Chapter One

Introduction

Model theory rarely deals directly with topology; the great exception is the theory of o-minimal structures, where the topology arises naturally from an ordered structure, especially in the setting of ordered fields. See [11] for a basic introduction. Our goal in this work is to create a framework of this kind for valued fields.

A fundamental tool, imported from stability theory, will be the notion of a definable type; it will play a number of roles, starting from the definition of a point of the fundamental spaces that will concern us. A definable type on a definable set V is a uniform decision, for each definable subset U (possibly defined with parameters from larger base sets), of whether $x \in U$; here x should be viewed as a kind of ideal element of V. A good example is given by any semi-algebraic function f from $\mathbb{R}$ to a real variety V. Such a function has a unique limiting behavior at ∞: for any semi-algebraic subset U of V, either $f(t) \in U$ for all large enough t, or $f(t) \notin U$ for all large enough t. In this way f determines a definable type.

One of the roles of definable types will be to be a substitute for the classical notion of a sequence, especially in situations where one is willing to refine to a subsequence. The classical notion of the limit of a sequence makes little sense in a saturated setting. In o-minimal situations it can often be replaced by the limit of a definable curve; notions such as definable compactness are defined using continuous definable maps from the field R into a variety V. Now to discuss the limiting behavior of f at ∞ (and thus to define notions such as compactness), we really require only the answer to this dichotomy—is $f(t) \in U$ for large t?—uniformly, for all U; i.e. knowledge of the definable type associated with f. For the spaces we consider, curves will not always be sufficiently plentiful to define compactness, but definable types will be, and our main notions will all be defined in these terms. In particular the limit of a definable type on a space with a definable topology is a point whose every neighborhood is large in the sense of the definable type.

A different example of a definable type is the generic type of the valuation ring $\mathcal{O}$, or of a closed ball B of K, for K a non-archimedean valued field, or of $V(\mathcal{O})$ where V is a smooth scheme over $\mathcal{O}$. Here again, for any definable subset U of $\mathbb{A}^1$, we have $v \in U$ for all sufficiently generic $v \in V$, or else $v \notin U$ for all sufficiently generic $v \in V$; where "sufficiently generic" means "having residue outside Z_U" for a certain proper Zariski closed subset Z_U of $V(\mathbf{k})$, depending only on U. Here $\mathbf{k}$ is the residue field. Note that the generic type of $\mathcal{O}$ is invariant under multiplication by $\mathcal{O}^*$ and addition by $\mathcal{O}$, and hence induces a definable type on any closed ball. Such definable types are *stably dominated*, being determined by a function into objects over the residue field, in this case the residue map into $V(\mathbf{k})$. They can also be characterized as *generically stable*. Their basic properties were developed in [20]; some results are now seen more easily using the general theory of NIP, [27].

Let V be an algebraic variety over a field K. A valuation or ordering on K induces a topology on K, hence on K^n, and finally on $V(K)$. We view this topology as an object of the definable world; for any model M, we obtain a topological space whose set of points is $V(M)$. In this sense, the topology is on V.

In the valuative case however, it has been recognized since the early days of the theory that this topology is inadequate for geometry. The valuation topology is totally disconnected, and does not afford a useful globalization of local questions. Various remedies have been proposed, by Krasner, Tate, Raynaud, Berkovich and Huber. Our approach can be viewed as a lifting of Berkovich's to the definable category. We will mention below a number of applications to classical Berkovich spaces, that indeed motivated the direction of our work.

The fundamental topological spaces we will consider will not live on algebraic varieties. Consider instead the set of semi-lattices in K^n. These are $\mathcal{O}^n$-submodules of K^n isomorphic to $\mathcal{O}^k \oplus K^{n-k}$ for some k. Intuitively, a sequence Λ_n of semi-lattices approaches a semi-lattice Λ if for any a, if $a \in \Lambda_n$ for infinitely many n then $a \in \Lambda$; and if $a \notin \mathcal{M}\Lambda_n$ for infinitely many n, then $a \notin \mathcal{M}\Lambda$. The actual definition is the same, but using definable types. A definable set of semi-lattices is *closed* if it is closed under limits of definable types. The set of closed balls in the affine line $\mathbb{A}^1$ can be viewed as a closed subset of the set of semi-lattices in K^2. In this case the limit of a decreasing sequence of balls is the intersection of these balls; the limit of the generic type of the valuation ring $\mathcal{O}$ (or of small closed balls around generic points of $\mathcal{O}$) is the closed ball $\mathcal{O}$. We also consider subspaces of these spaces of semi-lattices. They tend to be definably connected and compact, as tested by definable types. For instance the set of all semi-lattices in K^n cannot be split into two disjoint closed nonempty definable subsets.

To each algebraic variety V over a valued field K we will associate in a canonical way a projective limit $\widehat{V}$ of spaces of the type described above. A point of $\widehat{V}$ does not correspond to a point of V, but rather to a stably dominated definable type on V. We call $\widehat{V}$ the *stable completion* of V. For instance when $V = \mathbb{A}^1$, $\widehat{V}$ is the set of closed balls of V; the stably dominated type associated to a closed ball is just the generic type of that ball (which may be a point, or larger). In this case, and in general for curves, $\widehat{V}$ is definable (more precisely, a definable set of some imaginary sort), and no projective limit is needed.

While V admits no definable functions of interest from the value group Γ, there do exist definable functions from Γ to $\widehat{\mathbb{A}^1}$: for any point a of $\mathbb{A}^1$, one can consider the closed ball $B(a;\alpha) = \{x : \mathrm{val}(a-x) \geq \alpha\}$ as a definable function of $\alpha \in \Gamma$. These functions will serve to connect the space $\widehat{\mathbb{A}^1}$. In [19] the imaginary sorts were classified, and moreover the definable functions from Γ into them were classified; in the case of $\widehat{\mathbb{A}^1}$, essentially the only definable functions are the ones mentioned above. It is this kind of fact that is the basis of the geometry of imaginary sorts that we study here.

At present we remain in a purely algebraic setting. The applications to Berkovich spaces are thus only to Berkovich spaces of algebraic varieties. This limitation has the merit of showing that Berkovich spaces can be developed purely algebraically; historically, Krasner and Tate introduce analytic functions immediately even when interested in algebraic varieties, so that the name of the subject is rigid *analytic* geometry, but this is not necessary, a rigid algebraic geometry exists as well.

While we discussed o-minimality as an analogy, our real goal is a *reduction* of questions over valued fields to the o-minimal setting. The value group Γ of a valued field is o-minimal of a simple kind, where all definable objects are piecewise $\mathbb{Q}$-linear. Our main result is that for any quasi-projective variety V over K, $\widehat{V}$ admits a definable deformation retraction to a subset S, called a skeleton, which is definably homeomorphic to a space defined over Γ. There is a delicate point here: the definable homeomorphism is valid semi-algebraically, but if one stays in the (tropical) locally semi-linear setting, one must take into account subspaces of Γ_∞^n, where Γ_∞ is a partial completion of Γ by the addition of a point at ∞. The intersection of the space with the points at ∞ contains valuable additional information. In general, such a skeleton is non-canonical. At this point, o-minimal results such as triangulation can be quoted. As a corollary we obtain an equivalence of categories between the category of definable subsets of quasi-projective varieties over K, with homotopy classes of definable continuous maps $\widehat{U} \to \widehat{V}$ as morphisms $U \to V$, and a homotopy category of definable spaces over the o-minimal Γ.

In case the value group is $\mathbb{R}$, our results specialize to similar tameness theorems for Berkovich spaces. In particular we obtain local contractibility for Berkovich spaces associated to algebraic varieties, a result which was proved by Berkovich under smoothness assumptions [5], [6]. We also show that for projective varieties, the corresponding Berkovich space is homeomorphic to a projective limit of finite-dimensional simplicial complexes that are deformation retracts of itself. We further obtain finiteness statements that were not known classically; we refer to Chapter 14 for these applications.

We now present the contents of the chapters and a sketch of the proof of the main theorem.

Chapter 2 includes some background material on definable sets, definable types, orthogonality and domination, especially in the valued field context. In 2.11 we present the main result of [20] with a new insight regarding one point, that will be used in several critical points later in the paper. We know that every nonempty definable set over an algebraically closed substructure of a model of ACVF extends to a definable type. A definable type p can be decomposed into a definable type q on Γ^n, and a map f from this type to stably dominated definable types. In previous definitions of metastability, this decomposition involved an uncontrolled base change that prevented any canonicity. We note here that the q-germ of f is defined with no additional parameters, and that it is this germ that really determines p. Thus a general definable type is a function from a definable type on Γ^n to stably dominated definable types.

In Chapter 3 we introduce the space $\widehat{V}$ of stably dominated types on a definable set V. We show that $\widehat{V}$ is pro-definable; this is in fact true in any NIP theory, and not just in ACVF. We further show that $\widehat{V}$ is strict pro-definable, i.e. the image of $\widehat{V}$ under any projection to a definable set is definable. This uses metastability, and also a classical definability property of irreducibility in algebraically closed fields. In the case of curves, we note later that $\widehat{V}$ is in fact definable; for many purposes strict pro-definable sets behave in the same way. Still in Chapter 3, we define a topology on $\widehat{V}$, and study the connection between this topology and V. Roughly speaking, the topology on $\widehat{V}$ is generated by $\widehat{U}$, where U is a definable set cut out by *strict* valuation inequalities. The space V is a dense subset of $\widehat{V}$, so a continuous map $\widehat{V} \to \widehat{U}$ is determined by the restriction to V. Conversely, given a definable map $V \to \widehat{U}$, we explain the conditions for extending it to $\widehat{V}$. This uses the interpretation of $\widehat{V}$ as a set of definable types. We determine the Grothendieck topology on V itself induced from the topology on $\widehat{V}$; the closure or continuity of definable subsets or of functions on V can be described in terms of this Grothendieck topology without reference to $\widehat{V}$, but we will see that this viewpoint is more limited.

In Chapter 4 we define the central notion of definable compactness; we give a general definition that may be useful whenever one has definable topologies with enough definable types. The o-minimal formulation regarding limits of curves is replaced by limits of definable types. We relate definable compactness to being closed and bounded. We show the expected properties hold, in particular the image of a definably compact set under a continuous definable map is definably compact.

The definition of $\widehat{V}$ is a little abstract. In Chapter 5 we give a concrete representation of $\widehat{\mathbb{A}^n}$ in terms of spaces of semi-lattices. This was already alluded to in the first paragraphs of the introduction.

A major issue in this paper is the frontier between the definable and the topological categories. In o-minimality automatic continuity theorems play a role. Here we did not find such results very useful. At all events in 6.2 we characterize topologically those subspaces of $\widehat{V}$ that can be definably parameterized by Γ^n. They turn out to be o-minimal in the topological sense too. We use here in an essential way the construction of $\widehat{V}$ in terms of spaces of semi-lattices, and the characterization in [19] of definable maps from Γ into such spaces. We shall prove that our retractions provide skeleta lying in the subspace $V^{\#}$ of $\widehat{V}$ of *strongly stably dominated types* introduced in 8.1. This is another canonical space associated with V, ind-definable this time, admitting a natural continuous map into $\widehat{V}$ which restricts to a topological embedding on definable subsets. We study it further in Chapter 8; our uniformity results for $\widehat{V}$ depend on it.

Chapter 7 is concerned with the case of curves. We show that $\widehat{C}$ is definable (and not just pro-definable) when C is a curve. The case of $\mathbb{P}^1$ is elementary, and in equal characteristic zero it is possible to reduce everything to this case. But in general we use model-theoretic methods. We construct a definable deformation retraction from $\widehat{C}$ into a Γ-internal subset. We consider relative curves too, i.e. varieties V with maps $f : V \to U$, whose fibers are of dimension one. In this case we prove the existence of a deformation retraction of all fibers that is globally continuous and takes $\widehat{C}$ into a Γ-internal subset for almost all fibers C, i.e. all outside a proper subvariety of U. On curves lying over this variety, the motions on nearby curves do not converge to any continuous motion.

Chapter 9 contains some algebraic criteria for the verification of continuity. For the Zariski topology on algebraic varieties, the valuative criterion is useful: a constructible set is closed if it is invariant under specializations. Here we are led to doubly valued fields. These can be obtained from valued fields *either* by adding a valued field structure to the residue field, *or* by enriching the value group with a new convex subgroup. The functor $\widehat{X}$ is meaningful for definable sets of this theory as well, and interacts well with the various specializations. These criteria are used in Chapter 10 to verify the continuity of the relative homotopies of Chapter 7.

Chapter 10 includes some additional material on homotopies. In particular, for a smooth variety V, there exists an "inflation" homotopy, taking a simple point to the generic type of a small neighborhood of that point. This homotopy has an image that is properly a subset of $\widehat{V}$, and cannot be understood directly in terms of definable subsets of V. The image of this homotopy retraction has the merit of being contained in $\widehat{U}$ for any dense Zariski open subset U of V.

Chapter 11 contains the statement and proof of the main theorem. For any quasi-projective algebraic variety V, we prove the existence of a definable homotopy retraction from $\widehat{V}$ to an o-minimal subspace of the type described in 6.2. After some preliminary reductions, we may assume V fibers over a variety U of lower dimension and the fibers are curves. On each fiber, a homotopy retraction can be constructed with o-minimal image, as in Chapter

7; above a certain Zariski open subset U_1 of U, these homotopies can be viewed as the fibers of a single homotopy h_1. We require however a global homotopy. The homotopy h_1 itself does not extend to the complement of U_1; but in the smooth case, one can first apply an inflation homotopy whose image lies in $\widehat{V_1}$, where V_1 is the pullback of U_1. If V has singular points, a more delicate preparation is necessary. Let S_1 be the image of the homotopy h_1. Now a relative version of the results of 6.2 applies (Theorem 6.4.2); after pulling back the situation to a finite covering U' of U, we show that S_1 embeds topologically into $U' \times \Gamma_\infty^N$. Now any homotopy retraction of $\widehat{U}$, lifting to $\widehat{U'}$ and fixing certain functions into Γ^m, can be extended to a homotopy retraction of S_1 (Theorem 6.4.4). Using induction on dimension, we apply this to a homotopy retraction taking U to an o-minimal set; we obtain a retraction of V to a subset S_2 of S_1 lying over an o-minimal set, hence itself o-minimal. At this point o-minimal topology as in [9] applies to S_2, and hence to the homotopy type of $\widehat{V}$. In 11.7 we give a uniform version of Theorem 11.1.1 with respect to parameters. In Chapter 12 we examine the simplifications occuring in the proof of the main theorem in the smooth case and in Chapter 13 we deduce an equivalence of categories between a certain homotopy category of definable subsets of quasi-projective varieties over a given valued field and a suitable homotopy category of definable spaces over the o-minimal Γ.

Chapter 14 contains various applications to classical Berkovich spaces. Let V be a quasi-projective variety over a field F endowed with a non-archimedean norm and let V^{an} be the corresponding Berkovich space. We deduce from our main theorem several new results on the topology of V^{an} which were not known previously in such a level of generality. In particular we show that V^{an} admits a strong deformation retraction to a subspace homeomorphic to a finite simplicial complex and that V^{an} is locally contractible. We prove a finiteness statement for the homotopy type of fibers in families. We also show that if V is projective, V^{an} is homeomorphic to a projective limit of finite-dimensional simplicial complexes that are deformation retracts of V^{an}.

We do not assume any previous knowledge of Berkovich spaces, but highly recommend the survey [13], as well as [14] for an introduction to the model-theoretic viewpoint, and a sketch of proof of Theorem 11.1.1.

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