CHAPTER I

Representations of the fundamental group and the torsor of deformations. An overview

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I.1. Introduction

I.1.1. We develop a new approach to the p-adic Simpson correspondence, closely related to Faltings' original approach [**27**], and inspired by the work of Ogus and Vologodsky [**59**] on an analogue in characteristic p of the complex Simpson correspondence. Before giving the details of this approach in Chapters II and III, we give a summary in this introductory chapter.

I.1.2. Let K be a complete discrete valuation ring of characteristic 0, with perfect residue field of characteristic $p>0$, $\mathscr{O}_K$ the valuation ring of K, $\overline{K}$ an algebraic closure of K, and $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let X be a smooth $\mathscr{O}_K$-scheme of finite type with integral generic geometric fiber $X_{\overline{K}}$, $\overline{x}$ a geometric point of $X_{\overline{K}}$, and $\mathfrak{X}$ the formal scheme p-adic completion of $X\otimes_{\mathscr{O}_K}\mathscr{O}_{\overline{K}}$. In this work, we consider a more general smooth logarithmic situation (cf. II.6.2 and III.4.7). Nevertheless, to simplify the presentation, we restrict ourselves in this introductory chapter to the smooth case in the usual sense. We are looking for a functor from the category of p-adic representations of the geometric fundamental group $\pi_1(X_{\overline{K}},\overline{x})$ (that is, the finite-dimensional continuous linear $\mathbb{Q}_p$-representations of $\pi_1(X_{\overline{K}},\overline{x})$) to the category of Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundles (that is, the pairs $(\mathscr{M},\theta)$ consisting of a locally projective $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module of finite type $\mathscr{M}$ and an $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-linear morphism $\theta\colon \mathscr{M}\to\mathscr{M}\otimes_{\mathscr{O}_X}\Omega^1_{X/\mathscr{O}_K}$ such that $\theta\wedge\theta=0$). Following Faltings' strategy, which at present has been only partly achieved, this functor should extend to a strictly larger category than that of the p-adic representations of $\pi_1(X_{\overline{K}},\overline{x})$, called category of *generalized representations* of $\pi_1(X_{\overline{K}},\overline{x})$. It would then be an *equivalence of categories* between this new category and the category of Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundles. The main motivation for the present work is the construction of such an equivalence of categories. When X_K is a proper and smooth curve over K, Faltings shows that the Higgs bundles associated with the "true" p-adic representations of $\pi_1(X_{\overline{K}},\overline{x})$ are semi-stable of slope zero and expresses the hope that all semi-stable Higgs bundles of slope zero are obtained this way. This statement, which would correspond to the difficult part of Simpson's result in the complex case, seems out of reach at present.

I.1.3. The notion of generalized representations is due to Faltings. They are, in simplified terms, continuous p-adic *semi-linear* representations of $\pi_1(X_{\overline{K}},\overline{x})$ on modules over a certain p-adic ring endowed with a continuous action of $\pi_1(X_{\overline{K}},\overline{x})$. Faltings' approach in [**27**] to construct a functor $\mathscr{H}$ from the category of these generalized representations to the category of Higgs bundles consists of two steps. He first defines $\mathscr{H}$ for the generalized representations that are p-adically close to the trivial representation, which he calls *small*. He carries out this step in arbitrary dimension. In the second step, achieved only for curves, he extends the functor $\mathscr{H}$ to all generalized representations of $\pi_1(X_{\overline{K}}, \overline{x})$ by descent. Indeed, every generalized representation becomes small over a finite étale cover of $X_{\overline{K}}$.

I.1.4. Our new approach, which works in arbitrary dimension, allows us to define the functor $\mathscr{H}$ on the category of generalized representations of $\pi_1(X_{\overline{K}}, \overline{x})$ satisfying an admissibility condition à la Fontaine, called *Dolbeault* generalized representations. For this purpose, we introduce a family of period rings that we call *Higgs–Tate algebras*, and that are the main novelty of our approach compared to that of Faltings. We show that the admissibility condition for rational coefficients corresponds to the smallness condition of Faltings; but it is strictly more general for integral coefficients. Note that Faltings' construction for small rational coefficients is limited to curves and that it presents a number of difficulties that can be avoided with our approach.

I.1.5. We proceed in two steps. We first study in Chapter II the case of an affine scheme of a certain type, called also *small* by Faltings. We then tackle in Chapter III the global aspects of the theory. The general construction is obtained from the affine case using a gluing technique presenting unexpected difficulties. To do this we will use the *Faltings topos*, a fibered variant of Deligne's notion of *covanishing topos*, which we develop in Chapter VI.

I.1.6. This introductory chapter offers, in a geometric situation simplified for the clarity of the exposition, a detailed summary of the global steps leading to our main results. Let us take a quick look at its contents. We begin, in I.3, with a short aside on small generalized representations in the affine case, which will be used as intermediary for the study of Dolbeault representations. Section I.4 summarizes the local study conducted in Chapter II. We introduce the notion of generalized Dolbeault representation for a small affine scheme and the companion notion of solvable Higgs module, and then construct a natural equivalence between these two categories. We in fact develop two variants, an integral one and a more subtle rational one. We establish links between these notions and Faltings smallness conditions. We also link this to Hyodo's theory [**43**]. The global aspects of the theory developed in Chapter III are summarized in Sections I.5 and I.6. After a short introduction to Faltings' ringed topos in I.5, we introduce the *Higgs–Tate algebras* (I.5.13). The notion of *Dolbeault module* that globalizes that of generalized Dolbeault representation and the companion notion of *solvable Higgs bundle* are defined in I.6.13. Our main result (I.6.18) is the equivalence of these two categories. For the proof of this result, we need acyclicity statements for the Higgs–Tate algebras that we give in I.6.5 and I.6.8, which also allow us to show the compatibility of this equivalence with the relevant cohomologies on each side (I.6.19). We also study the functoriality of the various introduced properties by étale morphisms (I.6.21), as well as their local character for the étale topology (I.6.22, I.6.23, I.6.24). Finally, we return in this global situation to the logical links (I.6.26, I.6.27, I.6.28), for a Higgs bundle, between smallness (I.6.25) and solvability.

At the beginning of Chapters II and III, the reader will find a detailed description of their structure. Chapter VI, which is of separate interest, has its own introduction.

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I.2. Notation and conventions

All rings in this chapter have an identity element; all ring homomorphisms map the identity element to the identity element. We mostly consider commutative rings, and rings are assumed to be commutative unless stated otherwise; in particular, when we take a ringed topos (X, A), the ring A is assumed to be commutative unless stated otherwise.

I.2.1. In this introduction, K denotes a complete discrete valuation ring of characteristic 0, with perfect residue field k of characteristic $p > 0$, $\mathscr{O}_K$ the valuation ring of K, $\overline{K}$ an algebraic closure of K, $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$, $\mathscr{O}_C$ the p-adic Hausdorff completion of $\mathscr{O}_{\overline{K}}$, and C the field of fractions of $\mathscr{O}_C$. From I.5 on, we will assume that k is algebraically closed. We set $S = \mathrm{Spec}(\mathscr{O}_K)$, $\overline{S} = \mathrm{Spec}(\mathscr{O}_{\overline{K}})$, and $\check{\overline{S}} = \mathrm{Spec}(\mathscr{O}_C)$. We denote by s (resp. η, resp. $\overline{\eta}$) the closed point of S (resp. the generic point of S, resp. the generic point of $\overline{S}$). For any integer $n \geq 1$ and any S-scheme X, we set $S_n = \mathrm{Spec}(\mathscr{O}_K/p^n\mathscr{O}_K)$,

$$X_n = X \times_S S_n, \quad \overline{X} = X \times_S \overline{S}, \quad \text{and} \quad \check{\overline{X}} = X \times_S \check{\overline{S}}. \tag{I.2.1.1}$$

For any abelian group M, we denote by $\widehat{M}$ its p-adic Hausdorff completion.

I.2.2. Let G be a profinite group and A a topological ring endowed with a continuous action of G by ring homomorphisms. An *A-representation* of G consists of an A-module M and an A-semi-linear action of G on M, that is, such that for all $g \in G$, $a \in A$, and $m \in M$, we have $g(am) = g(a)g(m)$. We say that the A-representation is *continuous* if M is a topological A-module and if the action of G on M is continuous. Let M, N be two A-representations (resp. two continuous A-representations) of G. A morphism from M to N is a G-equivariant and A-linear (resp. G-equivariant, continuous, and A-linear) morphism from M to N.

I.2.3. Let (X, A) be a ringed topos and E an A-module. A *Higgs A-module with coefficients in E* is a pair (M, θ) consisting of an A-module M and an A-linear morphism $\theta\colon M \to M \otimes_A E$ such that $\theta \wedge \theta = 0$ (cf. II.2.8). Following Simpson ([**68**] p. 24), we call *Dolbeault* complex of (M, θ) and denote by $\mathbb{K}^\bullet(M, \theta)$ the complex of cochains of A-modules

$$M \to M \otimes_A E \to M \otimes_A \wedge^2 E \dots \tag{I.2.3.1}$$

deduced from θ (cf. II.2.8.2).

I.2.4. Let (X, A) be a ringed topos, B an A-algebra, M a B-module, and $\lambda \in \Gamma(X, A)$. A λ-*connection* on M with respect to the extension B/A consists of an A-linear morphism

$$\nabla\colon M \to \Omega^1_{B/A} \otimes_B M \tag{I.2.4.1}$$

such that for all local sections x of B and s of M, we have

$$\nabla(xs) = \lambda d(x) \otimes s + x\nabla(s). \tag{I.2.4.2}$$

It is *integrable* if $\nabla \circ \nabla = 0$ (cf. II.2.10). We will leave the extension B/A out of the terminology when there is no risk of confusion.

Let (M, ∇), (M', ∇') be two modules with λ-connections. A morphism from (M, ∇) to (M', ∇') is a B-linear morphism $u\colon M \to M'$ such that $(\mathrm{id} \otimes u) \circ \nabla = \nabla' \circ u$.

Classically, 1-connections are called *connections*. Integrable 0-connections are the Higgs B-fields with coefficients in $\Omega^1_{B/A}$.

Remark I.2.5. Let (X, A) be a ringed topos, B an A-algebra, $\lambda \in \Gamma(X, A)$, and (M, ∇) a module with λ-connection with respect to the extension B/A. Suppose that there exist an A-module E and a B-linear isomorphism $\gamma\colon E \otimes_A B \xrightarrow{\sim} \Omega^1_{B/A}$ such that for every local section ω of E, we have $d(\gamma(\omega \otimes 1)) = 0$. The λ-connection ∇ is integrable if and only if the morphism $\theta\colon M \to E \otimes_A M$ induced by ∇ and γ is a Higgs A-field on M with coefficients in E (cf. II.2.12).

I.2.6. If $\mathscr{C}$ is an additive category, we denote by $\mathscr{C}_{\mathbb{Q}}$ and call *category of objects of $\mathscr{C}$ up to isogeny* the category with the same objects as $\mathscr{C}$, and such that the set of morphisms between two objects is given by

$$\mathrm{Hom}_{\mathscr{C}_{\mathbb{Q}}}(E, F) = \mathrm{Hom}_{\mathscr{C}}(E, F) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{I.2.6.1}$$

The category $\mathscr{C}_{\mathbb{Q}}$ is none other than the localized category of $\mathscr{C}$ with respect to the multiplicative system of the *isogenies* of $\mathscr{C}$ (cf. III.6.1). We denote by

$$\mathscr{C} \to \mathscr{C}_{\mathbb{Q}}, \quad M \mapsto M_{\mathbb{Q}} \tag{I.2.6.2}$$

the localization functor. If $\mathscr{C}$ is an abelian category, the category $\mathscr{C}_{\mathbb{Q}}$ is abelian and the localization functor (I.2.6.2) is exact. Indeed, $\mathscr{C}_{\mathbb{Q}}$ identifies canonically with the quotient of $\mathscr{C}$ by the thick subcategory of objects of finite exponent (III.6.1.4).

I.2.7. Let (X, A) be a ringed topos. We denote by $\mathbf{Mod}(A)$ the category of A-modules of X and by $\mathbf{Mod}_{\mathbb{Q}}(A)$, instead of $\mathbf{Mod}(A)_{\mathbb{Q}}$, the category of A-modules up to isogeny (I.2.6). The tensor product of A-modules induces a bifunctor

$$\mathbf{Mod}_{\mathbb{Q}}(A) \times \mathbf{Mod}_{\mathbb{Q}}(A) \to \mathbf{Mod}_{\mathbb{Q}}(A), \quad (M, N) \mapsto M \otimes_{A_{\mathbb{Q}}} N \tag{I.2.7.1}$$

making $\mathbf{Mod}_{\mathbb{Q}}(A)$ into a symmetric monoidal category with $A_{\mathbb{Q}}$ as unit object. The objects of $\mathbf{Mod}_{\mathbb{Q}}(A)$ will also be called $A_{\mathbb{Q}}$-*modules*. This terminology is justified by considering $A_{\mathbb{Q}}$ as a monoid of $\mathbf{Mod}_{\mathbb{Q}}(A)$.

I.2.8. Let (X, A) be a ringed topos and E an A-module. We call *Higgs A-isogeny with coefficients in E* a quadruple

$$(M, N, u\colon M \to N, \theta\colon M \to N \otimes_A E) \tag{I.2.8.1}$$

consisting of two A-modules M and N and two A-linear morphisms u and θ satisfying the following property: there exist an integer $n \neq 0$ and an A-linear morphism $v\colon N \to M$ such that $v \circ u = n \cdot \mathrm{id}_M$, $u \circ v = n \cdot \mathrm{id}_N$, and that $(M, (v \otimes \mathrm{id}_E) \circ \theta)$ and $(N, \theta \circ v)$ are Higgs A-modules with coefficients in E (I.2.3). Note that u induces an isogeny of

Higgs modules from $(M, (v \otimes \mathrm{id}_E) \circ \theta)$ to $(N, \theta \circ v)$ (III.6.1), whence the terminology. Let (M, N, u, θ), (M', N', u', θ') be two Higgs A-isogenies with coefficients in E. A morphism from (M, N, u, θ) to (M', N', u', θ') consists of two A-linear morphisms $\alpha\colon M \to M'$ and $\beta\colon N \to N'$ such that $\beta \circ u = u' \circ \alpha$ and $(\beta \otimes \mathrm{id}_E) \circ \theta = \theta' \circ \alpha$. We denote by $\mathbf{HI}(A, E)$ the category of Higgs A-isogenies with coefficients in E. It is an additive category. We denote by $\mathbf{HI}_{\mathbb{Q}}(A, E)$ the category of objects of $\mathbf{HI}(A, E)$ up to isogeny.

I.2.9. Let (X, A) be a ringed topos, B an A-algebra, and $\lambda \in \Gamma(X, A)$. We call *λ-isoconnection with respect to the extension B/A* (or simply *λ-isoconnection* when there is no risk of confusion) a quadruple

$$(M, N, u\colon M \to N, \nabla\colon M \to \Omega^1_{B/A} \otimes_B N) \tag{I.2.9.1}$$

where M and N are B-modules, u is an isogeny of B-modules (III.6.1), and ∇ is an A-linear morphism such that for all local sections x of B and t of M, we have

$$\nabla(xt) = \lambda d(x) \otimes u(t) + x\nabla(t). \tag{I.2.9.2}$$

For every B-linear morphism $v\colon N \to M$ for which there exists an integer n such that $u \circ v = n \cdot \mathrm{id}_N$ and $v \circ u = n \cdot \mathrm{id}_M$, the pairs $(M, (\mathrm{id} \otimes v) \circ \nabla)$ and $(N, \nabla \circ v)$ are modules with $(n\lambda)$-connections (I.2.2), and u is a morphism from $(M, (\mathrm{id} \otimes v) \circ \nabla)$ to $(N, \nabla \circ v)$. We call the λ-isoconnection (M, N, u, ∇) *integrable* if there exist a B-linear morphism $v\colon N \to M$ and an integer $n \neq 0$ such that $u \circ v = n \cdot \mathrm{id}_N$, $v \circ u = n \cdot \mathrm{id}_M$, and that the $(n\lambda)$-connections $(\mathrm{id} \otimes v) \circ \nabla$ on M and $\nabla \circ v$ on N are integrable.

Let (M, N, u, ∇) and (M', N', u', ∇') be two λ-isoconnections. A morphism from (M, N, u, ∇) to (M', N', u', ∇') consists of two B-linear morphisms $\alpha\colon M \to M'$ and $\beta\colon N \to N'$ such that $\beta \circ u = u' \circ \alpha$ and $(\mathrm{id} \otimes \beta) \circ \nabla = \nabla' \circ \alpha$.

I.3. Small generalized representations

I.3.1. In this section, we fix a smooth affine S-scheme $X = \mathrm{Spec}(R)$ such that $X_{\overline{\eta}}$ is connected and X_s is nonempty, an integer $d \geq 1$, and an étale S-morphism

$$X \to \mathbb{G}^d_{m,S} = \mathrm{Spec}(\mathscr{O}_K[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]). \tag{I.3.1.1}$$

This is the typical example of a Faltings' small affine scheme. The assumption that $X_{\overline{\eta}}$ is connected is not necessary but allows us to simplify the presentation. The reader will recognize the logarithmic nature of the datum (I.3.1.1). Following [**27**], we consider in this work a more general smooth logarithmic situation, which turns out to be necessary even for defining the p-adic Simpson correspondence for a proper smooth curve over S. Indeed, in the second step of the descent, we will need to consider finite covers of its generic fiber, which brings us to the case of a semi-stable scheme over S. Nevertheless, to simplify the presentation, we will restrict ourselves in this introduction to the smooth case in the usual sense (cf. II.6.2 for the logarithmic smooth affine case). We denote by t_i the image of T_i in R $(1 \leq i \leq d)$, and we set

$$R_1 = R \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}}. \tag{I.3.1.2}$$

I.3.2. Let $\overline{y}$ be a geometric point of $X_{\overline{\eta}}$ and $(V_i)_{i \in I}$ a universal cover of $X_{\overline{\eta}}$ at $\overline{y}$. We denote by Δ the geometric fundamental group $\pi_1(X_{\overline{\eta}}, \overline{y})$. For every $i \in I$, we denote by $\overline{X}_i = \mathrm{Spec}(R_i)$ the integral closure of $\overline{X}$ in V_i, and we set

$$\overline{R} = \varinjlim_{i \in I} R_i. \tag{I.3.2.1}$$

In this context, the *generalized representations* of Δ are the continuous $\widehat{\overline{R}}$-representations of Δ with values in projective $\widehat{\overline{R}}$-modules of finite type, endowed with their p-adic topologies (I.2.2). Such a representation M is called *small* if M is a free $\widehat{\overline{R}}$-module of finite type having a basis made up of elements that are Δ-invariant modulo $p^{2\alpha}M$ for a rational number $\alpha > \frac{1}{p-1}$. The main property of the small generalized representations of Δ is their good behavior under descent for certain quotients of Δ isomorphic to $\mathbb{Z}_p(1)^d$. Let us fix such a quotient Δ_∞ by choosing, for every $1 \leq i \leq d$, a compatible system $(t_i^{(n)})_{n\in\mathbb{N}}$ of p^nth roots of t_i in $\overline{R}$. We define the notion of small $\widehat{R_1}$-representation of Δ_∞ similarly. The functor

$$M \mapsto M \otimes_{\widehat{R_1}} \widehat{\overline{R}} \tag{I.3.2.2}$$

from the category of small $\widehat{R_1}$-representations of Δ_∞ to that of small $\widehat{\overline{R}}$-representations of Δ is then an equivalence of categories (cf. II.14.4). This is a consequence of Faltings' almost purity theorem (cf. II.6.16; **[26]** § 2b).

I.3.3. If (M, φ) is a small $\widehat{R_1}$-representation of Δ_∞, we can consider the logarithm of φ, which is a homomorphism from Δ_∞ to $\mathrm{End}_{\widehat{R_1}}(M)$. By fixing a $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$, the latter can be written uniquely as

$$\log(\varphi) = \sum_{i=1}^{d} \theta_i \otimes \chi_i \otimes \zeta^{-1}, \tag{I.3.3.1}$$

where ζ^{-1} is the dual basis of $\mathbb{Z}_p(-1)$, χ_i is the character of Δ_∞ with values in $\mathbb{Z}_p(1)$ that gives its action on the system $(t_i^{(n)})_{n\in\mathbb{N}}$, and θ_i is an $\widehat{R_1}$-linear endomorphism of M. We immediately see that

$$\theta = \sum_{i=1}^{d} \theta_i \otimes d\log(t_i) \otimes \zeta^{-1} \tag{I.3.3.2}$$

is a Higgs $\widehat{R_1}$-field on M with coefficients in $\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{R}_1(-1)$ (I.2.3) (to simplify we will say with coefficients in $\Omega^1_{R/\mathscr{O}_K}(-1)$). The resulting correspondence $(M, \varphi) \mapsto (M, \theta)$ is in fact an *equivalence of categories* between the category of small $\widehat{R_1}$-representations of Δ_∞ and that of *small* Higgs $\widehat{R_1}$-modules with coefficients in $\Omega^1_{R/\mathscr{O}_K}(-1)$ (that is, the category of Higgs $\widehat{R_1}$-modules with coefficients in $\Omega^1_{R/\mathscr{O}_K}(-1)$ whose underlying $\widehat{R_1}$-module is free of finite type and whose Higgs field is zero modulo $p^{2\alpha}$ for a rational number $\alpha > \frac{1}{p-1}$). Combining this with the previous descent statement (I.3.2.2), we obtain an equivalence between the category of small $\widehat{\overline{R}}$-representations of Δ and that of small Higgs $\widehat{R_1}$-modules with coefficients in $\Omega^1_{R/\mathscr{O}_K}(-1)$. The disadvantage of this construction is its dependence on the $(t_i^{(n)})_{n\in\mathbb{N}}$ $(1 \leq i \leq d)$, which excludes any globalization. To remedy this defect, Faltings proposes another equivalent definition that depends on another choice that can be globalized easily. Our approach, which is the object of the remainder of this introduction, was inspired by this construction.

I.4. The torsor of deformations

I.4.1. In this section, we are given a smooth affine S-scheme $X = \mathrm{Spec}(R)$ such that $X_{\overline{\eta}}$ is connected, X_s is nonempty, and that there exist an integer $d \geq 1$ and an étale S-morphism $X \to \mathbb{G}^d_{m,S}$ (but we do not fix such a morphism). We also fix a geometric point $\overline{y}$ of $X_{\overline{\eta}}$ and a universal cover $(V_i)_{i\in I}$ of $X_{\overline{\eta}}$ at $\overline{y}$, and we use the notation of I.3.2: $\Delta = \pi_1(X_{\overline{\eta}}, \overline{y})$, $R_1 = R \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}}$, and $\overline{R}$ (I.3.2.1).

I.4.2. Recall that Fontaine associates functorially with each $\mathbb{Z}_{(p)}$-algebra A the ring

(I.4.2.1) $$\mathscr{R}_A = \varprojlim_{x\mapsto x^p} A/pA$$

and a homomorphism θ from the ring $\mathrm{W}(\mathscr{R}_A)$ of Witt vectors of $\mathscr{R}_A$ to the p-adic Hausdorff completion $\widehat{A}$ of A (cf. II.9.3). We set

(I.4.2.2) $$\mathscr{A}_2(A) = \mathrm{W}(\mathscr{R}_A)/\ker(\theta)^2$$

and denote also by $\theta\colon \mathscr{A}_2(\mathscr{R}_A) \to \widehat{A}$ the homomorphism induced by θ.

For the remainder of this chapter, we fix a sequence $(p_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{O}_{\overline{K}}$ such that $p_0 = p$ and $p_{n+1}^p = p_n$ for every $n \geq 0$. We denote by $\underline{p}$ the element of $\mathscr{R}_{\mathscr{O}_{\overline{K}}}$ induced by the sequence $(p_n)_{n\in\mathbb{N}}$ and set

(I.4.2.3) $$\xi = [\underline{p}] - p \in \mathrm{W}(\mathscr{R}_{\mathscr{O}_{\overline{K}}}),$$

where $[\ ]$ is the multiplicative representative. The sequence

(I.4.2.4) $$0 \longrightarrow \mathrm{W}(\mathscr{R}_{\mathscr{O}_{\overline{K}}}) \xrightarrow{\cdot\xi} \mathrm{W}(\mathscr{R}_{\mathscr{O}_{\overline{K}}}) \xrightarrow{\theta} \mathscr{O}_C \longrightarrow 0$$

is exact (II.9.5). It induces an exact sequence

(I.4.2.5) $$0 \longrightarrow \mathscr{O}_C \xrightarrow{\cdot\xi} \mathscr{A}_2(\mathscr{O}_{\overline{K}}) \xrightarrow{\theta} \mathscr{O}_C \longrightarrow 0,$$

where $\cdot\xi$ again denotes the morphism deduced from the morphism of multiplication by ξ in $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$. The ideal $\ker(\theta)$ of $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$ has square zero. It is a free $\mathscr{O}_C$-module with basis ξ. It will be denoted by $\xi\mathscr{O}_C$. Note that unlike ξ, this module does not depend on the choice of the sequence $(p_n)_{n\in\mathbb{N}}$. We denote by $\xi^{-1}\mathscr{O}_C$ the dual $\mathscr{O}_C$-module of $\xi\mathscr{O}_C$. For every $\mathscr{O}_C$-module M, we denote the $\mathscr{O}_C$-modules $M \otimes_{\mathscr{O}_C} (\xi\mathscr{O}_C)$ and $M \otimes_{\mathscr{O}_C} (\xi^{-1}\mathscr{O}_C)$ simply by ξM and $\xi^{-1}M$, respectively.

Likewise, we have an exact sequence (II.9.11.2)

(I.4.2.6) $$0 \longrightarrow \widehat{\overline{R}} \xrightarrow{\cdot\xi} \mathscr{A}_2(\overline{R}) \xrightarrow{\theta} \widehat{\overline{R}} \longrightarrow 0.$$

The ideal $\ker(\theta)$ of $\mathscr{A}_2(\overline{R})$ has square zero. It is a free $\widehat{\overline{R}}$-module with basis ξ, canonically isomorphic to $\xi\widehat{\overline{R}}$. The group Δ acts by functoriality on $\mathscr{A}_2(\overline{R})$.

We set $\mathscr{A}_2(\overline{S}) = \mathrm{Spec}(\mathscr{A}_2(\mathscr{O}_{\overline{K}}))$, $Y = \mathrm{Spec}(\overline{R})$, $\widehat{Y} = \mathrm{Spec}(\widehat{\overline{R}})$, and $\mathscr{A}_2(Y) = \mathrm{Spec}(\mathscr{A}_2(\overline{R}))$.

I.4.3. From now on, we fix a smooth $\mathscr{A}_2(\overline{S})$-deformation $\widetilde{X}$ of $\check{\overline{X}}$, that is, a smooth $\mathscr{A}_2(\overline{S})$-scheme $\widetilde{X}$ that fits into a Cartesian diagram

(I.4.3.1)
$$\begin{tikzcd} \check{\overline{X}} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ \check{\overline{S}} \arrow[r] & \mathscr{A}_2(\overline{S}) \end{tikzcd}$$

This additional datum replaces the datum of an étale S-morphism $X \to \mathbb{G}_{m,S}^d$; in fact, such a morphism provides a deformation.

We set

(I.4.3.2) $$\mathrm{T} = \mathrm{Hom}_{\widehat{\overline{R}}}(\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}}, \xi\widehat{\overline{R}}).$$

We identify the dual $\widehat{\overline{R}}$-module with $\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}}$ (I.4.2) and denote by $\mathbf{T}$ the associated $\widehat{Y}$-vector bundle, in other words,

$$\mathbf{T} = \mathrm{Spec}(\mathrm{Sym}_{\widehat{\overline{R}}}(\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}})). \tag{I.4.3.3}$$

Let U be an open subscheme of $\widehat{Y}$ and $\widetilde{U}$ the open subscheme of $\mathscr{A}_2(Y)$ defined by U. We denote by $\mathscr{L}(U)$ the set of morphisms represented by dotted arrows that complete the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \widetilde{U} \arrow[d, dotted] \arrow[dd, bend left] \\ \check{\overline{X}} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ \check{\overline{S}} \arrow[r] & \mathscr{A}_2(\overline{S}) \end{tikzcd} \tag{I.4.3.4}$$

in such a way that it remains commutative. The functor $U \mapsto \mathscr{L}(U)$ is a T-torsor for the Zariski topology of $\widehat{Y}$. We denote by $\mathscr{F}$ the $\widehat{\overline{R}}$-module of affine functions on $\mathscr{L}$ (cf. II.4.9). The latter fits into a canonical exact sequence (II.4.9.1)

$$0 \to \widehat{\overline{R}} \to \mathscr{F} \to \xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}} \to 0. \tag{I.4.3.5}$$

This sequence induces for every integer $n \geq 1$ an exact sequence

$$0 \to \mathrm{Sym}^{n-1}_{\widehat{\overline{R}}}(\mathscr{F}) \to \mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\mathscr{F}) \to \mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}}) \to 0. \tag{I.4.3.6}$$

The $\widehat{\overline{R}}$-modules $(\mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\mathscr{F}))_{n\in\mathbb{N}}$ therefore form a filtered direct system whose direct limit

$$\mathscr{C} = \varinjlim_{n\geq 0} \mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\mathscr{F}) \tag{I.4.3.7}$$

is naturally endowed with a structure of $\widehat{\overline{R}}$-algebra. By II.4.10, the $\widehat{Y}$-scheme

$$\mathbf{L} = \mathrm{Spec}(\mathscr{C}) \tag{I.4.3.8}$$

is naturally a principal homogeneous $\mathbf{T}$-bundle on $\widehat{Y}$ that canonically represents $\mathscr{L}$.

The natural action of Δ on the scheme $\mathscr{A}_2(Y)$ induces an $\widehat{\overline{R}}$-semi-linear action of Δ on $\mathscr{F}$, such that the morphisms in sequence (I.4.3.5) are Δ-equivariant. From this we deduce an action of Δ on $\mathscr{C}$ by ring automorphisms, compatible with its action on $\widehat{\overline{R}}$, which we call *canonical action.* These actions are continuous for the p-adic topologies (II.12.4). The $\widehat{\overline{R}}$-algebra $\mathscr{C}$, endowed with the canonical action of Δ, is called the *Higgs–Tate algebra* associated with $\widetilde{X}$. The $\widehat{\overline{R}}$-representation $\mathscr{F}$ of Δ is called the *Higgs–Tate extension* associated with $\widetilde{X}$.

I.4.4. Let (M, θ) be a *small* Higgs $\widehat{\overline{R_1}}$-module with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ (that is, a Higgs $\widehat{\overline{R_1}}$-module with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R_1}}$ whose underlying $\widehat{\overline{R_1}}$-module is free of finite type and whose Higgs field is zero modulo p^{α} for a rational number $\alpha > \frac{1}{p-1}$) and let $\psi \in \mathscr{L}(\widehat{Y})$. For every $\sigma \in \Delta$, we denote by ${}^{\sigma}\psi$ the section of $\mathscr{L}(\widehat{Y})$

defined by the commutative diagram

$$\begin{tikzcd} \mathbf{L} \arrow[r, "\sigma"] \arrow[d] & \mathbf{L} \arrow[d] \\ \widehat{Y} \arrow[r, "\sigma"] \arrow[u, bend left, "\psi"] & \widehat{Y} \arrow[u, bend right, "{}^{\sigma}\psi"'] \end{tikzcd} \tag{I.4.4.1}$$

The difference $D_\sigma = \psi - {}^\sigma\psi$ is an element of $\mathrm{Hom}_{\widehat{\overline{R}}}(\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}}, \widehat{\overline{R}})$. The endomorphism $\exp((D_\sigma \otimes \mathrm{id}_M) \circ \theta)$ of $M \otimes_{\widehat{R_1}} \widehat{\overline{R}}$ is well-defined, in view of the smallness of θ. We then obtain a small $\widehat{\overline{R}}$-representation of Δ on $M \otimes_{\widehat{R_1}} \widehat{\overline{R}}$. The resulting correspondence is in fact an equivalence of categories from the category of small Higgs $\widehat{R_1}$-modules with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ to that of small $\widehat{\overline{R}}$-representations of Δ. It is essentially a quasi-inverse of the equivalence of categories defined in I.3.3.

To avoid the choice of a section ψ of $\mathscr{L}(\widehat{Y})$, we can carry out the base change from $\widehat{\overline{R}}$ to $\mathscr{C}$ and use the diagonal embedding of $\mathbf{L}$. In this setting, the previous construction can be interpreted following the classic scheme of correspondences introduced by Fontaine (or even the more classic complex analytic Riemann–Hilbert correspondence) by taking for period ring making the link between generalized representations and Higgs modules a weak p-adic completion $\mathscr{C}^\dagger$ of $\mathscr{C}$ (the completion is made necessary by the exponential). With this ring is naturally associated a notion of admissibility; it is the notion of *generalized Dolbeault representation*. Before developing this approach, we will say a few words about the ring $\mathscr{C}$ that can itself play the role of period ring between the generalized representations and Higgs modules. Indeed, $\mathscr{C}$ is an *integral model of the Hyodo ring* (cf. (I.4.6.1) and II.15.6), which explains the link between our approach and that of Hyodo.

I.4.5. Recall that Faltings has defined a canonical extension of $\widehat{\overline{R}}$-representations of $\pi_1(X, \overline{y})$

$$0 \to \rho^{-1}\widehat{\overline{R}} \to \mathscr{E} \to \Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}}(-1) \to 0, \tag{I.4.5.1}$$

where ρ is an element of $\mathscr{O}_{\overline{K}}$ of valuation $\geq \frac{1}{p-1}$ that plays an important role in his approach to p-adic Hodge theory (cf. II.7.22). We show in II.10.19 that there exists a Δ-equivariant and $\widehat{\overline{R}}$-linear morphism

$$p^{-\frac{1}{p-1}}\mathscr{F} \to \mathscr{E} \tag{I.4.5.2}$$

that fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & p^{-\frac{1}{p-1}}\widehat{\overline{R}} \arrow[r] \arrow[d, hook] & p^{-\frac{1}{p-1}}\mathscr{F} \arrow[r] \arrow[d] & p^{-\frac{1}{p-1}}\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}} \arrow[r] \arrow[d, "-c"] & 0 \\ 0 \arrow[r] & \rho^{-1}\widehat{\overline{R}} \arrow[r] & \mathscr{E} \arrow[r] & \Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}}(-1) \arrow[r] & 0 \end{tikzcd} \tag{I.4.5.3}$$

where c is the isomorphism induced by the canonical isomorphism $\widehat{\overline{R}}(1) \xrightarrow{\sim} p^{\frac{1}{p-1}}\xi\widehat{\overline{R}}$ (II.9.18). The morphism (I.4.5.2) is canonical if we take for $\widetilde{X}$ the deformation induced by an étale S-morphism $X \to \mathbb{G}^d_{m,S}$. It is important to note that in the logarithmic setting that will be considered in this work, the Faltings extension changes form slightly because the factor $\rho^{-1}\widehat{\overline{R}}$ is replaced by $(\pi\rho)^{-1}\widehat{\overline{R}}$, where π is a uniformizer for R.

I.4.6. Taking Faltings extension $\mathscr{E}$ (I.4.5.1) as a starting point, Hyodo [**43**] defines an $\widehat{\overline{R}}$-algebra $\mathscr{C}_{\mathrm{HT}}$ using a direct limit analogous to (I.4.3.7). Note that p being invertible in $\mathscr{C}_{\mathrm{HT}}$, this is equivalent to beginning with $\mathscr{E}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, which corresponds to Hyodo's original definition. The morphism (I.4.5.2) therefore induces a Δ-equivariant isomorphism of $\widehat{\overline{R}}$-algebras

(I.4.6.1) $$\mathscr{C}[\frac{1}{p}] \xrightarrow{\sim} \mathscr{C}_{\mathrm{HT}}.$$

For every continuous $\mathbb{Q}_p$-representation V of $\Gamma = \pi_1(X,\overline{y})$ and every integer i, Hyodo defines the $\widehat{R}[\frac{1}{p}]$-module $\mathrm{D}^i(V)$ by setting

(I.4.6.2) $$\mathrm{D}^i(V) = (V \otimes_{\mathbb{Q}_p} \mathscr{C}_{\mathrm{HT}}(i))^{\Gamma}.$$

The representation V is called *Hodge–Tate* if it satisfies the following conditions:

(i) V is a $\mathbb{Q}_p$-vector space of finite dimension, endowed with the p-adic topology.
(ii) The canonical morphism

(I.4.6.3) $$\oplus_{i\in\mathbb{Z}}\,\mathrm{D}^i(V)\otimes_{\widehat{R}[\frac{1}{p}]}\mathscr{C}_{\mathrm{HT}}(-i) \to V\otimes_{\mathbb{Q}_p}\mathscr{C}_{\mathrm{HT}}$$

is an isomorphism.

I.4.7. For any rational number $r \geq 0$, we denote by $\mathscr{F}^{(r)}$ the $\widehat{\overline{R}}$-representation of Δ deduced from $\mathscr{F}$ by inverse image under the morphism of multiplication by p^r on $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}\otimes_R\widehat{\overline{R}}$, so that we have an exact sequence

(I.4.7.1) $$0 \to \widehat{\overline{R}} \to \mathscr{F}^{(r)} \to \xi^{-1}\Omega^1_{R/\mathscr{O}_K}\otimes_R\widehat{\overline{R}} \to 0.$$

For every integer $n \geq 1$, this sequence induces an exact sequence

(I.4.7.2) $$0 \to \mathrm{Sym}^{n-1}_{\widehat{\overline{R}}}(\mathscr{F}^{(r)}) \to \mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\mathscr{F}^{(r)}) \to \mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\xi^{-1}\Omega^1_{R/\mathscr{O}_K}\otimes_R\widehat{\overline{R}}) \to 0.$$

The $\widehat{\overline{R}}$-modules $(\mathrm{Sym}^{n}_{\widehat{\overline{R}}}(\mathscr{F}^{(r)}))_{n\in\mathbb{N}}$ therefore form a filtered direct system, whose direct limit

(I.4.7.3) $$\mathscr{C}^{(r)} = \varinjlim_{n\geq 0} \mathrm{S}^{n}_{\widehat{\overline{R}}}(\mathscr{F}^{(r)})$$

is naturally endowed with a structure of $\widehat{\overline{R}}$-algebra. The action of Δ on $\mathscr{F}^{(r)}$ induces an action on $\mathscr{C}^{(r)}$ by ring automorphisms, compatible with its action on $\widehat{\overline{R}}$, which we call *canonical action*. The $\widehat{\overline{R}}$-algebra $\mathscr{C}^{(r)}$ endowed with this action is called the *Higgs–Tate algebra of thickness* r associated with $\widetilde{X}$. We denote by $\widehat{\mathscr{C}}^{(r)}$ the p-adic Hausdorff completion of $\mathscr{C}^{(r)}$ that we always assume endowed with the p-adic topology.

For all rational numbers $r \geq r' \geq 0$, we have an injective and Δ-equivariant canonical $\widehat{\overline{R}}$-homomorphism $\alpha^{r,r'}\colon \mathscr{C}^{(r')} \to \mathscr{C}^{(r)}$. One easily verifies that the induced homomorphism $h^{r,r'}_{\alpha}\colon \widehat{\mathscr{C}}^{(r')} \to \widehat{\mathscr{C}}^{(r)}$ is injective. We set

(I.4.7.4) $$\mathscr{C}^{\dagger} = \varinjlim_{r\in\mathbb{Q}_{>0}} \widehat{\mathscr{C}}^{(r)},$$

which we identify with a sub-$\widehat{\overline{R}}$-algebra of $\widehat{\mathscr{C}} = \widehat{\mathscr{C}}^{(0)}$. The group Δ acts naturally on $\mathscr{C}^{\dagger}$ by ring automorphisms, in a manner compatible with its actions on $\widehat{\overline{R}}$ and on $\widehat{\mathscr{C}}$.

We denote by

(I.4.7.5) $$d_{\mathscr{C}^{(r)}}\colon \mathscr{C}^{(r)} \to \xi^{-1}\Omega^1_{R/\mathscr{O}_K}\otimes_R\mathscr{C}^{(r)}$$

the universal $\overline{\widehat{R}}$-derivation of $\mathscr{C}^{(r)}$ and by

$$d_{\widehat{\mathscr{C}}^{(r)}} : \widehat{\mathscr{C}}^{(r)} \to \xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\mathscr{C}}^{(r)} \tag{I.4.7.6}$$

its extension to the completions (note that the R-module $\Omega^1_{R/\mathscr{O}_K}$ is free of finite type). The derivations $d_{\mathscr{C}^{(r)}}$ and $d_{\widehat{\mathscr{C}}^{(r)}}$ are Δ-equivariant. They are also Higgs $\overline{\widehat{R}}$-fields with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ because $\xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \overline{\widehat{R}} = d_{\mathscr{C}^{(r)}}(\mathscr{F}^{(r)}) \subset d_{\mathscr{C}^{(r)}}(\mathscr{C}^{(r)})$ (cf. I.2.5).

For all rational numbers $r \geq r' \geq 0$, we have

$$p^{r'}(\mathrm{id} \times \alpha^{r,r'}) \circ d_{\mathscr{C}^{(r')}} = p^r d_{\mathscr{C}^{(r)}} \circ \alpha^{r,r'}. \tag{I.4.7.7}$$

The derivations $p^r d_{\widehat{\mathscr{C}}^{(r)}}$ therefore induce an $\overline{\widehat{R}}$-derivation

$$d_{\mathscr{C}^\dagger} : \mathscr{C}^\dagger \to \xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \mathscr{C}^\dagger, \tag{I.4.7.8}$$

that is none other than the restriction op $d_{\widehat{\mathscr{C}}}$ to $\mathscr{C}^\dagger$.

I.4.8. For any $\overline{\widehat{R}}$-representation M of Δ, we denote by $\mathbb{H}(M)$ the Higgs $\widehat{R_1}$-module with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ defined by

$$\mathbb{H}(M) = (M \otimes_{\overline{\widehat{R}}} \mathscr{C}^\dagger)^\Delta \tag{I.4.8.1}$$

and by the Higgs field induced by $d_{\mathscr{C}^\dagger}$. For every Higgs $\widehat{R_1}$-module (N, θ) with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$, we denote by $\mathbb{V}(N)$ the $\overline{\widehat{R}}$-representation of Δ defined by

$$\mathbb{V}(N) = (N \otimes_{\widehat{R_1}} \mathscr{C}^\dagger)^{\theta_{\mathrm{tot}}=0}, \tag{I.4.8.2}$$

where $\theta_{\mathrm{tot}} = \theta \otimes \mathrm{id} + \mathrm{id} \otimes d_{\mathscr{C}^\dagger}$, and by the action of Δ induced by its canonical action on $\mathscr{C}^\dagger$. In order to make the most of these functors we establish acyclicity results for $\mathscr{C}^\dagger \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for the Dolbeault cohomology (II.12.3) and for the continuous cohomology of Δ (II.12.5), slightly generalizing earlier results of Tsuji (cf. IV).

A continuous $\overline{\widehat{R}}$-representation M of Δ is called *Dolbeault* if it satisfies the following conditions (cf. II.12.11):

(i) M is a projective $\overline{\widehat{R}}$-module of finite type, endowed with the p-adic topology;
(ii) $\mathbb{H}(M)$ is a projective $\widehat{R_1}$-module of finite type;
(iii) the canonical $\mathscr{C}^\dagger$-linear morphism

$$\mathbb{H}(M) \otimes_{\widehat{R_1}} \mathscr{C}^\dagger \to M \otimes_{\overline{\widehat{R}}} \mathscr{C}^\dagger \tag{I.4.8.3}$$

is an isomorphism.

A Higgs $\widehat{R_1}$-module (N, θ) with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ is called *solvable* if it satisfies the following conditions (cf. II.12.12):

(i) N is a projective $\widehat{R_1}$-module of finite type;
(ii) $\mathbb{V}(N)$ is a projective $\overline{\widehat{R}}$-module of finite type;
(iii) the canonical $\mathscr{C}^\dagger$-linear morphism

$$\mathbb{V}(N) \otimes_{\overline{\widehat{R}}} \mathscr{C}^\dagger \to N \otimes_{\widehat{R_1}} \mathscr{C}^\dagger \tag{I.4.8.4}$$

is an isomorphism.

One immediately sees that the functors $\mathbb{V}$ and $\mathbb{H}$ induce equivalences of categories quasi-inverse to each other between the category of Dolbeault $\widehat{\overline{R}}$-representations of Δ and that of solvable Higgs $\widehat{R_1}$-modules with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ (II.12.15).

We show that small $\widehat{\overline{R}}$-representations of Δ are Dolbeault (II.14.6), that small Higgs $\widehat{R_1}$-modules are solvable (II.13.20), and that $\mathbb{V}$ and $\mathbb{H}$ induce equivalences of categories quasi-inverse to each other between the categories of these objects (II.14.7). We in fact recover the correspondence defined in I.3.3, up to renormalization (cf. II.13.18).

I.4.9. We define the notions of *Dolbeault* $\widehat{\overline{R}}[\frac{1}{p}]$-representation of Δ and *solvable* Higgs $\widehat{R_1}[\frac{1}{p}]$-module with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ by copying the definitions given in the integral case (cf. II.12.16 and II.12.18). We show that the functors $\mathbb{V}$ and $\mathbb{H}$ induce equivalences of categories quasi-inverse to each other between the category of Dolbeault $\widehat{\overline{R}}[\frac{1}{p}]$-representations of Δ and that of solvable Higgs $\widehat{R_1}[\frac{1}{p}]$-modules with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ (II.12.24). This result is slightly more delicate than its integral analogue (I.4.8).

Unlike the integral case, the rational admissibility conditions can be interpreted in terms of divisibility conditions. More precisely, we say that a continuous $\widehat{\overline{R}}[\frac{1}{p}]$-representation M of Δ is *small* if it satisfies the following conditions:

(i) M is a projective $\widehat{\overline{R}}[\frac{1}{p}]$-module of finite type, endowed with a p-adic topology (II.2.2);

(ii) there exist a rational number $\alpha > \frac{2}{p-1}$ and a sub-$\widehat{\overline{R}}$-module M° of M of finite type, stable under Δ, generated by a finite number of elements Δ-invariant modulo $p^\alpha M^\circ$, and that generates M over $\widehat{\overline{R}}[\frac{1}{p}]$.

We say that a Higgs $\widehat{R_1}[\frac{1}{p}]$-module (N,θ) with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ is *small* if it satisfies the following conditions:

(i) N is a projective $\widehat{R_1}[\frac{1}{p}]$-module of finite type;

(ii) there exist a rational number $\beta > \frac{1}{p-1}$ and a sub-$\widehat{R_1}$-module N° of N of finite type that generates N over $\widehat{R_1}[\frac{1}{p}]$, such that we have

(I.4.9.1) $$\theta(N^\circ) \subset p^\beta \xi^{-1} N^\circ \otimes_R \Omega^1_{R/\mathscr{O}_K}.$$

Proposition I.4.10 (cf. II.13.25)**.** *A Higgs $\widehat{R_1}[\frac{1}{p}]$-module with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$ is solvable if and only if it is small.*

Proposition I.4.11 (cf. II.13.26)**.** *Every Dolbeault $\widehat{\overline{R}}[\frac{1}{p}]$-representation of Δ is small.*

We prove that the converse implication is equivalent to a descent property for small $\widehat{\overline{R}}[\frac{1}{p}]$-representations of Δ (II.14.8).

Proposition I.4.12 (cf. II.12.26)**.** *Let M be a Dolbeault $\widehat{\overline{R}}[\frac{1}{p}]$-representation of Δ and $(\mathbb{H}(M),\theta)$ the associated Higgs $\widehat{R_1}[\frac{1}{p}]$-module with coefficients in $\xi^{-1}\Omega^1_{R/\mathscr{O}_K}$. We then have a functorial canonical isomorphism in $\mathbf{D}^+(\mathbf{Mod}(\widehat{R_1}[\frac{1}{p}]))$*

(I.4.12.1) $$\mathrm{C}^\bullet_{\mathrm{cont}}(\Delta, M) \xrightarrow{\sim} \mathbb{K}^\bullet(\mathbb{H}(M),\theta),$$

where $\mathrm{C}^\bullet_{\mathrm{cont}}(\Delta, M)$ is the complex of continuous cochains of Δ with values in M and $\mathbb{K}^\bullet(\mathbb{H}(M),\theta)$ is the Dolbeault complex (I.2.3).

This statement was proved by Faltings for small representations ([**27**] § 3) and by Tsuji (IV.5.3.2).

I.4.13. It follows from (I.4.6.1) that if V is a Hodge–Tate $\mathbb{Q}_p$-representation of Γ, then $V \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}$ is a Dolbeault $\widehat{\overline{R}}[\frac{1}{p}]$-representation of Δ; we have a functorial $\widehat{R_1}$-linear isomorphism

$$\mathbb{H}(V \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}) \xrightarrow{\sim} \oplus_{i \in \mathbb{Z}} \mathrm{D}^i(V) \otimes_{\widehat{R}} \widehat{R_1}(-1), \tag{I.4.13.1}$$

and the Higgs field on $\mathbb{H}(V \otimes_{\mathbb{Z}_p} \widehat{\overline{R}})$ is induced by the $\widehat{R}$-linear morphisms

$$\mathrm{D}^i(V) \to \mathrm{D}^{i-1}(V) \otimes_R \Omega^1_{R/\mathscr{O}_K} \tag{I.4.13.2}$$

deduced from the universal derivation of $\mathscr{C}_{\mathrm{HT}}$ over $\widehat{\overline{R}}[\frac{1}{p}]$ (cf. II.15.7). Moreover, the isomorphism (I.4.13.1) is canonical if we take for $\widetilde{X}$ the deformation induced by an étale S-morphism $X \to \mathbb{G}^d_{m,S}$.

I.4.14. Hyodo ([**43**] 3.6) has proved that if $f: Y \to X$ is a proper and smooth morphism, for every integer $m \geq 0$, the sheaf $\mathrm{R}^m f_{\eta*}(\mathbb{Q}_p)$ is Hodge–Tate of weight between 0 and m; for every $0 \leq i \leq m$, we have a canonical isomorphism

$$\mathrm{D}^i(\mathrm{R}^m f_{\eta*}(\mathbb{Q}_p)) \xrightarrow{\sim} (\mathrm{R}^{m-i} f_{\eta*}(\Omega^i_{Y/X})) \otimes_R \widehat{R}, \tag{I.4.14.1}$$

and the morphism (I.4.13.2) is induced by the Kodaira–Spencer class of f. It follows that the Higgs bundle associated with $\mathrm{R}^m f_{\eta*}(\mathbb{Q}_p)$ is equal to the vector bundle

$$\oplus_{0 \leq i \leq m} \mathrm{R}^{m-i} f_{\eta*}(\Omega^i_{Y/X}), \tag{I.4.14.2}$$

endowed with the Higgs field θ defined by the Kodaira–Spencer class of f.

I.5. Faltings ringed topos

I.5.1. We will tackle in Chapter III the global aspects of the theory in a logarithmic setting. However, in order to maintain a simplified presentation, we again restrict ourselves here to the smooth case in the usual sense (cf. III.4.7 for the smooth logarithmic case). In the remainder of this introduction, we suppose that k is *algebraically closed* and we denote by X a smooth S-scheme of finite type. From I.5.12 on, we will moreover suppose that there exists a smooth $\mathscr{A}_2(\overline{S})$-deformation $\widetilde{X}$ of $\check{\overline{X}}$ that we will fix.

I.5.2. The first difficulty we encounter in gluing the local construction described in I.4 is the sheafification of the notion of generalized representation. To do this, we use the *Faltings topos*, a fibered variant of Deligne's notion of *covanishing topos* that we develop in Chapter VI. We denote by E the category of morphisms of schemes $V \to U$ over the canonical morphism $X_{\overline{\eta}} \to X$, that is, the commutative diagrams

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ X_{\overline{\eta}} \arrow[r] & X \end{tikzcd} \tag{I.5.2.1}$$

such that the morphism $U \to X$ is étale and that the morphism $V \to U_{\overline{\eta}}$ is finite étale. It is fibered over the category $\mathbf{\acute{E}t}_{/X}$ of étale X-schemes, by the functor

$$\pi: E \to \mathbf{\acute{E}t}_{/X}, \quad (V \to U) \mapsto U. \tag{I.5.2.2}$$

The fiber of π over an étale X-scheme U is the category $\mathbf{\acute{E}t}_{\mathrm{f}/U_{\overline{\eta}}}$ of finite étale schemes over $U_{\overline{\eta}}$, which we endow with the étale topology. We denote by $U_{\overline{\eta},\mathrm{f\acute{e}t}}$ the topos of sheaves of sets on $\mathbf{\acute{E}t}_{\mathrm{f}/U_{\overline{\eta}}}$ (cf. VI.9.2). If $U_{\overline{\eta}}$ is connected and if $\overline{y}$ is a geometric point of $U_{\overline{\eta}}$, denoting by $\mathbf{B}_{\pi_1(U_{\overline{\eta}},\overline{y})}$ the classifying topos of the fundamental group $\pi_1(U_{\overline{\eta}},\overline{y})$, we have a canonical equivalence of categories (VI.9.8.4)

$$\nu_{\overline{y}}\colon U_{\overline{\eta},\mathrm{f\acute{e}t}} \xrightarrow{\sim} \mathbf{B}_{\pi_1(U_{\overline{\eta}},\overline{y})}. \tag{I.5.2.3}$$

We endow E with the *covanishing topology* generated by the coverings $\{(V_i \to U_i) \to (V \to U)\}_{i\in I}$ of the following two types:

(v) $U_i = U$ for every $i \in I$, and $(V_i \to V)_{i\in I}$ is a covering;
(c) $(U_i \to U)_{i\in I}$ is a covering and $V_i = U_i \times_U V$ for every $i \in I$.

The resulting covanishing site E is also called *Faltings site* of X. We denote by $\widetilde{E}$ and call *Faltings topos* of X the topos of sheaves of sets on E. We refer to Chapter VI for a detailed study of this topos. Let us give a practical and simple description of $\widetilde{E}$.

Proposition I.5.3 (cf. VI.5.10)**.** *Giving a sheaf F on E is equivalent to giving, for every object U of $\mathbf{\acute{E}t}_{/X}$, a sheaf F_U of $U_{\overline{\eta},\mathrm{f\acute{e}t}}$, and for every morphism $f\colon U' \to U$ of $\mathbf{\acute{E}t}_{/X}$, a morphism $F_U \to f_{\mathrm{f\acute{e}t}*}(F_{U'})$, these morphisms being subject to compatibility relations such that for every covering family $(f_n\colon U_n \to U)_{n\in\Sigma}$ of $\mathbf{\acute{E}t}_{/X}$, if for any $(m,n) \in \Sigma^2$, we set $U_{mn} = U_m \times_U U_n$ and denote by $f_{mn}\colon U_{mn} \to U$ the canonical morphism, the sequence of morphisms of sheaves of $U_{\overline{\eta},\mathrm{f\acute{e}t}}$*

$$F_U \to \prod_{n\in\Sigma} (f_{n,\overline{\eta}})_{\mathrm{f\acute{e}t}*}(F_{U_n}) \rightrightarrows \prod_{(m,n)\in\Sigma^2} (f_{mn,\overline{\eta}})_{\mathrm{f\acute{e}t}*}(F_{U_{mn}}) \tag{I.5.3.1}$$

is exact.

From now on, we will identify every sheaf F on E with the associated functor $\{U \mapsto F_U\}$, the sheaf F_U being the restriction of F to the fiber $\mathbf{\acute{E}t}_{\mathrm{f}/U_{\overline{\eta}}}$ of π over U.

I.5.4. The canonical injection functor $\mathbf{\acute{E}t}_{\mathrm{f}/X_{\overline{\eta}}} \to E$ is continuous and left exact (VI.5.32). It therefore defines a morphism of topos

$$\beta\colon \widetilde{E} \to Y_{\mathrm{f\acute{e}t}}. \tag{I.5.4.1}$$

Likewise, the functor

$$\sigma^+\colon \mathbf{\acute{E}t}_{/X} \to E, \quad U \mapsto (U_{\overline{\eta}} \to U) \tag{I.5.4.2}$$

is continuous and left exact (VI.5.32). It therefore defines a morphism of topos

$$\sigma\colon \widetilde{E} \to X_{\mathrm{\acute{e}t}}. \tag{I.5.4.3}$$

I.5.5. Let $\overline{x}$ be a geometric point of X and X' the strict localization of X at $\overline{x}$. We denote by E' the Faltings site associated with X', by $\widetilde{E}'$ the topos of sheaves of sets on E', and by

$$\beta'\colon \widetilde{E}' \to X'_{\overline{\eta},\mathrm{f\acute{e}t}} \tag{I.5.5.1}$$

the canonical morphism (I.5.4.1). We prove in VI.10.27 that the functor β'_* is exact. This property is crucial for the study of the main sheaves of the Faltings topos considered in this work. The canonical morphism $X' \to X$ induces, by functoriality, a morphism of topos (VI.10.12)

$$\Phi\colon \widetilde{E}' \to \widetilde{E}. \tag{I.5.5.2}$$

We denote by

$$\varphi_{\overline{x}}\colon \widetilde{E} \to X'_{\overline{\eta},\mathrm{f\acute{e}t}} \tag{I.5.5.3}$$

the composed functor $\beta'_* \circ \Phi^*$.

We denote by $\mathfrak{V}_{\overline{x}}$ the category of $\overline{x}$-pointed étale X-schemes, or, equivalently, the category of neighborhoods of $\overline{x}$ in the site $\mathbf{\acute{E}t}_{/X}$. For every object $(U, \xi\colon \overline{x} \to U)$ of $\mathfrak{V}_{\overline{x}}$, we denote also by $\xi\colon X' \to U$ the X-morphism induced by ξ. We prove in VI.10.37 that for every sheaf $F = \{U \mapsto F_U\}$ of $\widetilde{E}$, we have a functorial canonical isomorphism

(I.5.5.4) $$\varphi_{\overline{x}}(F) \xrightarrow{\sim} \varinjlim_{U \in \mathfrak{V}_{\overline{x}}^{\circ}} (\xi_{\overline{\eta}})^*_{\mathrm{f\acute{e}t}}(F_U).$$

Assume that $\overline{x}$ is over s. We prove (III.3.7) that $\overline{X}'$ is normal and strictly local (and in particular integral). Let $\overline{y}$ be a geometric point of $X'_{\overline{\eta}}$ (which is integral), $\mathbf{B}_{\pi_1(X'_{\overline{\eta}}, \overline{y})}$ the classifying topos of the fundamental group $\pi_1(X'_{\overline{\eta}}, \overline{y})$, and

(I.5.5.5) $$\nu_{\overline{y}}\colon X'_{\overline{\eta}, \mathrm{f\acute{e}t}} \xrightarrow{\sim} \mathbf{B}_{\pi_1(X'_{\overline{\eta}}, \overline{y})}$$

the fiber functor at $\overline{y}$ (VI.9.8.4). The composed functor

(I.5.5.6) $$\widetilde{E} \xrightarrow{\varphi_{\overline{x}}} X'_{\overline{\eta}, \mathrm{f\acute{e}t}} \xrightarrow{\nu_{\overline{y}}} \mathbf{B}_{\pi_1(X'_{\overline{\eta}}, \overline{y})} \longrightarrow \mathbf{Set},$$

where the last arrow is the forgetful functor of the action of $\pi_1(X'_{\overline{\eta}}, \overline{y})$, is a fiber functor (VI.10.31 and VI.9.9). It corresponds to a point of geometric origin of the topos $\widetilde{E}$, denoted by $\rho(\overline{y} \rightsquigarrow \overline{x})$ (cf. III.8.6).

Theorem I.5.6 (cf. VI.10.30)**.** *Under the assumptions of* I.5.5, *for every abelian sheaf F of $\widetilde{E}$ and every integer $i \geq 0$, we have a functorial canonical isomorphism* (I.5.4.3)

(I.5.6.1) $$\mathrm{R}^i\sigma_*(F)_{\overline{x}} \xrightarrow{\sim} \mathrm{H}^i(X'_{\overline{\eta}, \mathrm{f\acute{e}t}}, \varphi_{\overline{x}}(F)).$$

Corollary I.5.7. *We keep the assumptions of* I.5.5 *and moreover assume that x is over s. Then, for every abelian sheaf F of $\widetilde{E}$ and for every integer $i \geq 0$, we have a canonical functorial isomorphism*

(I.5.7.1) $$\mathrm{R}^i\sigma_*(F)_{\overline{x}} \xrightarrow{\sim} \mathrm{H}^i(\pi_1(X'_{\overline{\eta}}, \overline{y}), \nu_{\overline{y}}(\varphi_{\overline{x}}(F))).$$

Proposition I.5.8 (cf. VI.10.32)**.** *When $\overline{x}$ goes through the set of geometric points of X, the family of functors $\varphi_{\overline{x}}$* (I.5.5.3) *is conservative.*

I.5.9. For every object $(V \to U)$ of E, we denote by $\overline{U}^V$ the integral closure of $\overline{U} = U \times_S \overline{S}$ in V and we set

(I.5.9.1) $$\overline{\mathscr{B}}(V \to U) = \Gamma(\overline{U}^V, \mathscr{O}_{\overline{U}^V}).$$

We thus define a presheaf of rings on E, which turns out to be a *sheaf* (III.8.16). Note that $\overline{\mathscr{B}}$ is not in general a sheaf for the topology of E originally defined by Faltings in ([**26**] page 214) (cf. III.8.18). For every $U \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X})$, we denote by $\overline{\mathscr{B}}_U$ the restriction of $\overline{\mathscr{B}}$ to the fiber $\mathbf{\acute{E}t}_{\mathrm{f}/U_{\overline{\eta}}}$ of π over U, so that $\overline{\mathscr{B}} = \{U \to \overline{\mathscr{B}}_U\}$. In I.5.10 below, we give an explicit description of this sheaf. For any integer $n \geq 0$, we set

(I.5.9.2) $$\overline{\mathscr{B}}_n = \overline{\mathscr{B}}/p^n\overline{\mathscr{B}},$$

(I.5.9.3) $$\overline{\mathscr{B}}_{U,n} = \overline{\mathscr{B}}_U/p^n\overline{\mathscr{B}}_U.$$

Note that the correspondence $\{U \mapsto \overline{\mathscr{B}}_{U,n}\}$ naturally forms a presheaf on E whose associated sheaf is canonically isomorphic to $\overline{\mathscr{B}}_n$. It is in general difficult, if not impossible, to describe explicitly the restrictions of $\overline{\mathscr{B}}_n$ to the fibers of the functor π (I.5.2.2). However, its images by the fiber functors (I.5.5.6) are accessible (III.10.8.5).

We denote by $\hbar\colon \overline{X} \to X$ the canonical projection (I.2.1.1) and by

(I.5.9.4) $$\hbar_*(\mathscr{O}_{\overline{X}}) \to \sigma_*(\overline{\mathscr{B}})$$

the homomorphism defined for every $U \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X})$ by the canonical homomorphism

$$\Gamma(\overline{U}, \mathscr{O}_{\overline{U}}) \to \Gamma(\overline{U}^{U_{\overline{\eta}}}, \mathscr{O}_{\overline{U}^{U_{\overline{\eta}}}}). \tag{I.5.9.5}$$

Unless explicitly stated otherwise, we consider $\sigma \colon \widetilde{E} \to X_{\text{ét}}$ (I.5.4.3) as a morphism of ringed topos (by $\overline{\mathscr{B}}$ and $\hbar_*(\mathscr{O}_{\overline{X}})$, respectively).

I.5.10. Let U be an object of $\mathbf{\acute{E}t}_{/X}$, $\overline{y}$ a geometric point of $U_{\overline{\eta}}$, and V the connected component of $U_{\overline{\eta}}$ containing $\overline{y}$. We denote by $\mathbf{B}_{\pi_1(V,\overline{y})}$ the classifying topos of the fundamental group $\pi_1(V,\overline{y})$, by $(V_i)_{i\in I}$ the normalized universal cover of V at $\overline{y}$ (VI.9.8), and by

$$\nu_{\overline{y}} \colon V_{\text{fét}} \xrightarrow{\sim} \mathbf{B}_{\pi_1(V,\overline{y})}, \quad F \mapsto \varinjlim_{i\in I^\circ} F(V_i) \tag{I.5.10.1}$$

the fiber functor at $\overline{y}$. For every $i \in I$, $(V_i \to U)$ is naturally an object of E. We can therefore consider the inverse system of schemes $(\overline{U}^{V_i})_{i\in I}$. We set

$$\overline{R}^{\overline{y}}_U = \varinjlim_{i\in I^\circ} \Gamma(\overline{U}^{V_i}, \mathscr{O}_{\overline{U}^{V_i}}), \tag{I.5.10.2}$$

which is a ring of $\mathbf{B}_{\pi_1(V,\overline{y})}$. By III.8.15, we have a canonical isomorphism of $\mathbf{B}_{\pi_1(V,\overline{y})}$

$$\nu_{\overline{y}}(\overline{\mathscr{B}}_U|V) \xrightarrow{\sim} \overline{R}^{\overline{y}}_U. \tag{I.5.10.3}$$

I.5.11. Since X_η is a subobject of the final object X of $X_{\text{ét}}$, $\sigma^*(X_\eta)$ is a subobject of the final object of $\widetilde{E}$. We denote by

$$\gamma \colon \widetilde{E}_{/\sigma^*(X_\eta)} \to \widetilde{E} \tag{I.5.11.1}$$

the localization morphism of $\widetilde{E}$ at $\sigma^*(X_\eta)$. We denote by $\widetilde{E}_s$ the closed subtopos of $\widetilde{E}$ complement of $\sigma^*(X_\eta)$, that is, the full subcategory of $\widetilde{E}$ made up of the sheaves F such that $\gamma^*(F)$ is a final object of $\widetilde{E}_{/\sigma^*(X_\eta)}$, and by

$$\delta \colon \widetilde{E}_s \to \widetilde{E} \tag{I.5.11.2}$$

the canonical embedding, that is, the morphism of topos such that $\delta_* \colon \widetilde{E}_s \to \widetilde{E}$ is the canonical injection functor. There exists a morphism

$$\sigma_s \colon \widetilde{E}_s \to X_{s,\text{ét}}, \tag{I.5.11.3}$$

unique up to isomorphism, such that the diagram

$$\begin{array}{ccc} \widetilde{E}_s & \xrightarrow{\sigma_s} & X_{s,\text{ét}} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\iota_{\text{ét}}} \\ \widetilde{E} & \xrightarrow{\sigma} & X_{\text{ét}} \end{array} \tag{I.5.11.4}$$

where $\iota \colon X_s \to X$ is the canonical injection, is commutative up to isomorphism (cf. III.9.8).

For every integer $n \geq 1$, if we identify the étale topos of X_s and $\overline{X}_n$ (I.2.1.1) (k being algebraically closed), the morphism σ_s and the homomorphism (I.5.9.4) induce a morphism of ringed topos (III.9.9)

$$\sigma_n \colon (\widetilde{E}_s, \overline{\mathscr{B}}_n) \to (X_{s,\text{ét}}, \mathscr{O}_{\overline{X}_n}). \tag{I.5.11.5}$$

I.5.12. For the remainder of this introduction, we assume that there exists a smooth $\mathscr{A}_2(\overline{S})$-deformation $\widetilde{X}$ of $\check{\overline{X}}$ that we fix (cf. (I.2.1.1) and I.4.2 for the notation):

(I.5.12.1)
$$\begin{array}{ccc} \check{\overline{X}} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \check{\overline{S}} & \longrightarrow & \mathscr{A}_2(\overline{S}) \end{array}$$

Let $Y = \mathrm{Spec}(R)$ be a connected affine object of $\mathbf{\acute{E}t}_{/X}$ admitting an étale S-morphism to $\mathbb{G}^d_{m,S}$ for an integer $d \geq 1$ and such that $Y_s \neq \emptyset$ (in other words, Y satisfies the conditions of I.4.1) and $\widetilde{Y} \to \widetilde{X}$ the unique étale morphism that lifts $\check{\overline{Y}} \to \check{\overline{X}}$. For any geometric point $\overline{y}$ of $Y_{\overline{\eta}}$, we denote by $Y'_{\overline{\eta}}$ the connected component of $Y_{\overline{\eta}}$ containing $\overline{y}$, by

(I.5.12.2)
$$\nu_{\overline{y}}\colon \mathbf{Y}'_{\overline{\eta},\mathrm{f\acute{e}t}} \xrightarrow{\sim} \mathbf{B}_{\pi_1(Y'_{\overline{\eta}},\overline{y})}$$

the fiber functor at $\overline{y}$, and by $\mathscr{F}^{\overline{y}}_Y$ the Higgs–Tate $\widehat{\overline{R}}^{\overline{y}}_Y$-extension associated with $(Y,\widetilde{Y})$ (I.4.3).

For every integer $n \geq 0$, there exists a $\overline{\mathscr{B}}_{Y,n}$-module $\mathscr{F}_{Y,n}$, where $\overline{\mathscr{B}}_{Y,n} = \overline{\mathscr{B}}_Y/p^n\overline{\mathscr{B}}_Y$ (I.5.9.3), unique up to canonical isomorphism, such that for every geometric point $\overline{y}$ of $Y_{\overline{\eta}}$, we have a canonical isomorphism of $\overline{R}^{\overline{y}}_Y$-modules (I.5.10.3)

(I.5.12.3)
$$\nu_{\overline{y}}(\mathscr{F}_{Y,n}|Y'_{\overline{\eta}}) \xrightarrow{\sim} \mathscr{F}^{\overline{y}}_Y/p^n\mathscr{F}^{\overline{y}}_Y.$$

The exact sequence (I.4.3.5) induces a canonical exact sequence of $\overline{\mathscr{B}}_{Y,n}$-modules

(I.5.12.4)
$$0 \to \overline{\mathscr{B}}_{Y,n} \to \mathscr{F}_{Y,n} \to \xi^{-1}\Omega^1_{X/S}(Y) \otimes_{\mathscr{O}_X(Y)} \overline{\mathscr{B}}_{Y,n} \to 0.$$

For every rational number $r \geq 0$, we denote by $\mathscr{F}^{(r)}_{Y,n}$ the extension of $\overline{\mathscr{B}}_{Y,n}$-modules of $Y_{\overline{\eta},\mathrm{f\acute{e}t}}$ deduced from $\mathscr{F}_{Y,n}$ by inverse image under the morphism of multiplication by p^r on $\xi^{-1}\Omega^1_{X/S}(Y) \otimes_{\mathscr{O}_X(Y)} \overline{\mathscr{B}}_{Y,n}$, so that we have a canonical exact sequence of $\overline{\mathscr{B}}_{Y,n}$-modules

(I.5.12.5)
$$0 \to \overline{\mathscr{B}}_{Y,n} \to \mathscr{F}^{(r)}_{Y,n} \to \xi^{-1}\Omega^1_{X/S}(Y) \otimes_{\mathscr{O}_X(Y)} \overline{\mathscr{B}}_{Y,n} \to 0.$$

This induces, for every integer $m \geq 1$, an exact sequence of $\overline{\mathscr{B}}_{Y,n}$-modules

$$0 \to \mathrm{S}^{m-1}_{\overline{\mathscr{B}}_{Y,n}}(\mathscr{F}^{(r)}_{Y,n}) \to \mathrm{S}^{m}_{\overline{\mathscr{B}}_{Y,n}}(\mathscr{F}^{(r)}_{Y,n}) \to \mathrm{S}^{m}_{\overline{\mathscr{B}}_{Y,n}}(\xi^{-1}\Omega^1_{X/S}(Y) \otimes_{\mathscr{O}_X(Y)} \overline{\mathscr{B}}_{Y,n}) \to 0.$$

The $\overline{\mathscr{B}}_{Y,n}$-modules $(\mathrm{S}^{m}_{\overline{\mathscr{B}}_{Y,n}}(\mathscr{F}^{(r)}_{Y,n}))_{m\in\mathbb{N}}$ therefore form a direct system whose direct limit

(I.5.12.6)
$$\mathscr{C}^{(r)}_{Y,n} = \varinjlim_{m\geq 0} \mathrm{S}^{m}_{\overline{\mathscr{B}}_{Y,n}}(\mathscr{F}^{(r)}_{Y,n})$$

is naturally endowed with a structure of $\overline{\mathscr{B}}_{Y,n}$-algebra of $Y_{\overline{\eta},\mathrm{f\acute{e}t}}$.

For all rational numbers $r \geq r' \geq 0$, we have a canonical $\overline{\mathscr{B}}_{Y,n}$-linear morphism

(I.5.12.7)
$$\mathtt{a}^{r,r'}_{Y,n}\colon \mathscr{F}^{(r)}_{Y,n} \to \mathscr{F}^{(r')}_{Y,n}$$

that lifts the morphism of multiplication by $p^{r'-r}$ on $\xi^{-1}\Omega^1_{X/S}(Y) \otimes_{\mathscr{O}_X(Y)} \overline{\mathscr{B}}_{Y,n}$ and that extends the identity on $\overline{\mathscr{B}}_{Y,n}$ (I.5.12.5). It induces a homomorphism of $\overline{\mathscr{B}}_{Y,n}$-algebras

(I.5.12.8)
$$\alpha^{r,r'}_{Y,n}\colon \mathscr{C}^{(r)}_{Y,n} \to \mathscr{C}^{(r')}_{Y,n}.$$

Note that $\mathscr{C}^{(r)}_{Y,n}$ and $\mathscr{F}^{(r)}_{Y,n}$ depend on the choice of the deformation $\widetilde{X}$.

We extend the previous definitions to connected affine objects Y of $\mathbf{\acute{E}t}_{/X}$ such that $Y_s = \emptyset$ by setting $\mathscr{C}^{(r)}_{Y,n} = \mathscr{F}^{(r)}_{Y,n} = 0$.

I.5.13. Let n be an integer ≥ 0 and r a rational number ≥ 0. The correspondences $\{Y \mapsto \mathscr{F}^{(r)}_{Y,n}\}$ and $\{Y \mapsto \mathscr{C}^{(r)}_{Y,n}\}$ naturally form presheaves on the full subcategory of E made up of the objects $(V \to Y)$ such that Y is affine, connected, and admits an étale morphism to $\mathbb{G}^d_{m,S}$ for an integer $d \geq 1$ (cf. III.10.19). Since this subcategory is clearly topologically generating in E, we can consider the associated sheaves in $\widetilde{E}$

$$\mathscr{F}^{(r)}_n = \{Y \mapsto \mathscr{F}^{(r)}_{Y,n}\}^a, \tag{I.5.13.1}$$

$$\mathscr{C}^{(r)}_n = \{Y \mapsto \mathscr{C}^{(r)}_{Y,n}\}^a. \tag{I.5.13.2}$$

These are in fact a $\overline{\mathscr{B}}_n$-module and a $\overline{\mathscr{B}}_n$-algebra of $\widetilde{E}_s$ (III.10.22). We call $\mathscr{F}^{(r)}_n$ the *Higgs–Tate $\overline{\mathscr{B}}_n$-extension of thickness* r and call $\mathscr{C}^{(r)}_n$ the *Higgs–Tate $\overline{\mathscr{B}}_n$-algebra of thickness* r associated with $\widetilde{X}$. We have a canonical exact sequence of $\overline{\mathscr{B}}_n$-modules (I.5.11.5)

$$0 \to \overline{\mathscr{B}}_n \to \mathscr{F}^{(r)}_n \to \sigma^*_n(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n}) \to 0. \tag{I.5.13.3}$$

In III.10.29 we describe explicitly the images of $\mathscr{F}^{(r)}_n$ and $\mathscr{C}^{(r)}_n$ by the fiber functors (I.5.5.3).

For all rational numbers $r \geq r' \geq 0$, the homomorphisms (I.5.12.8) induce a homomorphism of $\overline{\mathscr{B}}_n$-algebras

$$\alpha^{r,r'}_n \colon \mathscr{C}^{(r)}_n \to \mathscr{C}^{(r')}_n. \tag{I.5.13.4}$$

For all rational numbers $r \geq r' \geq r'' \geq 0$, we have

$$\alpha^{r,r''}_n = \alpha^{r',r''}_n \circ \alpha^{r,r'}_n. \tag{I.5.13.5}$$

We have a canonical $\mathscr{C}^{(r)}_n$-linear isomorphism

$$\Omega^1_{\mathscr{C}^{(r)}_n/\overline{\mathscr{B}}_n} \xrightarrow{\sim} \sigma^*_n(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n}) \otimes_{\overline{\mathscr{B}}_n} \mathscr{C}^{(r)}_n. \tag{I.5.13.6}$$

The universal $\overline{\mathscr{B}}_n$-derivation of $\mathscr{C}^{(r)}_n$ corresponds through this isomorphism to the unique $\overline{\mathscr{B}}_n$-derivation

$$d^{(r)}_n \colon \mathscr{C}^{(r)}_n \to \sigma^*_n(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n}) \otimes_{\overline{\mathscr{B}}_n} \mathscr{C}^{(r)}_n \tag{I.5.13.7}$$

that extends the canonical morphism $\mathscr{F}^{(r)}_n \to \sigma^*_n(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n})$. For all rational numbers $r \geq r' \geq 0$, we have

$$p^{r-r'}(\mathrm{id} \otimes \alpha^{r,r'}_n) \circ d^{(r)}_n = d^{(r')}_n \circ \alpha^{r,r'}_n. \tag{I.5.13.8}$$

I.5.14. The inverse systems of objects of $\widetilde{E}_s$ (I.5.4) indexed by the ordered set of natural numbers $\mathbb{N}$ form a topos that we denote by $\widetilde{E}^{\mathbb{N}^\circ}_s$. Section III.7 contains useful results for this type of topos. We denote by $\breve{\overline{\mathscr{B}}}$ the ring $(\overline{\mathscr{B}}_{n+1})_{n\in\mathbb{N}}$ of $\widetilde{E}^{\mathbb{N}^\circ}_s$, by $\mathscr{O}_{\breve{\overline{X}}}$ the ring $(\mathscr{O}_{\overline{X}_{n+1}})_{n\in\mathbb{N}}$ of $X^{\mathbb{N}^\circ}_{s,\text{ét}}$, and by $\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}}$ the $\mathscr{O}_{\breve{\overline{X}}}$-module $(\xi^{-1}\Omega^1_{\overline{X}_{n+1}/\overline{S}_{n+1}})_{n\in\mathbb{N}}$ of $X^{\mathbb{N}^\circ}_{s,\text{ét}}$. The morphisms $(\sigma_{n+1})_{n\in\mathbb{N}}$ (I.5.11.5) induce a morphism of ringed topos

$$\breve{\sigma} \colon (\widetilde{E}^{\mathbb{N}^\circ}_s, \breve{\overline{\mathscr{B}}}) \to (X^{\mathbb{N}^\circ}_{s,\text{ét}}, \mathscr{O}_{\breve{\overline{X}}}). \tag{I.5.14.1}$$

We say that a $\breve{\overline{\mathscr{B}}}$-module $(M_{n+1})_{n\in\mathbb{N}}$ of $\widetilde{E}^{\mathbb{N}^\circ}_s$ is *adic* if for all integers m and n such that $m \geq n \geq 1$, the morphism $M_m \otimes_{\overline{\mathscr{B}}_m} \overline{\mathscr{B}}_n \to M_n$ deduced from the transition morphism $M_m \to M_n$ is an isomorphism.

Let r be a rational number ≥ 0. For all integers $m \geq n \geq 1$, we have a canonical $\overline{\mathscr{B}}_m$-linear morphism $\mathscr{F}_m^{(r)} \to \mathscr{F}_n^{(r)}$ and a canonical homomorphism of $\overline{\mathscr{B}}_m$-algebras $\mathscr{C}_m^{(r)} \to \mathscr{C}_n^{(r)}$ such that the induced morphisms

$$\mathscr{F}_m^{(r)} \otimes_{\overline{\mathscr{B}}_m} \overline{\mathscr{B}}_n \to \mathscr{F}_n^{(r)} \quad \text{and} \quad \mathscr{C}_m^{(r)} \otimes_{\overline{\mathscr{B}}_m} \overline{\mathscr{B}}_n \to \mathscr{C}_n^{(r)} \tag{I.5.14.2}$$

are isomorphisms. These morphisms form compatible systems when m and n vary, so that $(\mathscr{F}_{n+1}^{(r)})_{n\in\mathbb{N}}$ and $(\mathscr{C}_{n+1}^{(r)})_{n\in\mathbb{N}}$ are inverse systems. We call *Higgs–Tate $\breve{\overline{\mathscr{B}}}$-extension of thickness* r associated with $\widetilde{X}$, and denote by $\breve{\mathscr{F}}^{(r)}$, the $\breve{\overline{\mathscr{B}}}$-module $(\mathscr{F}_{n+1}^{(r)})_{n\in\mathbb{N}}$ of $\widetilde{E}_s^{\mathbb{N}^\circ}$. We call *Higgs–Tate $\breve{\overline{\mathscr{B}}}$-algebra of thickness* r associated with $\widetilde{X}$ and denote by $\breve{\mathscr{C}}^{(r)}$, the $\breve{\overline{\mathscr{B}}}$-algebra $(\mathscr{C}_{n+1}^{(r)})_{n\in\mathbb{N}}$ of $\widetilde{E}_s^{\mathbb{N}^\circ}$. These are adic $\breve{\overline{\mathscr{B}}}$-modules. We have an exact sequence of $\breve{\overline{\mathscr{B}}}$-modules

$$0 \to \breve{\overline{\mathscr{B}}} \to \breve{\mathscr{F}}^{(r)} \to \breve{\sigma}^*(\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}}) \to 0. \tag{I.5.14.3}$$

For all rational numbers $r \geq r' \geq 0$, the homomorphisms $(\alpha_n^{r,r'})_{n\in\mathbb{N}}$ induce a homomorphism of $\breve{\overline{\mathscr{B}}}$-algebras

$$\breve{\alpha}^{r,r'} : \breve{\mathscr{C}}^{(r)} \to \breve{\mathscr{C}}^{(r')}. \tag{I.5.14.4}$$

For all rational numbers $r \geq r' \geq r'' \geq 0$, we have

$$\breve{\alpha}^{r,r''} = \breve{\alpha}^{r',r''} \circ \breve{\alpha}^{r,r'}. \tag{I.5.14.5}$$

The derivations $(d_{n+1}^{(r)})_{n\in\mathbb{N}}$ define a morphism

$$\breve{d}^{(r)} : \breve{\mathscr{C}}^{(r)} \to \breve{\sigma}^*(\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}}) \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)} \tag{I.5.14.6}$$

that is none other than the universal $\breve{\overline{\mathscr{B}}}$-derivation of $\breve{\mathscr{C}}^{(r)}$. For all rational numbers $r \geq r' \geq 0$, we have

$$p^{r-r'}(\mathrm{id} \otimes \breve{\alpha}^{r,r'}) \circ \breve{d}^{(r)} = \breve{d}^{(r')} \circ \breve{\alpha}^{r,r'}. \tag{I.5.14.7}$$

I.6. Dolbeault modules

I.6.1. We keep the hypotheses and notation of I.5 in this section. We set $\mathscr{S} = \mathrm{Spf}(\mathscr{O}_C)$ and denote by $\mathfrak{X}$ the formal scheme p-adic completion of $\overline{X}$ and by $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ the p-adic completion of the $\mathscr{O}_{\overline{X}}$-module $\xi^{-1}\Omega^1_{\overline{X}/\overline{S}} = \xi^{-1}\Omega^1_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{\overline{X}}$. We denote by

$$\breve{u} : (X^{\mathbb{N}^\circ}_{s,\text{ét}}, \mathscr{O}_{\breve{\overline{X}}}) \to (X^{\mathbb{N}^\circ}_{s,\text{zar}}, \mathscr{O}_{\breve{\overline{X}}}) \tag{I.6.1.1}$$

the canonical morphism of ringed topos (I.5.14 and III.2.9), by

$$\lambda : (X^{\mathbb{N}^\circ}_{s,\text{zar}}, \mathscr{O}_{\breve{\overline{X}}}) \to (X_{s,\text{zar}}, \mathscr{O}_{\mathfrak{X}}) \tag{I.6.1.2}$$

the morphism of ringed topos whose corresponding direct image functor is the inverse limit (III.7.4), and by

$$\top : (\widetilde{E}_s^{\mathbb{N}^\circ}, \breve{\overline{\mathscr{B}}}) \to (\mathfrak{X}_{\text{zar}}, \mathscr{O}_{\mathfrak{X}}) \tag{I.6.1.3}$$

the composed morphism $\lambda \circ \breve{u} \circ \breve{\sigma}$ (I.5.14.1). For modules, we use the notation $\top^{-1}$ to denote the inverse image in the sense of abelian sheaves, and we keep the notation $\top^*$ for the inverse image in the sense of modules; we do likewise for $\breve{\sigma}$.

We denote by

$$\delta : \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}} \to \mathrm{R}^1\top_*(\breve{\overline{\mathscr{B}}}) \tag{I.6.1.4}$$

the $\mathscr{O}_{\mathfrak{X}}$-linear morphism of $X_{s,\mathrm{zar}}$ composed of the adjunction morphism (III.11.2.5)

$$\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}} \to \top_*(\breve{\sigma}^*(\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}})) \tag{I.6.1.5}$$

and the boundary map of the long exact sequence of cohomology deduced from the canonical exact sequence

$$0 \to \breve{\overline{\mathscr{B}}} \to \breve{\mathscr{F}} \to \breve{\sigma}^*(\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}}) \to 0. \tag{I.6.1.6}$$

Note that the morphism

$$\top^*(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \to \breve{\sigma}^*(\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}}), \tag{I.6.1.7}$$

adjoint to (I.6.1.5), is an isomorphism (III.11.2.6).

Theorem I.6.2 (cf. III.11.8)**.** *There exists a unique isomorphism of graded $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-algebras*

$$\wedge\,(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}[\frac{1}{p}]) \xrightarrow{\sim} \oplus_{i\geq 0}\mathrm{R}^i\top_*(\breve{\overline{\mathscr{B}}})[\frac{1}{p}] \tag{I.6.2.1}$$

whose component in degree one is the morphism $\delta\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ (I.6.1.4).

This statement is the key step in Faltings' approach in p-adic Hodge theory. We encounter it here and there in different forms. Its local Galois form (II.8.21) is a consequence of Faltings' almost purity theorem (II.6.16). The global statement has an integral variant (III.11.3) that follows from the local case by localization (I.5.7).

I.6.3. The canonical exact sequence (I.6.1.6) induces, for every integer $m \geq 1$, an exact sequence

$$0 \to \mathrm{Sym}^{m-1}_{\breve{\overline{\mathscr{B}}}}(\breve{\mathscr{F}}) \to \mathrm{Sym}^{m}_{\breve{\overline{\mathscr{B}}}}(\breve{\mathscr{F}}) \to \breve{\sigma}^*(\mathrm{Sym}^m_{\mathscr{O}_{\breve{\overline{X}}}}(\xi^{-1}\Omega^1_{\breve{\overline{X}}/\breve{\overline{S}}})) \to 0. \tag{I.6.3.1}$$

Proposition I.6.4 (cf. III.11.12)**.** *Let m be an integer ≥ 1. Then:*

(i) *The morphism*

$$\top_*(\mathrm{Sym}^{m-1}_{\breve{\overline{\mathscr{B}}}}(\breve{\mathscr{F}}))[\frac{1}{p}] \to \top_*(\mathrm{Sym}^{m}_{\breve{\overline{\mathscr{B}}}}(\breve{\mathscr{F}}))[\frac{1}{p}] \tag{I.6.4.1}$$

induced by (I.6.3.1) *is an isomorphism.*

(ii) *For every integer $q \geq 1$, the morphism*

$$\mathrm{R}^q\top_*(\mathrm{Sym}^{m-1}_{\breve{\overline{\mathscr{B}}}}(\breve{\mathscr{F}}))[\frac{1}{p}] \to \mathrm{R}^q\top_*(\mathrm{Sym}^{m}_{\breve{\overline{\mathscr{B}}}}(\breve{\mathscr{F}}))[\frac{1}{p}] \tag{I.6.4.2}$$

induced by (I.6.3.1) *is zero.*

The local Galois variant of this statement is due to Hyodo (**[43]** 1.2). It is the main ingredient in the definition of Hodge–Tate local systems.

Proposition I.6.5 (cf. III.11.18)**.** *The canonical homomorphism*

$$\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}] \to \varinjlim_{r\in\mathbb{Q}_{>0}} \top_*(\breve{\mathscr{C}}^{(r)})[\frac{1}{p}] \tag{I.6.5.1}$$

is an isomorphism, and for every integer $q \geq 1$,

$$\varinjlim_{r\in\mathbb{Q}_{>0}} \mathrm{R}^q\top_*(\breve{\mathscr{C}}^{(r)})[\frac{1}{p}] = 0. \tag{I.6.5.2}$$

The local Galois variant of this statement (II.12.5) is mainly due to Tsuji (IV.5.3.4).

I.6.6. We denote by $\mathbf{Mod}(\breve{\overline{\mathscr{B}}})$ the category of $\breve{\overline{\mathscr{B}}}$-modules of $\widetilde{E}_s^{\mathbb{N}^\circ}$, by $\mathbf{Mod}^{\mathrm{ad}}(\breve{\overline{\mathscr{B}}})$ (resp. $\mathbf{Mod}^{\mathrm{aft}}(\breve{\overline{\mathscr{B}}})$) the full subcategory made up of the adic $\breve{\overline{\mathscr{B}}}$-modules (resp. the adic $\breve{\overline{\mathscr{B}}}$-modules of finite type) (I.5.14), and by $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ (resp. $\mathbf{Mod}^{\mathrm{ad}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$, resp. $\mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$) the category of objects of $\mathbf{Mod}(\breve{\overline{\mathscr{B}}})$ (resp. $\mathbf{Mod}^{\mathrm{ad}}(\breve{\overline{\mathscr{B}}})$, resp. $\mathbf{Mod}^{\mathrm{aft}}(\breve{\overline{\mathscr{B}}})$) up to isogeny (I.2.6). The category $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ is abelian and the canonical functors

$$\mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{Mod}^{\mathrm{ad}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \tag{I.6.6.1}$$

are fully faithful. We denote by $\mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}})$ (resp. $\mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}])$) the category of coherent $\mathscr{O}_{\mathfrak{X}}$-modules (resp. $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules) of $X_{s,\mathrm{zar}}$ and by $\mathbf{Mod}^{\mathrm{coh}}_{\mathbb{Q}}(\mathscr{O}_{\mathfrak{X}})$ the category of coherent $\mathscr{O}_{\mathfrak{X}}$-modules up to isogeny. By III.6.16, the canonical functor

$$\mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}) \to \mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]), \quad \mathscr{F} \mapsto \mathscr{F}_{\mathbb{Q}_p} = \mathscr{F} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \tag{I.6.6.2}$$

induces an equivalence of abelian categories

$$\mathbf{Mod}^{\mathrm{coh}}_{\mathbb{Q}}(\mathscr{O}_{\mathfrak{X}}) \xrightarrow{\sim} \mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]). \tag{I.6.6.3}$$

I.6.7. For every rational number $r \geq 0$, we denote also by

$$\breve{d}^{(r)} \colon \breve{\mathscr{C}}^{(r)} \to \top^*(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)} \tag{I.6.7.1}$$

the $\breve{\overline{\mathscr{B}}}$-derivation induced by $\breve{d}^{(r)}$ (I.5.14.6) and the isomorphism (I.6.1.7), that we identify with the universal $\breve{\overline{\mathscr{B}}}$-derivation of $\breve{\mathscr{C}}^{(r)}$. This is a Higgs $\breve{\overline{\mathscr{B}}}$-field with coefficients in $\top^*(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ (I.2.5). We denote by $\mathbb{K}^\bullet(\breve{\mathscr{C}}^{(r)}, p^r\breve{d}^{(r)})$ the Dolbeault complex of the Higgs $\breve{\overline{\mathscr{B}}}$-module $(\breve{\mathscr{C}}^{(r)}, p^r\breve{d}^{(r)})$ (I.2.3) and by $\mathbb{K}^\bullet_{\mathbb{Q}}(\breve{\mathscr{C}}^{(r)}, p^r\breve{d}^{(r)})$ its image in $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$. By (I.5.14.7), for all rational numbers $r \geq r' \geq 0$, the homomorphism $\breve{\alpha}^{r,r'}$ (I.5.14.4) induces a morphism of complexes

$$\breve{\nu}^{r,r'} \colon \mathbb{K}^\bullet(\breve{\mathscr{C}}^{(r)}, p^r\breve{d}^{(r)}) \to \mathbb{K}^\bullet(\breve{\mathscr{C}}^{(r')}, p^{r'}\breve{d}^{(r')}). \tag{I.6.7.2}$$

Proposition I.6.8 (cf. III.11.24)**.** *The canonical morphism*

$$\breve{\overline{\mathscr{B}}}_{\mathbb{Q}} \to \varinjlim_{r \in \mathbb{Q}_{>0}} \mathrm{H}^0(\mathbb{K}^\bullet_{\mathbb{Q}}(\breve{\mathscr{C}}^{(r)}, p^r\breve{d}^{(r)})) \tag{I.6.8.1}$$

is an isomorphism, and for every integer $q \geq 1$,

$$\varinjlim_{r \in \mathbb{Q}_{>0}} \mathrm{H}^q(\mathbb{K}^\bullet_{\mathbb{Q}}(\breve{\mathscr{C}}^{(r)}, p^r\breve{d}^{(r)})) = 0. \tag{I.6.8.2}$$

Observe that filtered direct limits are not a priori representable in the category $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$.

I.6.9. The functor $\top_*$ (I.6.1.3) induces an additive and left exact functor that we denote also by

$$\top_* \colon \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]). \tag{I.6.9.1}$$

For every integer $q \geq 0$, we denote by

$$\mathrm{R}^q\top_* \colon \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]) \tag{I.6.9.2}$$

the qth right derived functor of $\top_*$. By (I.6.6.3), the inverse image functor $\top^*$ induces an additive functor that we denote also by

$$\top^*\colon \mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]) \to \mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}). \tag{I.6.9.3}$$

For every coherent $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module $\mathscr{F}$ and every $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\mathscr{G}$, we have a bifunctorial canonical homomorphism

$$\mathrm{Hom}_{\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}}(\top^*(\mathscr{F}),\mathscr{G}) \to \mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]}(\mathscr{F},\top_*(\mathscr{G})) \tag{I.6.9.4}$$

that is injective (III.12.1.5).

I.6.10. We denote by $\mathbf{HI}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ the category of Higgs $\mathscr{O}_{\mathfrak{X}}$-isogenies with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ (I.2.8) and by $\mathbf{HI}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ the full subcategory made up of the quadruples $(\mathscr{M},\mathscr{N},u,\theta)$ such that $\mathscr{M}$ and $\mathscr{N}$ are coherent $\mathscr{O}_{\mathfrak{X}}$-modules. These are additive categories. We denote by $\mathbf{HI}_{\mathbb{Q}}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ (resp. $\mathbf{HI}^{\mathrm{coh}}_{\mathbb{Q}}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$) the category of objects of $\mathbf{HI}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ (resp. $\mathbf{HI}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$) up to isogeny (I.2.6).

By Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$, we will mean a Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}[\frac{1}{p}]$ (I.2.3). We denote by $\mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}],\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ the category of such modules and by $\mathbf{HM}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}],\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ the full subcategory made up of the Higgs modules whose underlying $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module is coherent.

The functor

$$\begin{array}{rcl}\mathbf{HI}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) & \to & \mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}],\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \\ (\mathscr{M},\mathscr{N},u,\theta) & \mapsto & (\mathscr{M}_{\mathbb{Q}_p},(\mathrm{id}\otimes u^{-1}_{\mathbb{Q}_p})\circ\theta_{\mathbb{Q}_p})\end{array} \tag{I.6.10.1}$$

induces a functor

$$\mathbf{HI}_{\mathbb{Q}}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \to \mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}],\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}). \tag{I.6.10.2}$$

By III.6.21, this induces an equivalence of categories

$$\mathbf{HI}^{\mathrm{coh}}_{\mathbb{Q}}(\mathscr{O}_{\mathfrak{X}},\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \stackrel{\sim}{\to} \mathbf{HM}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}],\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}). \tag{I.6.10.3}$$

Definition I.6.11. We call *Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$* any Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ whose underlying $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module is locally projective of finite type (III.2.8).

I.6.12. Let r be a rational number ≥ 0. We denote by Ξ^r the category of integrable p^r-isoconnections with respect to the extension $\breve{\mathscr{C}}^{(r)}/\breve{\overline{\mathscr{B}}}$ (I.2.9) and by $\Xi^r_{\mathbb{Q}}$ the category of objects of Ξ^r up to isogeny (I.2.6). We denote by $\mathfrak{S}^r$ the functor

$$\mathfrak{S}^r\colon \mathbf{Mod}(\breve{\overline{\mathscr{B}}}) \to \Xi^r, \quad \mathscr{M} \mapsto (\breve{\mathscr{C}}^{(r)}\otimes_{\breve{\overline{\mathscr{B}}}}\mathscr{M},\breve{\mathscr{C}}^{(r)}\otimes_{\breve{\overline{\mathscr{B}}}}\mathscr{M},\mathrm{id},p^r\breve{d}^{(r)}\otimes\mathrm{id}). \tag{I.6.12.1}$$

This induces a functor that we denote also by

$$\mathfrak{S}^r\colon \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \Xi^r_{\mathbb{Q}}. \tag{I.6.12.2}$$

We denote by $\mathscr{K}^r$ the functor

$$\mathscr{K}^r\colon \Xi^r \to \mathbf{Mod}(\breve{\overline{\mathscr{B}}}), \quad (\mathscr{F},\mathscr{G},u,\nabla)\mapsto \ker(\nabla). \tag{I.6.12.3}$$

This induces a functor that we denote also by

$$\mathscr{K}^r\colon \Xi^r_{\mathbb{Q}} \to \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}). \tag{I.6.12.4}$$

It is clear that (I.6.12.1) is a left adjoint of (I.6.12.3). Consequently, (I.6.12.2) is a left adjoint of (I.6.12.4).

If $(\mathscr{N}, \mathscr{N}', v, \theta)$ is a Higgs $\mathscr{O}_{\mathfrak{X}}$-isogeny with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$,

$$(\breve{\mathscr{C}}^{(r)} \otimes_{\breve{\overline{\mathscr{B}}}} \top^*(\mathscr{N}), \breve{\mathscr{C}}^{(r)} \otimes_{\breve{\overline{\mathscr{B}}}} \top^*(\mathscr{N}'), \mathrm{id} \otimes_{\breve{\overline{\mathscr{B}}}} \top^*(v), \mathrm{id} \otimes \top^*(\theta) + p^r \breve{d}^{(r)} \otimes \top^*(v)) \tag{I.6.12.5}$$

is an object of Ξ^r (III.6.12). We thus obtain a functor (I.6.10)

$$\top^{r+} \colon \mathbf{HI}(\mathscr{O}_{\mathfrak{X}}, \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \to \Xi^r. \tag{I.6.12.6}$$

By (I.6.10.3), this induces a functor that we denote also by

$$\top^{r+} \colon \mathbf{HM}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \to \Xi^r_{\mathbb{Q}}. \tag{I.6.12.7}$$

Let $(\mathscr{F}, \mathscr{G}, u, \nabla)$ be an object of Ξ^r. By the projection formula (III.12.4), ∇ induces an $\mathscr{O}_{\mathfrak{X}}$-linear morphism

$$\top_*(\nabla) \colon \top_*(\mathscr{F}) \to \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \top_*(\mathscr{G}). \tag{I.6.12.8}$$

We immediately see that $(\top_*(\mathscr{F}), \top_*(\mathscr{G}), \top_*(u), \top_*(\nabla))$ is a Higgs $\mathscr{O}_{\mathfrak{X}}$-isogeny with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$. We thus obtain a functor

$$\top^r_+ \colon \Xi^r \to \mathbf{HI}(\mathscr{O}_{\mathfrak{X}}, \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \tag{I.6.12.9}$$

that is clearly a right adjoint of (I.6.12.6). The composition of the functors (I.6.12.9) and (I.6.10.1) induces a functor that we denote also by

$$\top^r_+ \colon \Xi^r_{\mathbb{Q}} \to \mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}). \tag{I.6.12.10}$$

Definition I.6.13 (cf. III.12.10)**.** Let $\mathscr{M}$ be an object of $\mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ and $\mathscr{N}$ a Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$.

(i) Let $r > 0$ be a rational number. We say that $\mathscr{M}$ and $\mathscr{N}$ are *r-associated* if there exists an isomorphism of $\Xi^r_{\mathbb{Q}}$

$$\alpha \colon \top^{r+}(\mathscr{N}) \xrightarrow{\sim} \mathfrak{S}^r(\mathscr{M}). \tag{I.6.13.1}$$

We then also say that the triple $(\mathscr{M}, \mathscr{N}, \alpha)$ is *r-admissible*.

(ii) We say that $\mathscr{M}$ and $\mathscr{N}$ are *associated* if there exists a rational number $r > 0$ such that $\mathscr{M}$ and $\mathscr{N}$ are r-associated.

Note that for all rational numbers $r \geq r' > 0$, if $\mathscr{M}$ and $\mathscr{N}$ are r-associated, they are also r'-associated.

Definition I.6.14 (cf. III.12.11)**.** (i) We call *Dolbeault $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module* any object of the category $\mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ for which there exists an associated Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$.

(ii) We say that a Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ is *solvable* if it admits an associated Dolbeault module.

We denote by $\mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ the full subcategory of $\mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ made up of the Dolbeault $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-modules and by $\mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ the full subcategory of $\mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ made up of the solvable Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundles with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$.

I.6.15. For every $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\mathscr{M}$ and all rational numbers $r \geq r' \geq 0$, we have a canonical morphism of $\mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$

$$\top_+^r(\mathfrak{S}^r(\mathscr{M})) \to \top_+^{r'}(\mathfrak{S}^{r'}(\mathscr{M})). \tag{I.6.15.1}$$

We thus obtain a filtered direct system $(\top_+^r(\mathfrak{S}^r(\mathscr{M})))_{r\in\mathbb{Q}_{\geq 0}}$. We denote by $\mathscr{H}$ the functor

$$\mathscr{H}\colon \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}), \quad \mathscr{M} \mapsto \varinjlim_{r\in\mathbb{Q}_{>0}} \top_+^r(\mathfrak{S}^r(\mathscr{M})). \tag{I.6.15.2}$$

For every object $\mathscr{N}$ of $\mathbf{HM}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$ and all rational numbers $r \geq r' \geq 0$, we have a canonical morphism of $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$

$$\mathscr{K}^r(\top^{r+}(\mathscr{N})) \to \mathscr{K}^{r'}(\top^{r'+}(\mathscr{N})). \tag{I.6.15.3}$$

We thus obtain a filtered direct system $(\mathscr{K}^r(\top^{r+}(\mathscr{N})))_{r\geq 0}$. Note, however, that filtered direct limits are not a priori representable in the category $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$.

Proposition I.6.16 (cf. III.12.18)**.** *For every Dolbeault $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\mathscr{M}$, $\mathscr{H}(\mathscr{M})$* (I.6.15.2) *is a solvable Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle associated with $\mathscr{M}$. In particular, $\mathscr{H}$ induces a functor that we denote also by*

$$\mathscr{H}\colon \mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}), \quad \mathscr{M} \mapsto \mathscr{H}(\mathscr{M}). \tag{I.6.16.1}$$

Proposition I.6.17 (cf. III.12.23)**.** *We have a functor*

$$\mathscr{V}\colon \mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \to \mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}), \quad \mathscr{N} \mapsto \varinjlim_{r\in\mathbb{Q}_{>0}} \mathscr{K}^r(\top^{r+}(\mathscr{N})). \tag{I.6.17.1}$$

Moreover, for every object $\mathscr{N}$ of $\mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$, $\mathscr{V}(\mathscr{N})$ is associated with $\mathscr{N}$.

Theorem I.6.18 (cf. III.12.26)**.** *The functors* (I.6.16.1) *and* (I.6.17.1)

$$\mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \underset{\mathscr{V}}{\overset{\mathscr{H}}{\rightleftarrows}} \mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\tfrac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \tag{I.6.18.1}$$

are equivalences of categories quasi-inverse to each other.

Theorem I.6.19 (cf. III.12.34)**.** *Let $\mathscr{M}$ be a Dolbeault $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module and $q \geq 0$ an integer. We denote by $\mathbb{K}^\bullet(\mathscr{H}(\mathscr{M}))$ the Dolbeault complex of the Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle $\mathscr{H}(\mathscr{M})$* (I.2.3)*. We then have a functorial canonical isomorphism of $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules* (I.6.9.2)

$$\mathrm{R}^q\top_*(\mathscr{M}) \xrightarrow{\sim} \mathrm{H}^q(\mathbb{K}^\bullet(\mathscr{H}(\mathscr{M}))). \tag{I.6.19.1}$$

I.6.20. Let $g\colon X' \to X$ be an étale morphism. There exists essentially a unique étale morphism $\widetilde{g}\colon \widetilde{X}' \to \widetilde{X}$ that fits into a Cartesian diagram (I.2.1.1)

$$\begin{tikzcd} \breve{\overline{X}}' \arrow[r] \arrow[d, "\breve{\overline{g}}"'] & \widetilde{X}' \arrow[d, "\widetilde{g}"] \\ \breve{\overline{X}} \arrow[r] & \widetilde{X} \end{tikzcd} , \tag{I.6.20.1}$$

so that $\widetilde{X}'$ is a smooth $\mathscr{A}_2(\overline{S})$-deformation of $\breve{\overline{X}}'$. We associate with $(X', \widetilde{X}')$ objects analogous to those defined earlier for $(X, \widetilde{X})$, which we will denote by the same symbols equipped with an exponent $'$. The morphism g defines by functoriality a morphism of ringed topos (III.8.20)

$$\Phi\colon (\widetilde{E}', \overline{\mathscr{B}}') \to (\widetilde{E}, \overline{\mathscr{B}}). \tag{I.6.20.2}$$

We prove in III.8.21 that Φ identifies with a localization morphism of $(\widetilde{E}, \overline{\mathscr{B}})$ at $\sigma^*(X')$. Furthermore, Φ induces a morphism of ringed topos

$$\breve{\Phi}\colon (\widetilde{E}_s'^{\mathbb{N}^\circ}, \breve{\overline{\mathscr{B}}}') \to (\widetilde{E}_s^{\mathbb{N}^\circ}, \breve{\overline{\mathscr{B}}}). \tag{I.6.20.3}$$

We denote by $\mathfrak{g}\colon \mathfrak{X}' \to \mathfrak{X}$ the extension of $\overline{g}\colon \overline{X}' \to \overline{X}$ to the p-adic completions.

Proposition I.6.21 (cf. III.14.9)**.** *Under the assumptions of* I.6.20*, let moreover* $\mathscr{M}$ *be a Dolbeault* $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$*-module and* $\mathscr{N}$ *a solvable Higgs* $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$*-bundle with coefficients in* $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$*. Then* $\breve{\Phi}^*(\mathscr{M})$ *is a Dolbeault* $\breve{\overline{\mathscr{B}}}'_{\mathbb{Q}}$*-module and* $\mathfrak{g}^*(\mathscr{N})$ *is a solvable Higgs* $\mathscr{O}_{\mathfrak{X}'}[\frac{1}{p}]$*-bundle with coefficients in* $\xi^{-1}\Omega^1_{\mathfrak{X}'/\mathscr{S}}$*. If, moreover,* $\mathscr{M}$ *and* $\mathscr{N}$ *are associated, then* $\breve{\Phi}^*(\mathscr{M})$ *and* $\mathfrak{g}^*(\mathscr{N})$ *are associated.*

We in fact prove that the diagrams of functors

$$\begin{tikzcd}
\mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \arrow[r, "\mathscr{H}"] \arrow[d, "\breve{\Phi}^*"'] & \mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \arrow[r, "\mathscr{V}"] \arrow[d, "\mathfrak{g}^*"] & \mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \arrow[d, "\breve{\Phi}^*"] \\
\mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}') \arrow[r, "\mathscr{H}'"] & \mathbf{HM}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}'}[\frac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}'/\mathscr{S}}) \arrow[r, "\mathscr{V}'"] & \mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}')
\end{tikzcd} \tag{I.6.21.1}$$

are commutative up to canonical isomorphisms (III.14.11).

I.6.22. There exists a unique morphism of topos

$$\psi\colon \widetilde{E}_s^{\mathbb{N}^\circ} \to X_{\text{ét}} \tag{I.6.22.1}$$

such that for every $U \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X})$, $\psi^*(U)$ is the constant inverse system $(\sigma_s^*(U_s))_{\mathbb{N}}$ (I.5.11.3). We denote by $\mathbf{\acute{E}t}_{\mathrm{coh}/X}$ the full subcategory of $\mathbf{\acute{E}t}_{/X}$ made up of étale schemes of finite presentation over X. We have a canonical fibered category

$$\mathrm{MOD}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{\acute{E}t}_{\mathrm{coh}/X} \tag{I.6.22.2}$$

whose fiber over an object U of $\mathbf{\acute{E}t}_{\mathrm{coh}/X}$ is the category $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}|\psi^*(U))$ and the inverse image functor under a morphism $U' \to U$ of $\mathbf{\acute{E}t}_{\mathrm{coh}/X}$ is the restriction functor (I.6.20.2)

$$\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}|\psi^*(U)) \to \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}|\psi^*(U')), \quad \mathscr{M} \mapsto \mathscr{M}|\psi^*(U'). \tag{I.6.22.3}$$

By I.6.21, it induces a fibered category

$$\mathrm{MOD}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{\acute{E}t}_{\mathrm{coh}/X} \tag{I.6.22.4}$$

whose fiber over an object U of $\mathbf{\acute{E}t}_{\mathrm{coh}/X}$ is the category $\mathbf{Mod}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}|\psi^*(U))$.

Proposition I.6.23 (cf. III.15.4)**.** *Let* $\mathscr{M}$ *be an object of* $\mathbf{Mod}^{\mathrm{aft}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ *and* $(U_i)_{i\in I}$ *a covering of* $\mathbf{\acute{E}t}_{\mathrm{coh}/X}$*. Then* $\mathscr{M}$ *is Dolbeault if and only if for every* $i \in I$*, the* $(\breve{\overline{\mathscr{B}}}|\psi^*(U_i))_{\mathbb{Q}}$*-module* $\mathscr{M}|\psi^*(U_i)$ *is Dolbeault.*

Proposition I.6.24 (cf. III.15.5)**.** *The following conditions are equivalent:*

(i) *The fibered category* (I.6.22.4)

$$\mathrm{MOD}^{\mathrm{Dolb}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \to \mathbf{\acute{E}t}_{\mathrm{coh}/X} \tag{I.6.24.1}$$

is a stack ([**35**] II 1.2.1).

(ii) *For every covering* $(U_i \to U)_{i\in I}$ *of* $\mathbf{\acute{E}t}_{\mathrm{coh}/X}$*, denoting by* $\mathscr{U}$ *(resp.* $\mathscr{U}_i$*, for* $i \in I$*) the formal p-adic completion of* $\overline{U}$ *(resp.* $\overline{U}_i$*), a Higgs* $\mathscr{O}_{\mathscr{U}}[\frac{1}{p}]$*-bundle* $\mathscr{N}$ *with coefficients in* $\xi^{-1}\Omega^1_{\mathscr{U}/\mathscr{S}}$ *is solvable if and only if for every* $i \in I$*, the Higgs* $\mathscr{O}_{\mathscr{U}_i}[\frac{1}{p}]$*-bundle* $\mathscr{N} \otimes_{\mathscr{O}_{\mathscr{U}}} \mathscr{O}_{\mathscr{U}_i}$ *with coefficients in* $\xi^{-1}\Omega^1_{\mathscr{U}_i/\mathscr{S}}$ *is solvable.*

Definition I.6.25 (cf. III.15.6)**.** Let $(\mathscr{N}, \theta)$ be Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle with coefficients in $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$.

(i) We say that $(\mathscr{N}, \theta)$ is *small* if there exist a coherent sub-$\mathscr{O}_{\mathfrak{X}}$-module $\mathfrak{N}$ of $\mathscr{N}$ that generates it over $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$ and a rational number $\varepsilon > \frac{1}{p-1}$ such that

$$\theta(\mathfrak{N}) \subset p^{\varepsilon}\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathfrak{N}. \tag{I.6.25.1}$$

(ii) We say that $(\mathscr{N}, \theta)$ is *locally small* if there exists an open covering $(U_i)_{i\in I}$ of X_s such that for every $i \in I$, $(\mathscr{N}|U_i, \theta|U_i)$ is small.

Proposition I.6.26 (cf. III.15.8)**.** *Every solvable Higgs* $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$*-bundle* $(\mathscr{N}, \theta)$ *with coefficients in* $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ *is locally small.*

Proposition I.6.27 (cf. III.15.9)**.** *Suppose that* X *is affine and connected, and that it admits an étale* S*-morphism to* $\mathbb{G}^d_{m,S}$ *for an integer* $d \geq 1$*. Then, every small Higgs* $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$*-bundle with coefficients in* $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ *is solvable.*

Corollary I.6.28 (cf. III.15.10)**.** *Under the conditions of* I.6.24*, every locally small Higgs* $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$*-bundle with coefficients in* $\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}$ *is solvable.*