic energy it had when it began its upward path. Now, during the fall, at every altitude, its potential energy is equal to what its potential energy was at the same altitude when it was going up. So, the remaining energy (its kinetic energy) at every altitude is less than it was when going up. That is, at every altitude as it falls the bullet is *always going slower* than it was when going upward. So, falling down takes longer than going upward.

## Solution to the Bobsledder Problem in the Preface

Looking back at Figure P4, we see that A has a horizontal speed component of $v_0$ (and no vertical speed component) *at every instant of time*. B, however, has an initial horizontal speed component of $v_0$ that *increases* whenever he travels downward, because he is accelerated. *Why* is he accelerated? A mass at rest on a horizontal surface exerts a force on that surface, and that surface exerts an equal but opposite (upward) reaction force back on the mass. If the reaction force weren't equal to the downward force, the mass would be accelerated and would *not* be at rest. These comments still hold when the mass moves, but as mass B moves up and down along its curved path, the reaction force has a horizontal component—to the right (which accelerates B) when moving down, and to the left (which decelerates B) when moving up. When B travels upward his horizontal speed component of course decreases back toward $v_0$, but it is never less than $v_0$ (remember, no friction). Thus, the horizontal speed component of B is, at every

instant, at least as large as A's, and so B wins the race. Notice that this conclusion is true independent of the details of B's path (assuming B's path is what mathematicians call *well behaved*; that is, it doesn't have such sharp angles that B crashes into a wall or jumps free of his path), even though that path is clearly the longer path.

## Solution to the Preface Problem in Endnote 14

Let $d$ be the distance between A and B, $s$ the speed of the airplane in still air, and $w$ the speed of the wind. Then, the total round-trip travel time $T$ is the sum of the times spent traveling with, and then against, the wind:

$$T = \frac{d}{s+w} + \frac{d}{s-w} = \frac{d\,(s-w) + d\,(s+w)}{(s+w)\,(s-w)}$$

$$= \frac{2sd}{s^2 - w^2} = \frac{2sd}{s^2\left(1 - \frac{w^2}{s^2}\right)} = \frac{2d}{s}\left[\frac{1}{1-\left(\frac{w}{s}\right)^2}\right].$$

When there is no wind ($w = 0$) then $T = \frac{2d}{s}$, and when $w > 0$ the denominator in the brackets gets smaller, and we have $T > \frac{2d}{s}$. So, a steady wind *always increases* the total travel time.

Here's a math-free way to see by inspection the special case of $w = s$. In that case the return part of the trip has the plane, with speed s, facing a headwind of the same speed. Thus, the plane *doesn't move* and so will *never* get back to A (that is, $T = \infty$ if $w = s$).

## Notes

1. Sydney Smith (1771–1845), an English cleric, was a witty commentator on life in general.

2. All we'll assume is the air resistance drag force law *f(v)*, where $v$ is the speed of the bullet, is *physically reasonable*. That means three conditions hold: (1) $f(v) > 0$ for $v > 0$, (2) $f(v) = 0$ for $v = 0$, and (3) $f(v)$ is monotonic increasing with increasing $v$.

3. I've written $N_1$ in factorial notation, where, if $n$ is a positive integer, then $n! = (n)(n-1)(n-2)\ldots(3)(2)(1)$. For example, $4! = 24$. Less obviously, if we notice $n! = n(n-1)!$, then we can conclude that $0! = 1$. Do you see this? (Try $n = 1$.)

4. Named after the Scottish mathematician James Stirling (1692–1770) but actually discovered (in 1733) by the French-born English mathematician Abraham de Moivre (1667–1754). The number $e$ is, of course, one of the most important in mathematics, with the value 2.7182818.... An asymptotic approximation has the property that while the approximation has an unbounded *absolute* error, its *relative* error approaches zero. That's why we use the $\sim$ symbol and not an equal sign. That is, if $E(n)$ is an asymptotic approximation for some function $f(n)$, then $lim_{n\to\infty} |E(n) - f(n)| = \infty$, but $lim_{n\to\infty} \frac{|E(n)-f(n)|}{f(n)} = 0$. You may say this is more than just arithmetic but, really, you can just look it up in any good book of math formulas and tables.

5. I won't ruin the story for you by revealing where Brown goes with this premise, but if you're wondering, you can find "Expedition" reprinted in *Fantasia Mathematica* (Clifton Fadiman, ed.), Simon and Schuster, 1958. I have long wondered if Brown's story was perhaps inspired by the 1954 hit tune "Thirteen Women and Only One Man in Town," by the great Bill Haley and the Comets (a fantasy about the lone male survivor of a nuclear war).

6. This vector description of the ray path can be thought of as the position vector of an individual photon in the ray.

7. The reason for such brief pulses is the enormous speed of light. Light travels 1 foot in 1 nanosecond, and so 1 inch of travel takes $\frac{1}{12}$ of a nanosecond. To make accurate Moon recession measurements, the timing must then be a *small* fraction of $\frac{1}{12}$ of a nanosecond.

8. Physicists, engineers, and other scientists rarely need to know $\pi$ to more than five or six digits, so why *trillions*? One example for the *why* comes from those mathematicians who wonder if the digits of pi are uniformly distributed. Crudely, that is, does each of the digits 0, 1, 2, ..., 8, 9 appear 10% of the time "at random"? Mathematicians need those trillions of digits to "experimentally" study this question. (As far as I know, the digits of pi *are* uniformly distributed).

9. It is due to the French mathematician Gottfried Leibniz (1646–1716), who discovered it in 1674. Leibniz was greatly taken by his formula, commenting on it that "The Lord loves odd numbers," obviously ignoring that leading *even* factor of 4.